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ORIGINAL RESEARCH

Extended Newton-type iteration for nonlinear ill-posed equations in Banach space

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Abstract In this paper, we study nonlinear ill-posed equations involving m-accretive mappings in Banach spaces. We produce an extended Newton-type iterative scheme that converges cubically to the solution which uses assumptions only on the first Fréchet derivative of the operator. Using general Hölder type source condition we obtain an error estimate. We also use the adaptive parameter choice strategy proposed by Pereverzev and Schock (SIAM J Numer Anal 43(5):2060–2076, 2005) for choosing the regularization parameter.

Keywords Extended Newton iterative scheme · Nonlinear ill-posed problem · Banach space · Lavrentiev regularization · m-Accretive mappings · Adaptive parameter choice strategy

Mathematics Subject Classification 47J06 · 47J05 · 65J20 · 47H06 · 49J30

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1 Introduction

Let E be a real Banach space with its dual space denoted by E^*. The norm of E and E^* are represented by $\| \cdot \|$. We also write $\langle u, j\rangle$ instead of $j(u)$ for $j \in E^*$ and $u \in E$. In this paper we discuss the problem of approximately solving the non linear ill-posed equation

$$F(u) = f, \quad f \in E, \tag{1.1}$$

where $F : D(F) \subseteq E \to E$ is an m-accretive, Fréchet differentiable and single valued nonlinear mapping. The Fréchet derivative of F at u is denoted by $F'(u)$. Note that F is an m-accretive and single valued in E means, F has the following properties [8,22]

1. $\langle F(u) - F(v), J(u - v)\rangle \geq 0$, where J is the dual mapping on E.
2. $R(F + \lambda I) = E$ for each $\lambda > 0$ where $R(F)$ and I denote the range of F and the identity mapping on E, respectively.

Therefore, if F is m-accretive, then for any fixed $f \in E$ and for all $\alpha > 0$ the equation

$$F(u) + \alpha(u - u_0) = f \tag{1.2}$$

has a unique solution u_α [1–7,22] where u_0 is the initial guess of the exact solution $\hat{u}$ (which is assumed to exist) for (1.1). But in practice, one has to deal with noisy data f^δ instead of f with,

$$\| f^\delta - f \| \leq \delta \longrightarrow 0. \tag{1.3}$$

So (1.2) must be changed to a practical form given by,

$$F(u) + \alpha(u - u_0) = f^\delta. \tag{1.4}$$

The above equation has a unique solution u_α^δ as F is m-accretive in E. This unique solution u_α^δ is called the Lavrentiev regularized solution [9,10,15,18,20,21] of (1.1).

In earlier studies such as [2,5,6,8,22,23], the order of convergence for $\|u_\alpha^\delta - \hat{u}\|$ is obtained under the assumption

$$u_0 - \hat{u} = F'(\hat{u})z, \tag{1.5}$$

for some $z \in E$. In this study we consider the Hölder type source condition

$$u_0 - \hat{u} = F'(u_0)^\nu z \quad 0 < \nu \leq 1, \tag{1.6}$$

where $z \in E$ and obtain an error estimate for $\|u_\alpha^\delta - \hat{u}\|$ in a Banach space setting. Since F is nonlinear, most of the solution methods for (1.4) are iterative. In this study we look at the iterative method considered in Xiao and Yin [16] for approximating solution $\hat{u}$ of the equation $F(u) = 0$, where $F : \mathbb{R}^n \to \mathbb{R}^n$, is properly modified to approximate u_α^δ. In [17], Xiao and Yin considered the method defined iteratively for

$k = 0, 1, 2, \ldots$ by

$$v_k = u_k - aF'(u_k)^{-1} F(u_k)$$
$$w_k = u_k - \frac{1}{2}\left\{\left(\frac{1}{a}F'(v_k) + \left(1 - \frac{1}{a}\right)F'(u_k)\right)^{-1} + F'(u_k)^{-1}\right\} F(u_k),$$
$$u_{k+1} = w_k - \left\{\frac{1}{a}F'(v_k) + \left(1 - \frac{1}{a}\right)F'(u_k)\right\}^{-1} F(w_k).$$

In [17], Xiao and Yin proved that the method defined above is well defined and converges cubically to $\hat{u}$.

Recall that a sequence u_k in E with $\lim u_k = \hat{u}$ is said to be cubically convergent to $\hat{u}$, if there exists positive reals C and γ such that for all $k \in \mathbb{N}$

$$\| u_k - \hat{u} \| \leq Ce^{-\gamma 3^k}. \tag{1.7}$$

For a detailed discussion of convergence rates, see [11,13].

We modified the above method of Xiao and Yin [17] to solve the ill-posed Eq. (1.1). Precisely, we consider the iteration defined for each $k = 0, 1, 2, \ldots$ by

$$v_k = u_k - aR'_\alpha(u_k)^{-1} R_\alpha(u_k), \tag{1.8}$$
$$w_k = u_k - \frac{1}{2}\left\{\left(\frac{1}{a}R'_\alpha(v_k) + \left(1 - \frac{1}{a}\right)R'_\alpha(u_k)\right)^{-1} + R'_\alpha(u_k)^{-1}\right\} R_\alpha(u_k), \tag{1.9}$$
$$u_{k+1} = w_k - \left\{\frac{1}{a}R'_\alpha(v_k) + \left(1 - \frac{1}{a}\right)R'_\alpha(u_k)\right\}^{-1} R_\alpha(w_k), \tag{1.10}$$

where,

$$R_\alpha(u) := F(u) + \alpha(u - u_0) - f^\delta, \tag{1.11}$$
$$R'_\alpha(.)h := F'(.)h + \alpha h, \tag{1.12}$$

where $\alpha > 0$ is the regularization parameter and the scalar parameter a will be defined later.

In this study we use assumptions only on the first Fréchet derivative of F to obtain the error estimate for $\|u_k - \hat{u}\|$ under the general source condition (1.6) for $0 < \nu \leq 1$. The advantage of the source condition (1.6) is that it depends on the known u_0.

