Ball comparison for three optimal eight order methods under weak conditions

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Abstract. We considered three optimal eighth order method for solving nonlinear equations. In earlier studies Taylors expansions and hypotheses reaching up to the eighth derivative are used to prove the convergence of these methods. These hypotheses restrict the applicability of the methods. In our study we use hypotheses on the first derivative. Numerical examples illustrating the theoretical results are also presented in this study.

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1. Introduction

In this paper we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq S \to T$ is a Fréchet-differentiable operator defined on a convex set D, where S, T are subsets of $\mathbb{R}$ or $\mathbb{C}$.

Equation of the form (1.1) is used to study problems in Computational Sciences and other disciplines [3, 7, 14, 16, 20]. Newton-like iterative methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] are famous for approximating a solution of the equation (1.1).

In this paper, we study the local convergence analysis of the methods defined for each $n = 0, 1, 2, \cdots$ by Siyyam et al. [19]

$$y_n = x_n - \frac{1}{F'(x_n)} F(x_n),$$

$$z_n = x_n + (1+\beta)\frac{1}{F'(x_n)}(F(x_n) + F(y_n)),$$

$$-\frac{1}{F'(x_n)}F(x_n)(F(x_n) - F(y_n))^{-1}F(x_n)$$

$$-\beta(\frac{1}{F'(x_n)}F(x_n) + (F'(x_n) + F^2(x_n)F'(x_n))^{-1}F(y_n)) \qquad (1.2)$$

$$x_{n+1} = z_n - A_n^{-1}F(z_n),$$

where $x_0 \in D$ is an initial point, $\beta \in S$,

$$A_n = F'(x_n) + ([x_n, y_n, z_n; F] + [x_n, x_n, y_n; F])(z_n - x_n)$$
$$+ 2([x_n, y_n, z_n; F] - [x_n, x_n, y_n; F])(z_n - y_n)$$

and $[\cdot, \cdot, \cdot; F]$ denotes a divided difference of order two for function F on D. The second and third method are due to Wang et. al. [23] and are defined, respectively as

$$y_n = x_n - \frac{1}{F'(x_n)}F(x_n),$$

$$z_n = x_n - \frac{1}{F'(x_n)}F(x_n)(F(x_n) - 2F(y_n))^{-1}(F(x_n) - F(y_n)),$$

$$x_{n+1} = z_n - \frac{1}{F'(x_n)}F(z_n) \qquad (1.3)$$

$$\times \left[\frac{1}{2} + \frac{1 + \frac{8F(y_n)}{5F(x_n)} + \frac{2}{5}(\frac{F(y_n)}{F(x_n)})^2}{1 - \frac{12}{5}\frac{1}{F'(x_n)}F(y_n)}(1 + F'(y_n)^{-1}F(z_n))\right],$$

and

$$y_n = x_n - \frac{1}{F'(x_n)}F(x_n),$$

$$z_n = x_n - \frac{1}{F'(x_n)}F(x_n)(F(x_n) - 2F(y_n))^{-1}(F(x_n) - F(y_n)),$$

$$x_{n+1} = z_n - F(x_n)^{-1}F(x_n)\left[\frac{1 - \frac{2}{5}\frac{1}{F'(x_n)}F(y_n) + \frac{1}{5}(F(x_n)^{-1}F(y_n))^2}{1 - \frac{12}{5}\frac{1}{F'(x_n)}F(y_n)}\right.$$

$$\left. + (1 + 4\frac{1}{F'(x_n)}F(y_n))F'(y_n)^{-1}F(z_n)\right]. \qquad (1.4)$$

Convergence ball of high convergence order methods is usually very small and in general decreases as the convergence order increases. The approach in this paper establishes the local convergence result under hypotheses only on the first derivative and give a larger convergence ball than the earlier studies, under weaker hypotheses. Notice that in earlier studies [19, 23] the convergence is shown under hypotheses on the eighth derivative. The same technique can be used to other methods. As a motivational example, define function f on $D = [-\frac{1}{2}, \frac{3}{2})$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad (1.5)$$

Choose $x^* = 1$. We also have that

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$
$$f''(x) = 6x \ln x^2 + 20x^3 + 12x^2 + 10x$$

and

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Notice that $f'''(x)$ is unbounded on D. Hence, the results in [19, 23], cannot apply to show the convergence of method (1.2) (see also the numerical examples).

