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Frames for Operators in Banach Spaces

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Abstract A family of local atoms in a Banach space has been introduced and it has been generalized to an atomic system for operators in Banach spaces, which has been further led to introduce new frames for operators by Dastourian and Janfada, by making use of semi-inner products. Unlike the traditional way of considering sequences in the dual space, sequences in the original space are considered to study them. Appropriate changes have been made in the definitions of atomic systems and frames for operators to fit them for sequences in the dual space without using semi-inner products so that the new notion for Banach spaces can be thought of as a generalization of Banach frames. With some crucial assumptions, we show that frames for operators in Banach spaces share nice properties of frames for operators in Hilbert spaces.

Keywords X_d-atomic system · X_d-K-frame

Mathematics Subject Classification (2010) 47B32 · 42C15

1 Introduction

Frames are a tool for the construction of series expansions in Hilbert spaces. Frames provide stable expansions, quite in contrast to orthogonal expansions — they may be overcomplete and the coefficients in the frame expansion therefore need not be unique. The redundancy

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and flexibility offered by frames has spurred their applications in a variety of areas throughout mathematics and engineering, such as operator theory [11], harmonic analysis [9], pseudo-differential operators [10], quantum computing [5], signal and image processing [4], and wireless communication [13].

Theoretical research of frames for Banach spaces is quite different from that for Hilbert spaces. Due to the lack of an inner product, the properties of Hilbert frames usually do not transfer automatically to Banach spaces. Gröchenig [8] generalized Banach frames with respect to certain sequence spaces. The main feature of frames that Gröchenig was trying to capture in a general Banach space was the unique association of a vector in a Hilbert space with the natural set of frame coefficients. After the work of Gröchenig, frames in Banach spaces have become topic of investigation for many mathematicians.

A sequence space X_d is called a *BK-space* if it is a Banach space and the coordinate functionals are continuous on X_d. If the canonical unit vectors form a Schauder basis for X_d, then X_d is called a *CB-space* and its canonical basis is denoted by $\{e_n\}$. If X_d is reflexive and a CB-space, then X_d is called an *RCB-space*. Also, the dual of X_d is denoted by X_d^*. When X_d^* is a CB-space, then its canonical basis is denoted by $\{e_n^*\}$.

We denote by $\mathcal{B}(X)$ the space of all bounded linear operators on a Banach space X. For $T \in \mathcal{B}(X)$, we denote $D(T)$, $R(T)$, and $N(T)$ for *domain*, *range* and *nullspace* of T, respectively. The set of all natural numbers is denoted by N. For simplicity, a sequence $\{f_n : n \in N\}$ indexed by N will be abbreviated as $\{f_n\}$ throughout the paper.

The results in this paper are organized as follows. In Section 2, we recall basic definitions, known results on K-frames in Hilbert spaces and X_d-frames in Banach spaces. Two new notions, "atomic systems" and "frames for operators" are defined in Section 3, without using semi-inner products. Operators preserving them and generating new such frames using old ones have been discussed. In the end, it is shown that the frames for operators in Banach spaces share few nice properties of frames for operators in Hilbert spaces, under some crucial assumptions. Throughout the paper, all spaces are nontrivial; operators are non-zero, and X is a reflexive separable Banach space.

2 Notations and Preliminaries

Găvruţa [7] introduced two notions, "atomic systems" and "K-frames" in a separable Hilbert space H, as a generalization of families of local atoms [6], where $K \in \mathcal{B}(H)$.

Definition 1 [7] A sequence $\{f_n\}$ in H is called an atomic system for K, if the following conditions are satisfied :

1. the series $\sum_n c_n f_n$ converges for all $c = \{c_n\} \in \ell_2$;
2. there exists $C > 0$ such that for every $f \in H$ there exists $a_f = \{a_n\} \in \ell_2$ such that $\|a_f\|_{\ell_2} \leq C\|f\|$ and $Kf = \sum_n a_n f_n$.

The condition 1 in Definition 1 says that $\{f_n\}$ is a Bessel sequence.

Definition 2 [7] Let $K \in \mathcal{B}(H)$. A sequence $\{f_n\}$ in H is called a K-frame for H if there exist two constants $0 < \lambda \leq \mu < \infty$ such that

$$\lambda\|K^* f\|^2 \leq \sum_n |\langle f, f_n\rangle|^2 \leq \mu\|f\|^2 \text{ for all } f \in H.$$

The constants λ and μ are called the lower and upper bounds respectively, for the K-frame $\{f_n\}$. If the above inequalities hold only for $f \in \overline{span}\{f_n\}$, then $\{f_n\}$ is said to be a K-frame sequence.