The rest of the paper is organized as follows. The convergence analysis of the iterative scheme is given in Sect. 2. Error estimate using Hölder-type source condition is given in Sect. 3. In Sect. 4 we present an algorithm for implementing the adaptive rule. Section 5 contains a numerical example. The paper ends with a conclusion given in Sect. 6.

2 Iterative method and convergence analysis

In order for us to present the convergence analysis, it is convenient to introduce some notations. Let,

$$e_k = u_k - u_\alpha^\delta, \tag{2.1}$$
$$\hat{e_k} = v_k - u_\alpha^\delta, \tag{2.2}$$
$$\bar{e_k} = w_k - u_\alpha^\delta. \tag{2.3}$$

Let $r = \|\hat{u} - u_0\|$ and $r_0 \leq 2r + 1$. Next, we define some scalar parameters: For $0 < k_0 < \frac{\sqrt{17}-3}{4}$, let

$$\hat{R} = \frac{1}{1 - k_0 r_0}, \quad C^{k_0,a} = |1-a| + ak_0 + ak_0\,(1+k_0)\,\hat{R},$$
$$C = \frac{k_0[C_{k_0,a} + |1-a|]}{a}, \quad \bar{R} = \frac{1}{1 - Cr_0},$$
$$\tilde{C} = k_0 + (1+k_0)\left(\frac{\bar{R}C}{2} + \frac{\hat{R}k_0}{2}\right) \quad \text{and}$$
$$\Lambda = \tilde{C}C\bar{R}\left(1 + k_0\tilde{C}\right) + k_0\tilde{C}^2.$$

The preceding constants depend on k_0, r_0 and a. We shall replace them with constants depending on k_0 and a which constitute part of the initial data. Choose $r_0 \in \left(0, \frac{1}{2k_0}\right)$. Then, $\hat{R} \leq \hat{R}_1 := 2$. Define

$$C_1^{k_0,a} = |1-a| + ak_0 + 2ak_0(1+k_0),$$
$$C_1 = \frac{k_0[C_1^{k_0,a} + |1-a|]}{a}$$

and

$$\tilde{R}_1 = \frac{1}{1 - C_1 r_0}.$$

Then, we have
$C^{k_0,a} \leq C_1^{k_0,a}$ and $C \leq C_1$. Choose $r_0 \in \left(0, \min\left\{\frac{1}{2k_0}, \frac{1}{2C_1}\right\}\right)$. Then, we have

$$\bar{R} \leq \tilde{R}_1 \leq \hat{R}_1 = 2.$$

Moreover, define $\tilde{C}_1 = k_0 + (1+k_0)(C_1 + k_0)$ and $\Lambda_1 = 2\tilde{C}_1 C_1(1 + k_0\tilde{C}_1) + k_0\tilde{C}_1^2$. Then, we have

$$\tilde{C} \leq \tilde{C}_1$$

and

$$\Lambda \leq \Lambda_1.$$

Hereafter, we assume that

$$\delta \in \left(\min\left\{\alpha, \frac{\alpha}{k_0}, \frac{\alpha}{C_1}, \frac{\alpha}{\tilde{C}_1}, \frac{\alpha}{\Lambda_1}\right\}, \alpha_0 \right), \tag{2.4}$$

for some $\alpha_0 > \min\left\{\alpha, \frac{\alpha}{k_0}, \frac{\alpha}{C_1}, \frac{\alpha}{\tilde{C}_1}, \frac{\alpha}{\Lambda_1}\right\}$. Moreover, we assume that

$$0 < a < \frac{2}{2k_0^2 + 3k_0 + 1}. \tag{2.5}$$

Furthermore, we assume that

$$r < r_1 := \frac{1}{2}\min\left\{1 - \frac{\delta}{\alpha}, \frac{1}{k_0} - \frac{\delta}{\alpha}, \frac{1}{C_1} - \frac{\delta}{\alpha}, \frac{1}{\tilde{C}_1} - \frac{\delta}{\alpha}, \frac{1}{\Lambda_1} - \frac{\delta}{\alpha}\right\}, \tag{2.6}$$

where δ is as in (2.4). Notice that r_1 depends only on the initial data α, a, k_0.

Remark 2.1 Note that by (2.5) and (2.6) we have

$$r_0 < \bar{r}_0 := \min\left\{1, \frac{1}{k_0}, \frac{1}{C_1}, \frac{1}{\tilde{C}_1}, \frac{1}{\Lambda_1}\right\} \quad \text{and} \quad C_1^{k_0, a} < 1. \tag{2.7}$$

We shall assume that

$$0 < r_0 < \min\left\{2r_1 + 1, \bar{r}_0, \frac{1}{2k_0}, \frac{1}{2C_1}\right\}. \tag{2.8}$$

Notice that r_0 depends on α, a and k_0. Next, we see that the Lipschitz-type constant k_0 depends on $D(F)$ which is part of the initial data.

By $B(w, d)$, we denote the open ball in E with center $w \in E$ and radius $d > 0$. The ball $\bar{B}(w, d)$ denote the closure of $B(w, d)$. The following assumption is used to prove the results in this paper.

Assumption 2.2 (*c.f.* [2,10,15,20,21]) There exists a constants $0 \leq l_0, l_1 < \frac{\sqrt{17}-3}{4}$ such that for every $u_1, u_2 \in D(F)$ and $v \in E$ there exists an element $\Phi(u_2, u_1, v) \in E$ such that $[F'(u_2) - F'(u_1)]v = F'(u_1)\Phi(u_2, u_1, v)$, $\|\Phi(u_2, u_1, v)\| \leq l_0\|v\|\|u_2 - u_1\|$, $\|\frac{d}{dv}\Phi(u_2 + tv, u_2, v)\| \leq l_1\|v\|$ for $t \in [0, 1]$ and $B(u_\alpha^\delta, r_0) \subseteq D(F)$.