The rest of the paper is organized as follows. In Section 2 we present the local convergence analysis of methods (1.2)–(1.4). The numerical examples are given in the concluding Section 3.

2. Local convergence

The local convergence of method (1.2), method (1.3) and method (1.4) is based on some functions and parameters. Let $K_0 > 0, K > 0, L_0 > 0, L > 0, M \geq 1$ and $\beta \in S$ be given parameters. Let g_1, p_1, h_{p_1}, p_2 and h_{p_2} be functions defined on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)}$$
$$p_1(t) = \frac{L_0 t}{2} + M g_1(t)$$
$$h_{p_1}(t) = p_1(t) - 1,$$
$$p_2(t) = L_0 t + \frac{M^2 t^2}{1 - L_0 t}$$
$$h_{p_2}(t) = p_2(t) - 1$$

and parameter r_1 by

$$r_1 = \frac{2}{2L_0 + L}. \tag{2.1}$$

We have that $g_1(r_1) = 1$ and for each $t \in [0, r_1) : 0 \leq g_1(t) < 1$. We also get that $h_{p_1}(0) = h_{p_2}(0) = -1 < 0$ and $h_{p_1}(t) \to +\infty, h_{p_2}(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It then follows from the intermediate value theorem that functions p_1 and p_2 have zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_{p_1} and r_{p_2} the smallest such zeros of functions h_{p_1} and r_{p_2}, respectively. Let $\bar{r} = \min\{r_{p_1}, r_{p_2}\}$. Define functions g_2 and h_2 on the interval $[0, \bar{r})$ by

$$g_2(t) = \frac{Lt}{2(1 - L_0 t)} + \frac{2M^2 g_1(t)}{(1 - L_0 t)(1 - p_1(t))}$$
$$+ \frac{|1 + \beta| M g_1(t)}{1 - L_0 t} + \frac{M|\beta| g_1(t)}{1 - p_2(t)}$$

and $h_2(t) = g_2(t) - 1$. We have that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to \bar{r}^-$. Denote by r_2 the smallest zero of function h_2 in the interval $(0, \bar{r})$. Moreover, define

functions q and h_q on the interval $[0, \bar{r})$ by $q(t) = L_0 t + (K + K_0)(1 + g_2(t))t + 2(K_0 + K)(g_1(t) + g_2(t))t$ and $h_q(t) = q(t) - 1$. We get that $h_q(0) = -1 < 0$ and $h_q(t) \to +\infty$ as $t \to \bar{r}_-$. Denote by r_q the smallest zero of function h_q on the interval $(0, \bar{r})$. Let $\bar{r}_0 = \min\{\bar{r}, r_q\}$.

Finally, define functions g_3 and h_3 on the interval $[0, \bar{r}_0)$ by

$$g_3(t) = (1 + \frac{M}{1 - q(t)})g_2(t)$$

and $h_3(t) = g_3(t) - 1$. We get that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \bar{r}_0^-$. Denote by r_3 the smallest zero of function h_3 on the interval $(0, \bar{r}_0)$. Define the radius of convergence r by

$$r = \min\{r_i\}, \ i = 1, 2, 3. \tag{2.2}$$

Then, we have that

$$0 < r < r_1 < \frac{1}{L_0} \tag{2.3}$$

and for each $t \in [0, r)$

$$0 \leq g_i(t) < 1, \ i = 1, 2, 3 \tag{2.4}$$

$$0 \leq p_j(t) < 1, \ j = 1, 2 \tag{2.5}$$

and

$$0 \leq q(t) < 1. \tag{2.6}$$

Let us denote by $U(v, \rho), \overline{U}(v, \rho)$ the open and closed balls in S with center $v \in S$ and of radius $\rho > 0$.