If K is equal to I, the identity operator on H, then K-frames and K-frame sequences are just ordinary frames and frame sequences, respectively. It is proved that these two concepts are equivalent [7]. Because of the higher generality of K-frames, many properties for ordinary frames may not hold for K-frames, such as the corresponding synthesis operator for K-frames is not surjective, the frame operator for K-frames is not isomorphic, the alternate dual reconstruction pair for K-frames is not interchangeable in general. Also, the frame operator S for a K-frame is semidefinite, so there is also $S^{1/2}$, but not positive. In general, it is not invertible. For more details on K-frames, see [7, 12, 14, 15, 17].

The concept of a family of local atoms in a Banach space X with respect to a BK-space X_d was introduced by Dastourian and Janfada [3] using a semi-inner product. This concept was generalized to an atomic system for an operator $K \in \mathcal{B}(X)$ called *X_d^*-atomic system* and it has been led to the definition of a new frame with respect to the operator K, called *X_d^*-K-frame.* Unlike the traditional way of considering sequences in the dual space X^*, sequences in the original space X are considered in [3] to study a family of X_d^*-local atoms and X_d^*-atomic systems by making use of semi-inner products.

Appropriate changes have been made in the definitions of X_d^*-atomic systems and X_d^*-K-frames to fit them for sequences in the dual space without using semi-inner products, called *X_d-atomic systems* and *X_d-K-frames*, respectively. Thus, the notion of X_d-K-frames for Banach spaces can be thought of a generalization of X_d-frames. We start with the definition of an X_d-frame defined by Casazza, Christensen, and Stoeva [2] which is a natural generalization of Hilbert frames to Banach frames.

Definition 3 Let X be a Banach space and let X_d be a BK-space. A sequence $\{g_n\}$ of elements in X^*, which satisfies

1. $\{g_n(f)\} \in X_d$ for all $f \in X$,
2. There are constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in X$

$$\lambda \|f\|_X \leq \|\{g_n(f)\}\|_{X_d} \leq \mu \|f\|_X \tag{1}$$

is called an X_d-frame for X. The constants λ and μ are called lower and upper bounds respectively for $\{g_n\}$. When $\{g_n\}$ satisfies the condition 1 and the upper inequality in (1) for all $f \in X$, $\{g_n\}$ is called an $\boldsymbol{X_d}$-Bessel sequence for X.

Note that the definition of X_d-frame is a part of the definition of a Banach frame introduced by Gröchenig [8]. If X is a Hilbert space and $X_d = \ell_2$, the X_d-frame inequalities in (1) mean that $\{g_n\}$ is a frame, and in this case it is well-known that there exists a sequence $\{f_n\}$ in X such that for each $f \in X$,

$$f = \sum_n \langle f, f_n\rangle\, g_n = \sum_n \langle f, g_n\rangle\, f_n.$$

Similar reconstruction formulas are not always available in the Banach space setting.

Lemma 1 [2] *Let X_d be a BK-space for which the canonical unit vectors $\{e_n\}$ form a Schauder basis. Then the space $Y_d = \{F(e_n) : F \in X_d^*\}$ with the norm $\|\{F(e_n)\}\|_{Y_d} = \|F\|_{X_d^*}$ is a BK-space isometrically isomorphic to X_d^*. Also, every continuous linear* *functional F on X_d has the form $F(c) = \sum_n c_n d_n$, where $\{d_n\} = F(e_n)$, is uniquely determined by $d_n = F(e_n)$, and $\|F\|_{X_d^*} = \|\{d_n\}\|_{Y_d}$.*

Lemma 2 [2] *Let X_d be a BK-space and let X_d^* be a CB-space. If $\{g_n\} \subseteq X^*$ is an X_d-Bessel sequence for X with bound μ, then the operator $L : \{d_n\} \mapsto \sum_n d_n g_n$ is well-defined (hence bounded) from X_d^* into X^* and $\|L\| \leq \mu$. If X_d is reflexive, the converse is also true.*

Let X_d be a BK-space and let $\{g_n\}$ be a sequence in X^*. If $\{g_n\}$ satisfies only the upper inequality in (1), the analysis operator U from X to X_d mapped by $f \mapsto \{g_n(f)\}$, is well-defined and linear, having domain $D(U) = \{f \in X : \{g_n(f)\} \in X_d\}$. The domain $D(U)$ is a subspace (not necessarily closed) of X. If $\{g_n\}$ is an X_d-Bessel sequence for X, then $D(U) = X$ and U is bounded with the norm $\|U\| \leq \mu$.