Let $k_0 = \max\{l_0, 2l_1\}$. Notice that $k_0 = k_0(D(F))$, i.e., k_0 depends on the initial data. Then, knowing the rest of the initial data a and α we can compute all the preceding introduced parameters. Since F is m-accretive and Fréchet differentiable on E, for any real number $\alpha > 0$ and $u \in E$, $F'(u) + \alpha I$ is invertible (see [22]),

$$\|(F'(u) + \alpha I)^{-1}\| \leq \frac{1}{\alpha} \tag{2.9}$$

and

$$\|(F'(u) + \alpha I)^{-1} F'(u)\| \leq 2. \tag{2.10}$$

Let,

$$R_\alpha(u_k) = F(u_k) + \alpha(u_k - u_0) - f^\delta \tag{2.11}$$

and $\Gamma = F'(u_\alpha^\delta) + \alpha I$. Then since $R_\alpha(u_\alpha^\delta) = F(u_\alpha^\delta) + \alpha(u_\alpha^\delta - u_0) - f^\delta = 0$, we have by Assumption 2.2,

$$\begin{aligned} R_\alpha(u_k) &= F(u_k) - F\left(u_\alpha^\delta\right) + \alpha\left(u_k - u_\alpha^\delta\right) \\ &= \int_0^1 F'\left(u_\alpha^\delta + te_k\right) e_k dt + \alpha e_k \\ &= \left[F'\left(u_\alpha^\delta\right) + \alpha I\right] e_k + \int_0^1 \left[F'\left(u_\alpha^\delta + te_k\right) - F'\left(u_\alpha^\delta\right)\right] e_k dt \\ &= \Gamma\{e_k + \Gamma^{-1} \int_0^1 \left[F'\left(u_\alpha^\delta + te_k\right) - F'\left(u_\alpha^\delta\right)\right] e_k dt\} \\ &= \Gamma\{e_k + \int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right) \phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right) dt\}. \end{aligned} \tag{2.12}$$

Differentiating (2.12) with respect to e_k we obtain,

$$R'_\alpha(u_k)(h) = \Gamma\left\{I + \frac{d}{de_k}\left\{\int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right) \phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right) dt\right\}\right\}(h). \tag{2.13}$$

Let $M_k(e_k) = \int_0^1 \Gamma^{-1} F'(u_\alpha^\delta)\phi(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k)dt$ and $\bar{M}_k = \frac{d}{de_k} M_k(e_k)$, then

$$R'_\alpha(u_k)(h) = \Gamma\left\{I + \bar{M}_k\right\}(h). \tag{2.14}$$

Suppose that $u_k \in B(u_\alpha^\delta, r_0)$. Then, we have

$$\begin{aligned} \|\bar{M}_k\| &= \left\|\int_0^1 \frac{d}{de_k}\left\{\Gamma^{-1} F'\left(u_\alpha^\delta\right) \phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right) dt\right\}\right\| \\ &\leq \int_0^1 \left\|\Gamma^{-1} F'\left(u_\alpha^\delta\right)\right\| \left\|\frac{d}{de_k}\left\{\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)\right\| dt\right\} \\ &\leq 2l_1\|e_k\| \leq k_0\|e_k\| \\ &\leq k_0 r_0 < 1. \end{aligned} \tag{2.15}$$

The last inequality follows from (2.8) and Assumption 2.2. Therefore $(I + \bar{M}_k)$ is invertible and its inverse is given by

$$\left(I + \bar{M}_k\right)^{-1} = I - \bar{M}_k + \bar{M}_k^2 \cdots . \tag{2.16}$$

So by (2.14), we have

$$R'_\alpha(u_k)^{-1} = \left(I - \bar{M}_k + \bar{M}_k^2 \cdots\right)\Gamma^{-1}. \tag{2.17}$$

Now by replacing e_k by $\hat{e_k}$ and u_k by v_k in (2.13) we get

$$R'_\alpha(v_k)(h) = \Gamma\left\{I + \frac{d}{d\hat{e_k}}\left\{\int_0^1 \Gamma^{-1}F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + t\hat{e_k}, u_\alpha^\delta, \hat{e_k}\right)dt\right\}\right\}(h). \tag{2.18}$$

We obtain again by (1.8),

$$\begin{aligned}
\hat{e_k} &= e_k - aR'_\alpha(u_k)^{-1}R_\alpha(u_k)\\
&= e_k - a\left\{\left\{I - \bar{M}_k + M_k^2 \cdots\right\}\Gamma^{-1}\Gamma\left\{e_k + \int_0^1 \Gamma^{-1}F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)dt\right\}\right\}\\
&= (1-a)e_k - a\int_0^1 \Gamma^{-1}F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)dt + a\bar{M}_k\left(I - \bar{M}_k + \bar{M}_k^2 \cdots\right)e_k\\
&\quad + a\bar{M}_k\left(I - \bar{M}_k + \bar{M}_k^2 \cdots\right)\int_0^1 \Gamma^{-1}F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)dt.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|\hat{e_k}\| &= \|(1-a)e_k - a\int_0^1 \Gamma^{-1}F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)dt\\
&\qquad + a\bar{M}_k\left(I - \bar{M}_k + \bar{M}_k^2 \cdots\right)e_k\\
&\qquad + a\bar{M}_k\left(I - \bar{M}_k + \bar{M}_k^2 \cdots\right)\int_0^1 \Gamma^{-1}F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)dt\|\\
&\le |1-a|\|e_k\| + ak_0\|e_k\|^2 + a\|e_k\|\frac{\|\bar{M}_k\|}{1-\|\bar{M}_k\|} + ak_o\|e_k\|^2\frac{\|\bar{M}_k\|}{1-\|\bar{M}_k\|}\\
&\le |1-a|\|e_k\| + ak_0\|e_k\|^2 + a\|e_k\|^2k_0\hat{R} + ak_o\|e_k\|^3k_0\hat{R}\\
&\le \|e_k\|\left\{|1-a| + ak_0 + ak_0(1+k_0)\hat{R}\right\}\\
&= \|e_k\|C_1^{k_0,a}.
\end{aligned} \tag{2.19}$$

In the last, but one step we use the fact that $\|e_k\| \le r_0 < 1$. Therefore by (2.19) and (2.7) we get $v_k \in B(u_\alpha^\delta, r_0)$.