Next, we present the local convergence analysis of method (1.2) using the preceding notation.

Theorem 2.1. *Let $F : D \subset S \to T$ be a differentiable function. Let also $[., ., .; F]$ denote a divided difference of order two for function F on D. Suppose that there exist $x^* \in D$*

$$F(x^*) = 0, \ F'(x^*) \neq 0 \tag{2.7}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x^*\|. \tag{2.8}$$

Moreover, suppose that there exist $L > 0$ and $M \geq 1$ and $K > 0$ such that for each $x, y, z \in D_0 = D \cap U(x^*, \frac{1}{L_0})$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \tag{2.9}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M, \tag{2.10}$$

$$\|F'(x^*)^{-1}[x, x, y; F]\| \leq K_0, \ \|F'(x^*)^{-1}[x, y, z; F]\| \leq K \tag{2.11}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.12}$$

where the radius of convergence r is defined by (2.2). Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \cdots$, and converges to x^*. Moreover, the following estimates hold

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.13}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.15}$$

where the "g" functions are defined previously. Furthermore, for $T \in [r, \frac{2}{L_0})$ the limit point x^* is the only solution of the equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x^*, T)$.

Proof. We shall show that method (1.2) is well defined in $U(x^*, r)$ remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$, and converges to x^* so that estimates (2.13)–(2.15) are satisfied. Using hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.3) and (2.8), we have that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| \leq L_0 r < 1. \tag{2.16}$$

It follows from (2.16) and the Banach Lemma on invertible functions [3, 7, 14] that $F'(x_0) \neq 0$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.17}$$

Hence, y_0 is well defined. By the first sub-step of method (1.2) for $n = 0$, (2.3), (2.4), (2.7), (2.9) and (2.17), we get in turn that

$$\begin{aligned}
\|y_0 - x^*\| &= \|x_0 - x^* - F'(x_0)^{-1}F'(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x^*)\| \\
&\quad \times \|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(x_0 - x^*)) - F'(x_0))(x_0 - x^*)d\theta\| \\
&\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\
&\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \tag{2.18}
\end{aligned}$$

which shows (2.13) for $n = 0$ and $y_0 \in U(x^*, r)$.

We can write by (2.7) that

$$F(y_0) = F(y_0) - F(x^*) = \int_0^1 F'(x^* + \theta(y_0 - x^*))(y_0 - x^*)d\theta. \tag{2.19}$$

Notice that $\|x^* + \theta(y_0 - x^*) - x^*\| = \theta\|y_0 - x^*\| < r$, so $x^* + \theta(y_0 - x^*) \in U(x^*, r)$ for each $\theta \in [0, 1]$. Then, by (2.10), (2.18) and (2.19), we get that

$$\|F(y_0)F'(x^*)^{-1}\| \leq M\|y_0 - x^*\| \leq Mg_1(\|x_0 - x^*\|)\|x_0 - x^*\|. \tag{2.20}$$

We must show in turn that $F(x_0) - F(y_0) \neq 0$ and $F'(x_0) + \frac{F^2(x_0)}{F'(x_0)} \neq 0$. We have by (2.3), (2.5), (2.8) and (2.20) that

$$\begin{aligned}
&\|(F'(x^*)(x_0 - x^*))^{-1}(F(x) - F(x^*) - F'(x^*)(x_0 - x^*) - F(y_0))\| \\
&\leq \|x_0 - x^*\|^{-1}(\frac{L_0}{2}\|x_0 - x^*\|^2 + M\|y_0 - x^*\|) \\
&\leq p_1(\|x_0 - x^*\|) < p_1(r) < 1, \tag{2.21}
\end{aligned}$$

so

$$\|(F(x_0) - F(y_0))^{-1}F'(x^*)\| \leq \frac{1}{\|x_0 - x^*\|(1 - p_1(\|x_0 - x^*\|))}. \tag{2.22}$$