If only the lower inequality in (1) is satisfied by $\{g_n\}$, then U is bounded below on $D(U)$. Thus if $\{g_n\}$ satisfies the X_d-frame inequalities in (1), we get that U is bounded and bounded below on $D(U)$. Hence $R(U)$ is closed in X_d and the inverse $U^{-1} : R(U) \to D(U)$ is also bounded with the norm $\|U^{-1}\| \leq \frac{1}{\lambda}$. We can conclude that given an X_d-frame $\{g_n\} \subseteq X^*$ for X, the analysis operator $U : X \to X_d$ defined by $Uf = \{g_n(f)\}$ is an isomorphism of X onto $R(U)$.

Given a sequence $\{g_n\}$ in X^*, we now consider a function $L : X_d^* \to X^*$, called the synthesis operator, mapped as $\{d_n\} \mapsto \sum_n d_n g_n$ is well-defined and linear on the domain $D(L) = \left\{\{d_n\} \in X_d^* : \sum_n d_n g_n \text{ converges in } X^*\right\}$. If $\{g_n\} \subseteq X^*$ is an X_d-Bessel sequence in X with bound λ and if X_d^* is a CB-space, then L is bounded from X_d^* to X^* and $\|L\| \leq \mu$, by Lemma 2. If X_d is a CB-space, then $U^* = L$. If X_d is reflexive and $\{g_n\}$ is an X_d-frame for X, then $U = L^*$ because X is isomorphic to a closed subspace of X_d and every closed subspace of a reflexive space is reflexive. Hence X is also reflexive. The section ends with a result connecting majorization, factorization and range inclusion for operators on Banach spaces.

Theorem 1 [1] *Let X, Y, Z be Banach spaces and let $A \in \mathcal{B}(X, Y)$, $B \in \mathcal{B}(Z, Y)$. Then the following statements hold:*

1. *If $A = BT$ for some $T \in \mathcal{B}(X, Z)$, then B^* majorizes A^*. The converse is true when $N(B)$ is complemented in Z, and Z is reflexive. Note that B^* majorizes A^* if there exists $C > 0$ such that for each $f \in Y$, $\|A^* f\|_X \leq C\|B^* f\|_Z$.*
2. *If $R(A) \subseteq R(B)$, then B^* majorizes A^*. The converse is true when Z is reflexive.*

3 Atomic System for Banach Spaces

Definition 4 Let X be a Banach space and let X_d be a BK-space. Let $K \in \mathcal{B}(X^*)$ and $\{g_n\} \subseteq X^*$. We say that $\{g_n\}$ is an $\boldsymbol{X_d}$-atomic system for $\boldsymbol{X}$ with respect to K if the following statements hold:

1. $\sum\limits_n d_n g_n$ converges in X^* for all $d = \{d_n\}$ in X_d^* and there exists $\mu > 0$ such that $\left\| \sum\limits_n d_n g_n \right\|_{X^*} \leq \mu \|d\|_{X_d^*}$;
2. there exists $C > 0$ such that for every $g \in X^*$ there exists $a_g = \{a_n\} \in X_d$ such that $\|a_g\|_{X_d} \leq C\|g\|_{X^*}$ and $Kg = \sum_n a_n g_n$.

When X_d is reflexive, the condition 1 in Definition 4 actually says that $\{g_n\}$ is an X_d-Bessel sequence for X with bound μ, by Lemma 2. We find a necessary condition for a sequence $\{g_n\} \subseteq X^*$ to be an X_d-atomic system for X with respect to a given operator K if the associated sequence space satisfies the following crucial property : For each $\{f_n\}, \{h_n\} \in X_d$,

$$\Big|\sum_n f_n h_n\Big| \leq \|\{f_n\}\|_{X_d}\|\{h_n\}\|_{X_d}. \tag{2}$$

For instance, let $\{f_n\}, \{h_n\} \in \ell_p$ and $p \in (1, 2]$. Then the conjugate of p, q lies in $[2, \infty)$. Hence by Hölder's inequality, the sequence space ℓ_p for $1 < p \leq 2$ satisfies the inequality (2).