Let $N_k(\hat{e_k}) = \int_0^1 \Gamma^{-1} F'(u_\alpha^\delta)\phi(u_\alpha^\delta + t(\hat{e_k}), u_\alpha^\delta, \hat{e_k})dt$ and $\bar{N}_k = \frac{d}{d\hat{e_k}} N_k(\hat{e_k})$. Then,

$$R'_\alpha(v_k)(h) = \Gamma\{I + \bar{N}_k\}(h). \tag{2.20}$$

We also have,

$$\begin{aligned}\|\bar{N}_k\| &= \left\| \int_0^1 \frac{d}{d\hat{e_k}} \left\{ \Gamma^{-1} F'\left(u_\alpha^\delta\right) \phi\left(u_\alpha^\delta + t\hat{e_k}, u_\alpha^\delta, \hat{e_k}\right) dt \right\} \right\| \\ &\leq \int_0^1 \left\| \Gamma^{-1} F'\left(u_\alpha^\delta\right) \right\| \left\| \frac{d}{d\hat{e_k}} \left\{ \phi\left(u_\alpha^\delta + t\hat{e_k}, u_\alpha^\delta, \hat{e_k}\right) \right\| dt \right\} \\ &\leq 2l_1 \|\hat{e_k}\| \leq k_0 \|\hat{e_k}\|.\end{aligned}$$

Let $H_k = \frac{1}{a} R'_\alpha(v_k) + (1 - \frac{1}{a}) R'_\alpha(u_k)$.
Then,

$$\begin{aligned}H_k &= \Gamma \left\{ \frac{1}{a}\{I + \bar{N}_k\} + \left(1 - \frac{1}{a}\right) \{I + \bar{M}_k\} \right\} \\ &= \Gamma \left\{ I + \frac{1}{a}\bar{N}_k + \left(1 - \frac{1}{a}\right) \bar{M}_k \right\} \\ &= \Gamma\{I + P_k\}\end{aligned} \tag{2.21}$$

where $P_k = \frac{1}{a}\bar{N}_k + (1 - \frac{1}{a})\bar{M}_k$. Now,

$$\begin{aligned}\|P_k\| &= \|\frac{1}{a}\bar{N}_k + \left(1 - \frac{1}{a}\right)\bar{M}_k\| \\ &\leq \frac{\|\hat{e_k}\|k_0}{a} + \frac{|a-1|}{a}\|e_k\|k_0 \\ &\leq \|e_k\| \left\{ \frac{k_0 C_1^{k_0,a} + |a-1|k_0}{a} \right\} \\ &< r_0 \frac{k_0 \left[C_1^{k_0,a} + |1-a| \right]}{a} = r_0 C_1 < 1.\end{aligned} \tag{2.22}$$

The last inequality follows from (2.7). This implies H_k is invertible and its inverse is given by:

$$H_k^{-1} = \left\{ I - P_k + P_k^2 \dots \right\} \Gamma^{-1}. \tag{2.23}$$

From (1.9) we have

$$
\begin{aligned}
\bar{e_k} &= e_k - \frac{1}{2}\left\{H_k^{-1} + R_\alpha\left(u_k\right)^{-1}\right\} R_\alpha\left(u_k\right) \\
&= e_k - \frac{1}{2}\left\{\left\{\left(I - P_k + P_k^2 \cdots\right) + \left(I - \bar{M}_k + \bar{M}_k^2 \cdots\right)\right\}\Gamma^{-1}\Gamma\right. \\
&\quad \left. \times \left\{e_k + \int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right) dt\right\}\right\} \\
&= -\int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right) dt + \frac{1}{2} P_k\left(I - P_k + P_k^2 \cdots\right) e_k \\
&\quad + \frac{1}{2}\bar{M}_k\left(I - \bar{M}_k + \bar{M}_k^2 \cdots\right) e_k \\
&\quad + \frac{1}{2} P_k\left(I - P_k + P_k^2 \cdots\right)\int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right) dt \\
&\quad + \frac{1}{2}\bar{M}_k\left(I - \bar{M}_k + \bar{M}_k^2 \cdots\right)\int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right)\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right) dt.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\bar{e_k}\| &\leq \int_0^1 \|\Gamma^{-1} F'\left(u_\alpha^\delta\right)\|\|\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)\| dt + \frac{1}{2}\|e_k\|\frac{\|P_k\|}{1 - \|P_k\|} \\
&\quad + \frac{1}{2}\|e_k\|\frac{\|\bar{M}_k\|}{1 - \|\bar{M}_k\|} \\
&\quad + \frac{1}{2}\frac{\|P_k\|}{1 - \|P_k\|}\int_0^1 \|\Gamma^{-1} F'\left(u_\alpha^\delta\right)\|\|\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)\| dt \\
&\quad + \frac{1}{2}\frac{\|\bar{M}_k\|}{1 - \|\bar{M}_k\|}\int_0^1 \|\Gamma^{-1} F'\left(u_\alpha^\delta\right)\|\|\phi\left(u_\alpha^\delta + te_k, u_\alpha^\delta, e_k\right)\| dt \\
&\leq \|e_k\|^2\left\{k_0 + \frac{1}{2}C_1\tilde{R}_1 + \frac{1}{2}k_0\hat{R}_1 + k_0\frac{1}{2}C_1\tilde{R}_1 + \frac{1}{2}k_0^2\hat{R}_1\right\} \\
&= \tilde{C}_1\|e_k\|^2. \qquad (2.24)
\end{aligned}
$$

Therefore, by (2.24) and (2.7) we get $w_k \in B(u_\alpha^\delta, r_0)$.

Next, using the preceding notation we prove our main result of this section.

Theorem 2.3 *Let R_α be as in (1.11) and suppose that u_k, v_k and $w_k \in B(u_\alpha^\delta, r_0)$. Further let the first derivative of F exists in $B(u_\alpha^\delta, r_0)$. Then $u_{k+1} \in B(u_\alpha^\delta, r_0)$ and the iteration defined in (1.8)–(1.10) converges cubically to u_α^δ. Moreover*

$$\|u_{k+1,\alpha}^\delta - u_\alpha^\delta\| = O\left(e^{-\gamma 3^k}\right),$$

where $\gamma = -ln(\|e_0\|)$.