Similarly, by (2.3), (2.5), (2.8) and (2.20) (for $x_0 = y_0$) that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*) + \frac{F^2(x_0)}{F'(x_0)})\|$$

$$\leq \quad L_0\|x_0 - x^*\| + \frac{M^2\|x_0 - x^*\|^2}{1 - L_0\|x_0 - x^*\|} = p_2(\|x_0 - x^*\|)$$

$$< \quad p_2(r) < 1, \tag{2.23}$$

so

$$\|(F'(x_0) + \frac{F^2(x_0)}{F'(x_0)})^{-1}F'(x^*)\| \leq \frac{1}{1 - p_2(\|x_0 - x^*\|)}. \tag{2.24}$$

and z_0 is well defined. Using the second substep of method (1.2), (2.3), (2.17), (2.18), (2.20), (2.22) and (2.24) we obtain in turn that

$$
\begin{aligned}
z_0 - x^* \quad = \quad & x_0 - x^* - F'(x_0)^{-1}F(x_0) + (2 + \beta)F'(x_0)^{-1}F(x_0) \\
& + (1 + \beta)F'(x_0)^{-1}F(y_0) - 2\frac{F^2(x_0)}{F'(x_0)(F(x_0) - F(y_0))} \\
& - \beta F'(x_0)F(x_0) - \beta\frac{F(y_0)}{F'(x_0) + \frac{F^2(x_0)}{F'(x_0)}} \\
= \quad & y_0 - x^* - 2[F'(x^*)^{-1}F(x_0)][F'(x_0)^{-1}F'(x^*)] \\
& \times [(F(x_0) - F(y_0))^{-1}F'(x^*)][F'(x^*)^{-1}F(y_0)] \\
& + (1 + \beta)[F'(x_0)^{-1}F'(x^*)][F'(x^*)^{-1}F(y_0)] \\
& - \beta[F'(x^*)^{-1}F(y_0)][(F'(x_0) + \frac{F^2(x_0)}{F'(x_0)})^{-1}F'(x^*)], \tag{2.25}
\end{aligned}
$$

so

$$
\begin{aligned}
\|z_0 - x^*\| \quad \leq \quad & \|y_0 - x^*\| \\
& + \frac{2M^2\|y_0 - x^*\|\|x_0 - x^*\|}{\|x_0 - x^*\|(1 - L_0\|x_0 - x^*\|)(1 - p_1(\|x_0 - x^*\|))} \\
& + \frac{|1 + \beta|M\|y_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} + \frac{|\beta|M\|y_0 - x^*\|}{1 - p_2(\|x_0 - x^*\|)} \\
= \quad & g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \tag{2.26}
\end{aligned}
$$

which shows (2.14) for $n = 0$ and $z_0 \in U(x^*, r)$. Next, we must show that $A_0 \neq 0$. Using (2.3), (2.6), (2.8), (2.11), (2.18) and (2.26), we get in turn that

$$
\begin{aligned}
& \|F'(x^*)^{-1}(A_0 - F'(x^*))\| \\
\leq \quad & L_0\|x_0 - x^*\| \\
& + (K_0 + K)[\|z_0 - x^*\| + \|x_0 - x^*\|] + 2(K_0 + K)[\|z_0 - x^*\| + \|y_0 - x^*\|] \\
\leq \quad & L_0\|x_0 - x^*\| + (K_0 + K)(1 + g_2(\|x_0 - x^*\|))\|x_0 - x^*\| \\
& 2(K_0 + K)(g_1(\|x_0 - x^*\|) + g_2(\|x_0 - x^*\|))\|x_0 - x^*\| \\
= \quad & q(\|x_0 - x^*\|) < q(r) < 1,
\end{aligned}
$$