Theorem 2 *Let X_d be a BK-space. Let $\{g_n\}$ be a sequence in X^* and $K \in \mathcal{B}(X^*)$. If $\{g_n\}$ is an X_d-atomic system for X with respect to K and the sequence space X_d satisfies the inequality (2), then there exists a constant $\lambda > 0$ such that*

$$\|K^* f\|_X \leq \lambda \|\{g_n(f)\}\|_{X_d} \text{ for each } f \in X.$$

Proof Suppose $\{g_n\}$ is an X_d-atomic system for X with respect to K. Then there is some $C > 0$ such that for every $g \in X^*$ there exists $a_g = \{a_n\} \in X_d$ such that $\|a_g\|_{X_d} \leq C\ \|g\|_{X^*}$ and $Kg = \sum_n a_n g_n$. Hence for each $f \in X$,

$$\begin{aligned}
\|K^* f\|_X &= \sup_{g \in X^*,\ \|g\|=1} |g(K^* f)| \\
&= \sup_{g \in X^*,\ \|g\|=1} |(Kg)(f)| \\
&= \sup_{g \in X^*,\ \|g\|=1} \Big|\sum_n a_n g_n(f)\Big| \\
&\leq \sup_{g \in X^*,\ \|g\|=1} \|\{a_n\}\|_{X_d}\ \|\{g_n(f)\}\|_{X_d} \\
&= \sup_{g \in X^*,\ \|g\|=1} \|a_g\|_{X_d}\ \|\{g_n(f)\}\|_{X_d} \\
&\leq C \sup_{g \in X^*,\ \|g\|=1} \|g\|_{X^*}\ \|\{g_n(f)\}\|_{X_d} \quad [\text{using } \|a_g\|_{X_d} \leq C\ \|g\|_{X^*}].
\end{aligned}$$

Thus for some $C > 0$, $\|K^* f\|_X \leq C\ \|\{g_n(f)\}\|_{X_d}$ for each $f \in X$. □

Definition 5 Let X be a Banach space and let X_d be a BK-space. Let $K \in \mathcal{B}(X^*)$ and $\{g_n\} \subseteq X^*$. We say that $\{g_n\}$ is an $\boldsymbol{X_d}$**-**$\boldsymbol{K}$-frame for $\boldsymbol{X}$ if the following statements hold:

1. $\{g_n(f)\} \in X_d$ for each $f \in X$;
2. there exist two constants $0 < \lambda \leq \mu < \infty$ such that

$$\lambda \|K^* f\|_X \leq \|\{g_n(f)\}\|_{X_d} \leq \mu \|f\|_X \text{ for each } f \in X.$$

The elements λ and μ are called the lower and upper X_d-K-frame bounds.

We say that an X_d-frame for X is an X_d-I-frame for X, where I is the identity operator on X^*. The set of all X_d-frames for X can be considered as a subset of X_d-K-frames for X. Thus X_d-K-frame is a generalization of X_d-frame for a Banach space X. We present an example for an X_d-K-frame which is not an X_d-frame for X.

Example 1 Let X be the space of all triplets $(\alpha_1, \alpha_2, \alpha_3)$ with complex scalars and having 3/2-norm, denoted by $\ell_{3/2}(3)$. Let $\{g_n\} \subseteq X^* = \ell_3(3)$ be such that for $n = 1, 2, 3$,

$g_n(e_m) = \delta_{nm}$, where e_m's are vectors in X, having 1 in m^{th} place and 0 elsewhere, and $g_n = 0$ for all $n \geq 4$. Define $K : X^* \to X^*$ as follows:

$$Kg_1 = 0, \ \ Kg_2 = g_3, \ \text{ and } Kg_3 = g_2.$$

For any $f \in X$, we have $f = \sum_{n=1}^{3} \alpha_n e_n$ and

$$\|K^* f\|_X = \|\alpha_2 e_3 + \alpha_3 e_2\|_{3/2} = \left(|\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} = \|\{g_n(f)\}_{n=2}^{\infty}\|_{\ell_{3/2}}.$$

Then $\{g_n\}_{n=2}^{\infty}$ is an X_d-K frame for X. But it is not an X_d-frame because there is no constant λ such that for any scalar α_1,

$$\lambda\|f\|_X = \left(|\alpha_1|^{3/2} + |\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} \leq \left(|\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} = \|\{g_n(f)\}_{n=2}^{\infty}\|_{\ell_{3/2}}.$$

We can generate new X_d-K-frames for X from each X_d-frame for X and each operator $K \in \mathcal{B}(X^*)$, by the following proposition.