Proof Since, $u_0 \in B(u_\alpha^\delta, r_0)$, by (2.19), (2.24) and Remark 3.2, we have $v_0, w_0 \in B(u_\alpha^\delta, r_0)$. Suppose $u_k \in B(u_\alpha^\delta, r_0)$. Then by (2.19), (2.24) and Remark 3.2, we have $v_k, w_k \in B(u_\alpha^\delta, r_0)$. Then from (1.8)–(1.10), we have

$$\begin{aligned} e_{k+1} &= \bar{e_k} - \{H_k\}^{-1} R_\alpha(w_k) \\ &= \bar{e_k} - \left\{I - P_k + P_k^2 \cdots\right\} \Gamma^{-1}\Gamma \left\{\bar{e_k} + \int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right) \phi\left(u_\alpha^\delta + t|\bar{e_k}|, u_\alpha^\delta, \bar{e_k}\right) dt\right\} \\ &= -\int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right) \phi\left(u_\alpha^\delta + t|\bar{e_k}|, u_\alpha^\delta, \bar{e_k}\right) dt\} + P_k\left(I - P_k + P_k^2 \cdots\right)\bar{e_k} \\ &\quad + P_k\left(I - P_k + P_k^2 \cdots\right) \int_0^1 \Gamma^{-1} F'\left(u_\alpha^\delta\right) \phi\left(u_\alpha^\delta + t|\bar{e_k}|, u_\alpha^\delta, \bar{e_k}\right) dt. \end{aligned}$$

Thus,

$$\begin{aligned} \|e_{k+1}\| &\leq k_0\|\bar{e_k}\|^2 + \|\bar{e_k}\| \frac{\|P_k\|}{1 - \|P_k\|} + k_0\|\bar{e_k}\|^2 \frac{\|P_k\|}{1 - \|P_k\|} \\ &\leq k_0 \tilde{C}_1^2 \|e_k\|^4 + \|e_k\|^3 \tilde{C}_1 C_1 \bar{R} + k_0 \|e_k\|^5 \tilde{C}_1^2 C_1 \tilde{R}_1 \\ &\leq \|e_k\|^3 \left\{C_1 \tilde{C}_1 \tilde{R}_1 \left(1 + k_0 \tilde{C}_1\right) + k_0 \tilde{C}_1^2\right\} \\ &= \Lambda_1 \|e_k\|^3. \end{aligned} \tag{2.25}$$

Therefore by (2.25) and (2.7) we get $u_{k+1} \in B(u_\alpha^\delta, r_0)$.

Repeated application of (2.25) above leads to

$$\|e_{k+1}\| \leq \Lambda_1^{\frac{3^k - 1}{2}} \|e_0\|^{3^k} = \Lambda_1^{\frac{3^k - 1}{2}} e^{-\gamma 3^k}, \tag{2.26}$$

where $\gamma = -log\|e_0\|$. □

3 Error estimates using Hölder type source condition

Let u_α^δ and u_α be the unique solution of (1.4) and (1.2) respectively. The following results can be found in [22],

$$\|u_\alpha^\delta - u_\alpha\| \leq \frac{\delta}{\alpha} \tag{3.1}$$

and

$$\|u_\alpha - \hat{u}\| \leq \|u_0 - \hat{u}\|. \tag{3.2}$$

By (2.1), $F'(u)$ is positive type, so for $0 < \nu < 1$, we have (see [12, p. 287]),

$$F'(u)^\nu w = \frac{\sin \pi\nu}{\pi\nu} \int_0^\infty t^\nu (F'(u) + tI)^{-2} F'(u) w dt. \tag{3.3}$$

Lemma 3.1 (c.f. [14]) *Let $F : E \to E$ be a Fréchet differentiable and monotone operator. Then for $u \in E$ and $0 < \nu < 1$,*

$$\|\alpha(F' + \alpha I)^{-1} F'(u)^{\nu}\| \leq 4\frac{\sin(\pi\nu)}{\pi\nu}\left(\frac{\nu}{1-\nu}\right)^{\nu}\alpha^{\nu}. \tag{3.4}$$

Proof By (3.3) we have

$$\begin{aligned}(F' + \alpha I)^{-1}F'(u)^{\nu}w &= \frac{\sin\pi\nu}{\pi\nu}\int_0^{\infty} t^{\nu}(F'(u)+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)w dt\\ &= \frac{\sin\pi\nu}{\pi\nu}[\int_0^{\rho} t^{\nu}(F'(u)+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)w dt\\ &\quad + \int_{\rho}^{\infty} t^{\nu}(F'(u)+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)w dt]\\ &= \frac{\sin\pi\nu}{\pi\nu}[H_1 + H_2],\end{aligned} \tag{3.5}$$

where $H_1 = \int_0^{\rho} t^{\nu}(F'(u)+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)w dt$ and $H_2 = \int_{\rho}^{\infty} t^{\nu}(F'(u)+\alpha I)^{-1}(F'(u)+tI)^{-2}F'(u)w dt$. So, by (1.11) and (1.12) we have

$$\begin{aligned}\|H_1\| &= \left\|\int_0^{\rho} t^{\nu}(F'(u)+tI)^{-2}(F'(u)+\alpha I)^{-1}F'(u)w dt\right\|\\ &\leq \int_0^{\rho} t^{\nu}\|F'(u)+tI)^{-1}\|\|F'(u)+tI)^{-1}F'(u)\|\|(F'(u)+\alpha I)^{-1}w\|dt\\ &\leq 2\int_0^{\rho}\frac{t^{\nu-1}}{\alpha}\|w\|dt = \frac{\rho^{\nu}}{\nu\alpha}\|w\|\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\|H_2\| &= \left\|\int_{\rho}^{\infty} t^{\nu}(F'(u)+tI)^{-2}(F'+\alpha I)^{-1}F'(u)w dt\right\|\\ &\leq 2\int_{\rho}^{\infty} t^{\nu-2}\|w\|dt\\ &= 2\frac{\rho^{\nu-1}}{1-\nu}\|w\|.\end{aligned} \tag{3.7}$$