so

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1 - q(\|x_0 - x^*\|)} \tag{2.27}$$

and x_1 is well defined. Then, from (2.3), (2.4), (2.18), (2.20) (for $y_0 = z_0$), (2.27), and the last substep of method (1.2) for $n = 0$, we have that

$$
\begin{aligned}
\|x_1 - x^*\| &\leq \|z_0 - x^*\| + \frac{M\|z_0 - x^*\|}{1 - q(\|x_0 - x^*\|)} \\
&= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \tag{2.28}
\end{aligned}
$$

which implies (2.15) holds for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at (2.13)–(2.15). Using the estimate $\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|, c = g_3(\|x_0 - x^*\|) \in [0, 1)$, we deduce that $\lim_{k \to \infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$. The proof of the uniqueness part is standard [5]. $\qquad \square$

Next, we introduce the needed functions as the corresponding ones above Theorem 2.1 but for method (1.3). Define functions $\varphi_1, \varphi_2, \varphi_3, h_{\varphi_1}, h_{\varphi_2}, h_{\varphi_3}$ on the interval $[0, \frac{1}{L_0})$ by

$$\varphi_1(t) = \frac{12}{5}\frac{M g_1(t)}{1 - \frac{L_0}{2}t}, \quad h_{\varphi_1}(t) = \varphi_1(t) - 1,$$

$$\varphi_2(t) = \frac{L_0}{2}t + 2M g_1(t), \quad h_{\varphi_2}(t) = \varphi_2(t) - 1,$$

$$\varphi_3(t) = \frac{L_0}{2}g_1(t)t \text{ and } h_{\varphi_3}(t) = \varphi_3(t) - 1.$$

We have that $h_{\varphi_1}(0) = h_{\varphi_2}(0) = h_{\varphi_3}(0) = -1 < 0$ and $h_{\varphi_1}(t) \to +\infty$, $h_{\varphi_2}(t) \to +\infty$, $h_{\varphi_3}(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. Denote by $r_{\varphi_1}, r_{\varphi_2}, r_{\varphi_3}$ the smallest zero of functions $h_{\varphi_1}, h_{\varphi_2}, h_{\varphi_3}$, respectively on the interval $(0, \frac{1}{L_0})$. Moreover, define functions g_2 and h_2 on the interval $[0, r_{\varphi_2})$ by

$$g_2(t) = (1 + \frac{M^2}{(1 - L_0 t)(1 - \varphi_2(t))})g_1(t)$$

and $h_2(t) = g_2(t) - 1$. We get that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_{\varphi_2}$. Denote by r_2 the smallest such zero. Finally, for

$$\bar{r} = \min\{r_{\varphi_1}, r_{\varphi_2}, r_{\varphi_3}\}$$

define functions g_3 and h_3 on the interval $[0, \bar{r})$ by

$$g_3(t) = \left[1 + \frac{M}{1 - L_0 t} \left(\frac{1}{2} + \frac{1 + \frac{8M g_1(t)}{5\left(1 - \frac{L_0}{2}t\right)} + \frac{2}{5}\left(\frac{M g_1(t)}{1 - \frac{L_0}{2}t}\right)^2}{1 - \varphi_1(t)} \right) \left(\frac{1}{2} + \frac{M \bar{g}_2(t)}{1 - \frac{L_0}{2}t} \right) g_2(t) \right],$$

$$h_3(t) = g_3(t) - 1$$

and

$$\bar{g}_2(t) = 1 + \frac{M^2}{(1 - L_0 t)(1 - \varphi_2(t))}.$$

We have that $h_3(0) = -1 < 0$ and $h_3(t) \to +\infty$ as $t \to \bar{r}^-$. Denote by r_3 the smallest zero of function g_3 on the interval $(0, \bar{r})$. Define the radius of convergence ρ_1 by