Proposition 1 *If $\{g_n\}$ is an X_d-frame for X and $K \in \mathcal{B}(X^*)$, then $\{Kg_n\}$ is an X_d-K-frame for X.*

Proof Suppose $\{g_n\}$ is an X_d-frame for X. Then $\{g_n(f)\} \in X_d$, for all $f \in X$ and there are constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in X$

$$\lambda\|f\|_X \leq \|\{g_n(f)\}\|_{X_d} \leq \mu\|f\|_X.$$

Let $f \in X$ be fixed. Since $(Kg_n)(f) = g_n(K^* f)$ and $K^* f \in X$, we have $\{(Kg_n)(f)\} \in X_d$. Also, $\|K^* f\|_X \leq \|K\| \, \|f\|_X$ gives that for each $f \in X$,

$$\lambda\|K^* f\|_X \leq \|\{(Kg_n)(f)\}\|_{X_d} \leq \mu\|K\| \, \|f\|_X.$$

Thus $\{Kg_n\}$ is an X_d-K-frame for X. □

The following example illustrates that an X_d-Bessel sequence is an X_d-K-frame but it is not the same for the other operator T.

Example 2 Let $X = \ell_{3/2}(3)$. Let $\{g_n\} \subseteq X^* = \ell_3(3)$ be such that for $n = 1, 2, 3$, $g_n(e_m) = \delta_{nm}$, and $g_n = 0$ for all $n \geq 4$. Define K and T from X^* to X^* as follows: $Kg_1 = 0$, $Kg_2 = g_3$, and $Kg_3 = g_2$, and $Tg_1 = g_1$, $Tg_2 = g_3$, and $Tg_3 = g_2$. Then $\{g_n\}_{n=2}^{\infty}$ is an X_d-K frame but it is not an X_d-T-frame for X.

Theorem 3 *Let $\{g_n\}$ be an X_d-K-frame for X. Let $T \in \mathcal{B}(X^*)$ be such that $R(T) \subseteq R(K)$. Then $\{g_n\}$ is an X_d-T-frame for X.*

Proof Suppose $\{g_n\}$ is an X_d-K-frame for X. Then there are constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in X$

$$\lambda\|K^* f\|_X \leq \|\{g_n(f)\}\|_{X_d} \leq \mu\|f\|_X. \tag{3}$$

Since $R(T) \subseteq R(K)$, by Theorem 1, there exists $C > 0$ such that for each $f \in X$, $\|T^* f\|_X \leq C\|K^* f\|_X$. From the second inequality in (3), we have for each $f \in X$

$$\frac{\lambda}{C}\|T^* f\|_X \leq \|\{g_n(f)\}\|_{X_d} \leq \mu\|f\|_X.$$

Hence $\{g_n\}$ is an X_d-T-frame for X. □

Theorem 4 *Let X_d be a reflexive space and let $\{g_n\} \subseteq X^*$. Let $\{e_n\}$ be the canonical unit vectors for X_d and X_d^*. Then $\{g_n\}$ is an X_d-K-frame for X if and only if there exists a bounded linear operator $L : X_d^* \to X^*$ such that $Le_n = g_n$ and $R(K) \subseteq R(L)$.*

Proof Since $\{g_n\}$ is an X_d-K-frame for X, there exist constants $0 < \lambda \le \mu < \infty$ such that for each $f \in X$,

$$\lambda \|K^* f\|_X \le \|\{g_n(f)\}\|_{X_d} \le \mu \|f\|_X.$$

Hence the operator $U : X \to X_d$ defined by $Uf = \{g_n(f)\}$ is bounded and $\|U\| \le \mu$. The adjoint of U, $U^* : X_d^* \to X^*$ satisfies $(U^* e_n)(f) = e_n(Uf) = g_n(f)$. Since X_d is an RCB-space, $U^* = L$, hence we get $Le_n = g_n$. Also we have

$$\lambda \|K^* f\|_X \le \|\{g_n(f)\}\|_{X_d} = \|L^* f\|_{X_d} \text{ for each } f \in X.$$

Thus by Theorem 1, $R(K) \subseteq R(L)$.