Thus by (3.5), (3.6) and (3.7), we have

$$\|(F' + \alpha I)^{-1}F'(u)^{\nu}w\| \leq 2\frac{\sin(\pi\nu)}{\pi\nu}\left[\frac{\rho^{\nu}}{\nu\alpha} + \frac{\rho^{\nu-1}}{1-\nu}\right]\|w\|.$$

Now the result follows by taking minimum of the right side of the above expression $\left(\text{i.e., } \rho = \frac{\nu\alpha}{1-\nu}\right)$. □

Remark 3.2 Note that for $\nu = 1$, we have by (2.10),

$$\|\alpha(F'(u) + \alpha I)^{-1} F'(u)\| \leq 2\alpha. \tag{3.8}$$

Therefore, by Lemma 3.1 and (3.8), for $0 \leq \nu \leq 1$ we can write

$$\|\alpha(F'(u) + \alpha I)^{-1} F'(u)^{\nu}\| = O(\alpha^{\nu}). \tag{3.9}$$

Theorem 3.3 *Let Assumption 2.2 and (1.6) hold. If $6k_0 r < 1$, then*

$$\|u_\alpha - \hat{u}\| \leq \hat{C}\alpha^{\nu},$$

where $\hat{C} = \begin{cases} 4\frac{\frac{\sin \pi\nu}{\pi\nu}(\frac{\nu}{(1-\nu)})^{\nu}\|z\|}{1-3k_0 r} & 0 < \nu < 1 \\ \frac{2\|z\|}{1-3k_0 r} & \nu = 1. \end{cases} \leq \hat{C}_1 := \begin{cases} 8\frac{\sin \pi\nu}{\pi\nu}(\frac{\nu}{(1-\nu)})^{\nu}\|z\| & 0<\nu<1 \\ 4\|z\| & \nu=1. \end{cases}$

Proof We have

$$F(u_\alpha) - F(\hat{u}) + \alpha(u_\alpha - u_0) = 0.$$

Thus by mean value theorem of integral calculus, we have

$$\begin{aligned}(F'(u_0) + \alpha I)(u_\alpha - \hat{u}) &= \alpha(u_0 - \hat{u}) \\ &\quad - \int_0^1 [F'(\hat{u} + t(u_\alpha - \hat{u})) - F'(u_0)](u_\alpha - \hat{u})dt.\end{aligned}$$

Therefore by (1.5), (3.9), Assumption 2.2, (3.2), we have in turn

$$\begin{aligned}\|u_\alpha - \hat{u}\| &\leq \|\alpha\left(F'(u_0) + \alpha I\right)^{-1} F'(u_0)^{\nu} v\| \\ &\quad + \|\left(F'(u_0) + \alpha I\right)^{-1} \int_0^1 [F'\left(\hat{u} + t\left(u_\alpha - \hat{u}\right)\right) - F'(u_0)]\left(u_\alpha - \hat{u}\right) dt\| \\ &\leq \hat{C}\alpha^{\nu} + 2\int_0^1 \|\varphi\left(\hat{u} + t\left(u_\alpha - \hat{u}\right), u_0, u_\alpha - \hat{u}\right) dt\| \\ &\leq \hat{C}\alpha^{\nu} + 2k_0\left(\|\hat{u} - u_0\| + \frac{1}{2}\|u_\alpha - \hat{u}\|\right)\|u_\alpha - \hat{u}\| \\ &\leq \hat{C}\alpha^{\nu} + 2k_0\left(\|\hat{u} - u_0\| + \frac{1}{2}\|u_0 - \hat{u}\|\right)\|u_\alpha - \hat{u}\| \\ &\leq \hat{C}\alpha^{\nu} + 3k_0\|\hat{u} - u_0\|\|u_\alpha - \hat{u}\| \\ &\leq \hat{C}_1\alpha^{\nu} + 3k_0 r\|u_\alpha - \hat{u}\|.\end{aligned}$$

□

Combining Theorems 2.3 and 3.3, we have the following:

Theorem 3.4 *Let u_k be as in (1.8) and let the assumptions in Theorems 2.3 and 3.3 be satisfied. Let*

$$k_\delta := \min\left\{k : e^{-\gamma 3^k} \leq \frac{\delta}{\alpha}\right\}. \tag{3.10}$$

Then we have the following;

$$\|u_k - \hat{u}\| \leq \bar{C}_1\left(\alpha^\nu + \frac{\delta}{\alpha}\right), \tag{3.11}$$

where $\bar{C}_1 = \max\left\{\Lambda_1^{\frac{3^k-1}{2}} + 1, \hat{C}_1\right\}$.

Note that the error $\alpha^\nu + \frac{\delta}{\alpha}$ in (3.11) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\alpha_\delta^{1+\nu} = \delta$. That is $\alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Hence by (3.11) we have the following Theorem.

Theorem 3.5 *Let the assumptions in Theorem 3.4 holds. For $\delta > 0$, let $\alpha := \alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Let k_δ be as in (3.10). Then*

$$\|u_k - \hat{u}\| = O(\delta^{\frac{\nu}{1+\nu}}).$$

In order to obtain the above order, without knowing ν, we use the adaptive selection of the parameter strategy considered by Pereverzev and Schock [19], modified suitably for the situation for choosing the parameter α. For convenience, take $u_i := u_{k_i}$. Let $i \in \{0, 1, 2, \ldots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := \max\left\{i : \alpha_i^\nu \leq \frac{\delta}{\alpha_i}\right\} < N \quad \text{and} \tag{3.12}$$

$$k := \max\left\{i : \|u_i - u_j\| \leq 4\bar{C}_1\frac{\delta}{\alpha_j}, \; j = 0, 1, 2, \ldots, i-1\right\} \tag{3.13}$$

where $\bar{C}_1$ is as in Theorem 3.4. Now we have the following Theorem.