$$\rho_1 = \min\{r_i\}, \ i = 1, 2, 3. \tag{2.29}$$

Finally, for method (1.4), define functions g_1 and g_2 as in method (1.3) but define function g_3 and h_3 by

$$g_3(t) = \left[1 + \frac{M}{1 - L_0 t} \frac{1 + \frac{2Mg_1(t)}{5\left(1 - \frac{L_0}{2}t\right)} + \frac{1}{5}\left(\frac{Mg_1(t)}{1 - \frac{L_0}{2}t}\right)^2}{1 - \varphi_1(t)} \left(1 + \frac{4Mg_1(t)}{1 - \frac{L_0}{2}t}\right) \frac{M\bar{g}_2(t)}{1 - \varphi_3(t)} \right] g_2(t),$$

$$h_3(t) = g_3(t) - 1$$

and radius of convergence ρ_2 by

$$\rho_2 = \min\{r_i\}, i = 1, 2, 3. \tag{2.30}$$

Next, drop the hypotheses on the divided differences and K from Theorem 2.1 and exchange the "g" functions and r with the corresponding "g" functions for method (1.3), ρ_1 and method (1.4), ρ_2. Call the resulting hypotheses (C) and (H), respectively. Then, we obtain the corresponding results.

Theorem 2.2. *Under the (C) hypotheses the conclusions of Theorem* 2.1 *hold for method* (1.3) *with ρ_1 replacing r.*

Theorem 2.3. *Under the (H) hypotheses the conclusions of Theorem* 2.1 *hold for method* (1.4) *with ρ_2 replacing r.*

Remark 2.4. (a) The radius r_1 was obtained by Argyros in [2] as the convergence radius for Newton's method under condition (2.13)-(2.15). Notice that the convergence radius for Newton's method given independently by Rheinboldt [18] and Traub [21] is given by

$$\rho = \frac{2}{3L} < r_1.$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$.
Set $D = U(0, 1)$. Then, we have that $L_0 = e - 1 < l = e$, so

$$\rho = 0.24252961 < r_1 = 0.3827.$$

Moreover, the new error bounds [2, 3, 6, 7] are:

$$\|x_{n+1} - x^*\| \le \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [14, 16]

$$\|x_{n+1} - x^*\| \le \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r to be larger than r_1 (see (2.4)) .

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2, 3, 6, 7].

(c) The results can be also be used to solve equations where the operator F' satisfies the autonomous differential equation [3, 7, 14, 16]:

$$F'(x) = p(F(x)),$$

where p is a known continuous operator. Since $F'(x^*) = p(F(x^*)) = p(0)$, we can apply the results without actually knowing the solution x^*. Let as an example $F(x) = e^x - 1$. Then, we can choose $p(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) are not changing if we use the new instead of the old conditions [23]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{\ln \frac{\|x_{n+2} - x_{n+1}\|}{\|x_{n+1} - x_n\|}}{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}, \quad \text{for each } n = 1, 2, \ldots$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2} - x^*\|}{\|x_{n+1} - x^*\|}}{\ln \frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}}, \quad \text{for each } n = 0, 1, 2, \ldots$$

instead of the error bounds obtained in Theorem 2.1. Related work on convergence orders can be found in [8].

(e) In view of (2.9) and the estimate

$$\begin{aligned} \|F'(x^*)^{-1} F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\| \end{aligned}$$

condition (2.11) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t$$

or

$$M(t) = M = 2,$$

since $t \in [0, \frac{1}{L_0})$.

3. Numerical Example

We present a numerical example in this section.

Example 3.1. Returning back to the motivation example at the introduction on this paper, we have $L_0 = L = 96.662907$, $M = 1.0631$, $K = K_0 = \frac{L}{2}, \beta = -1$. Then, the parameters for method (1.2) are

$$r_1 = 0.0069, \ r_2 = 0.0051 = r, \ r_3 = 0.1217.$$

We have $ACOC = 1.7960$ and $COC = 1.8371$.

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