Theorem 3.6 (cf. [10]) *Assume that there exists $i \in \{0, 1, \ldots, N\}$ such that $\alpha_i^\nu \leq \frac{\delta}{\alpha_i}$. Let assumptions of Theorem 3.4 be fulfilled, and let l and k be as in (3.12) and (3.13) respectively. Then $l \leq k$; and*

$$\|\hat{u} - u_k\| \leq 6\bar{C}_1\mu\delta^{\frac{\nu}{1+\nu}}.$$

Proof To prove $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \ldots N\}$, $\alpha_i^{\nu} \leq \frac{\delta}{\alpha_i} \implies \|u_i - u_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}$, $\forall j = 0, 1, 2, \ldots i-1$. For $j < i$, we have

$$\begin{aligned}
\| u_i - u_j \| &\leq \| u_i - \hat{u} \| + \| \hat{u} - u_j \| \\
&\leq \bar{C}_1 \left(\alpha_i^{\nu} + \frac{\delta}{\alpha_i} \right) + \bar{C}_1 \left(\alpha_j^{\nu} + \frac{\delta}{\alpha_j} \right) \\
&\leq 2\bar{C}_1 \frac{\delta}{\alpha_i} + 2\bar{C}_1 \frac{\delta}{\alpha_j} \\
&\leq 4\bar{C}_1 \frac{\delta}{\alpha_j}.
\end{aligned}$$

Thus the relation $l \leq k$ is proved. Observe that

$$\| \hat{u} - u_k \| \leq \| \hat{u} - u_l \| + \| u_k - u_l \|,$$

where

$$\| \hat{u} - u_l^{\delta} \| \leq \bar{C}_1 \left(\alpha_l^{\nu} + \frac{\delta}{\alpha_l} \right) \leq 2\bar{C}_1 \frac{\delta}{\alpha_l}.$$

Now since $l \leq k$, we have

$$\| u_k - u_l \| \leq 4\bar{C}_1 \frac{\delta}{\alpha_l}.$$

Hence

$$\| \hat{u} - u_k \| \leq 6\bar{C}_1 \frac{\delta}{\alpha_l}$$

It follows again, since $\alpha_\delta = \delta^{\frac{1}{1+\nu}} \leq \alpha_{l+1} \leq \mu\alpha_l$, that

$$\frac{\delta}{\alpha_l} \leq \frac{\mu\delta}{\alpha_\delta} = \mu\delta^{\frac{\nu}{1+\nu}}.$$ □

4 Implementation of adaptive choice rule

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 3.6 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0$, $i = 0, 1, 2, \ldots, N$.

4.1 Algorithm

1. Set $i = 0$.
2. Choose $k_i := \min \left\{ k : e^{-\gamma 3^k} \leq \frac{\delta}{\alpha_i} \right\}$.
3. Solve $u_i := u_{k_i}$ by using the iteration (1.8).
4. If $\|u_i - u_j\| > 4\bar{C}_1 \frac{\delta}{\alpha_j}$, $j < i$, then take $k = i - 1$ and return u_k.
5. Else set $i = i + 1$ and go to 2.

5 Numerical examples

We apply the algorithm by choosing a sequence of finite dimensional subspace (V_M) of $L^2(0, 1)$ with $dim\ V_M = M + 1$. Precisely we choose V_M as the linear span of $\{v_1, v_2, v_3, \ldots, v_{M+1}\}$ where v_i, $i = 1, 2, \ldots, M + 1$ are linear splines in a uniform grid of $M + 1$ points in [0, 1].

Example 5.1 (see [15, Sect. 4.3]) Let $F : D(F) \subseteq L^2(0, 1) \longrightarrow L^2(0, 1)$ defined by

$$F(u) := \int_0^1 k(t, s)u^3(s)ds,$$

where

$$k(t, s) = \begin{cases} (1 - t)s, & 0 \leq s \leq t \leq 1 \\ (1 - s)t, & 0 \leq t \leq s \leq 1 \end{cases}.$$

Then for all $u(t), v(t) : u(t) > v(t)$:

$$\langle F(u) - F(v), u - v \rangle = \int_0^1 \left[\int_0^1 k(t, s)(u^3 - v^3)(s)ds \right] \times (u - v)(t)dt \geq 0.$$

Thus the operator F is monotone. The Fréchet derivative of F is given by

$$F'(u)w = 3 \int_0^1 k(t, s)u^2(s)w(s)ds. \tag{5.1}$$

Note that for $u, v > 0$,

$$\begin{aligned} [F'(v) - F'(u)]w &= 3 \int_0^1 k(t, s)u^2(s) \frac{[v^2(s) - u^2(s)]w(s)ds}{u^2(s)} \\ &:= F'(u)\Phi(v, u, w) \end{aligned}$$

where $\Phi(v, u, w) = \frac{[v^2 - u^2]w}{u^2}$. Observe that

$$\Phi(v, u, w) = \frac{[v^2 - u^2]w}{u^2} = \frac{[u + v][v - u]w}{u^2}$$

and

$$\begin{aligned}\|\frac{d}{dw}\Phi(u + tw, u, w)\| &= \left\| \frac{d}{dw} \frac{\left[2tuw + t^2 w^2\right] w}{u^2} \right\| \\ &= \left\| \frac{4tuw + 3t^2 w^2}{u^2} \right\| \\ &\leq \left\| \frac{4tu + 3t^2 w}{u^2} \right\| \|w\|.\end{aligned}$$

So Assumption 2.2 satisfies with $k_0 \geq \max\left\{ \left\| \frac{u+v}{u^2} \right\|, 2 \left\| \frac{4tu+3t^2 w}{u^2} \right\| \right\}$. In our computation, we take $f(t) = \frac{6\sin(\pi t)+\sin^3(\pi t)}{9\pi^2}$ and $f^\delta = f + \delta$. Then the exact solution

$$\hat{u}(t) = \sin(\pi t).$$

We use

$$u_0(t) = \sin(\pi t) + \frac{3\left[t\pi^2 - t^2\pi^2 + \sin^2(\pi t)\right]}{4\pi^2}$$

as our initial guess, so that the function $u_0 - \hat{u}$ satisfies the source condition

$$u_0 - \hat{u} = \varphi\left(F'\left(\hat{u_0}\right)\right)\left(\frac{\hat{u}^2}{4u_0^2}\right)$$

where $\varphi(\lambda) = \lambda$. Thus we expect to obtain the rate of convergence $O\left((\delta)^{\frac{1}{2}}\right)$.

We choose $a = 1.5$, $\alpha_0 = \mu\delta$ and $\mu = 1.01$. The results of the computation are presented in Table 1. The plots of the exact solution and the approximate solution obtained are given in Fig. 1.

Table 1 Iterations and corresponding error estimates

M	$\delta = 0.1$				$\delta = 0.01$			
	k	α_k	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|u_k-\hat{u}\|\|}{\delta^{\frac{1}{2}}}$	k	α_k	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|u_k-\hat{u}\|\|}{\delta^{\frac{1}{2}}}$
8	30	0.1363454479	0.0503927259	0.1125936299	30	0.01382598145	0.05039707647	0.3536092805
16	30	0.1136185917300361	0.1061109950797	0.113661932322	30	0.013666450897	0.06110993160	0.43127096521
32	30	0.113614603466	0.1063664784762	0.1142351835898	30	0.013626568258	0.063664794104	0.44995842369
64	30	0.11361360640	0.1064305298623	0.114378926380	30	0.0136165976	0.064305284639	0.45465153223
128	30	0.11361335714	0.1064465606055	0.114414903744	30	0.01361410493	0.064465606142	0.45582676204
256	30	0.113613294817	0.1064505677544	0.114423896988	30	0.01361348177	0.064505677538	0.45612054021
512	30	0.113613279238	0.1064515697139	0.114426145690	30	0.013613325975	0.064515697139	0.45619399928
1024	30	0.113613275343	0.1064518202046	0.114426707868	30	0.013613287027	0.064518202046	0.45621236423
M	$\delta = 0.004$				$\delta = 0.002$			
	k	α_k	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|u_k-\hat{u}\|\|}{\delta^{\frac{1}{2}}}$	k	α_k	$\frac{\|\|u_k-\hat{u}\|\|}{\|\|\hat{u}\|\|}$	$\frac{\|\|u_k-\hat{u}\|\|}{\delta^{\frac{1}{2}}}$
8	30	0.00565801702480	0.050397933465	0.552773098978	30	0.0029353622159	0.0503989450005	0.76746201827
16	30	0.0054984864697	0.061109929996	0.67991794194	30	0.0027758316607	0.06110992901	0.95693295815
32	30	0.0054586038309	0.063664794728	0.71092674246	30	0.0027359490219	0.063664794936	1.0041801675
64	30	0.0054486331712	0.064305283708	0.71873562117	30	0.0027259783622	0.064305283398	1.0161357947
128	30	0.0054461405062	0.06446560615	0.72069240541	30	0.0027234856972	0.064465606149	1.0191352350
256	30	0.005445517340	0.064505677538	0.72118164423	30	0.0027228625311	0.064505677538	1.0198854104
512	30	0.0054453615485	0.064515697139	0.72130398266	30	0.0027227067395	0.06451569714	1.0200730108
1024	30	0.0054453226006	0.064518202046	0.72133456791	30	0.0027226677916	0.064518202047	1.0201199129

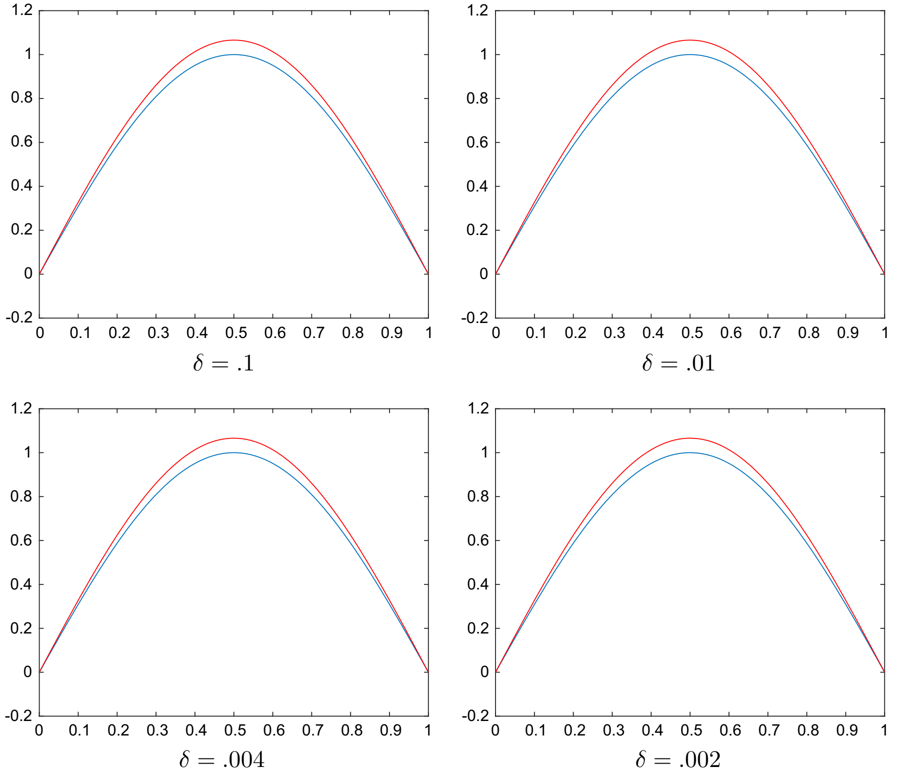

Fig. 1 Curves of the exact (lower curve) and approximate (upper curve) solutions with M = 1024

6 Conclusion

In this paper we are producing an extended Newton iterative scheme that converges cubically to the solution which uses assumptions only on first Fréchet derivative of the operator. We obtained an error estimate under a general Hölder type source condition. Also we considered the adaptive parameter choice strategy considered by Pereverzev and Schock [19], for choosing the regularization parameter.

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