On the other hand, suppose there exists a bounded linear operator $L : X_d^* \to X^*$ such that $Le_n = g_n$ and $R(K) \subseteq R(L)$. Then by Theorem 1, there exists $\lambda > 0$ such that for each $f \in X$, $\lambda \|K^* f\|_X \le \|L^* f\|_{X_d}$. Thus for each $f \in X$,

$$\lambda \|K^* f\|_X \le \|\{g_n(f)\}\|_{X_d} = \|L^* f\|_{X_d} \le \|L\| \, \|f\|_X. \tag{4}$$

□

Corollary 1 *Let X_d be a reflexive space and let $\{g_n\} \subseteq X^*$. Let $\{e_n\}$ be the canonical unit vectors for X_d and X_d^*. Let $N(L)$ be complemented in X_d^*. Then $\{g_n\}$ is an X_d-K-frame for X if and only if $L = KV$ for some $V \in \mathcal{B}(X_d^*, K^*)$.*

Zhang and Zhang [16] defined frames in Banach spaces via a compatiable semi-inner product, which is a natural substitute for inner products on Hilbert spaces. As assumed in the paper [16], we assume that X_d is reflexive, the canonical unit vectors $\{e_n\}$ form a Schauder basis for X_d and X_d^*; the following crucial requirement is also imposed as in [16]: If $d = \{d_n\}$ is a sequence of scalars satisfying $\sum_n c_n d_n$ converges for every $c = \{c_n\} \in X_d$, then $d \in X_d^*$, and if the above series converges for all $d \in X_d^*$, then $c \in X_d$.

For instance, if $X_p = \ell_p$, $1 < p < \infty$, then $X_d^* = \ell_q$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, it satisfies all of our requirements on X_d and X_d^*. The above requirements about the spaces X and X_d are assumed in the rest of the paper. We now prove the converse of Theorem 2 with the above assumptions.

Theorem 5 *Let X be a Banach space and let X_d be a BK-space. Let $\{g_n\} \subseteq X^*$ be an X_d-Bessel sequence for X, and $K \in \mathcal{B}(X^*)$. If $N(L)$ is complemented, and if there exists a constant $\lambda > 0$ such that for each $f \in X$*

$$\|K^* f\|_X \le \lambda \|\{g_n(f)\}\|_{X_d},$$

then $\{g_n\}$ is an X_d-atomic system for X with respect to K.

Proof Using the synthesis operator L, the given inequality in hypothesis can be written as

$$\|K^* f\|_X \le \lambda \|L^* f\|_{X_d} \text{ for all } f \in X.$$

By Theorem 1, $K = LT$ for some $T \in \mathcal{B}(X^*, X_d^*)$. Let $g \in X^*$ be fixed. Then $Tg \in X_d^*$. Since X_d has the canonical unit vectors $\{e_n\}$ as a Schauder basis, the continuous linear functional Tg on X_d has the form $Tg(c) = \sum_n c_n d_n$, where $\{d_n\} \in X_d$ is uniquely determined

$d_n = F(e_n)$, and $\|Tg\|_{X_d^*} = \|\{d_n\}\|_{X_d}$. Since T is bounded, the sequence $\{d_n\}$ associated for $g \in X^*$ satisfies

$$\|\{d_n\}\|_{X_d} = \|Tg\|_{X_d^*} \leq \|T\| \; \|g\|_{X^*}.$$

Also, we have

$$Kg = LTg = L(\{d_n\}) = \sum_n d_n g_n.$$

Thus $\{g_n\}$ is an X_d-atomic system for X with respect to K. □

Theorem 6 *Let $K_1, K_2 \in \mathcal{B}(X^*)$. Let $\{g_n\}$ be an X_d-atomic system for X with respect to K_1, K_2, and let α, β be scalars. If $N(L)$ is complemented, then $\{g_n\}$ is an X_d-atomic system for $\alpha K_1 + \beta K_2$.*

Proof Suppose $\{g_n\}$ is an X_d-atomic system for X with respect to K_1, K_2 and α, β are any scalars. Then there are constants $0 < \lambda_i \leq \mu_i < \infty$ $(i = 1, 2)$ such that for each $f \in X$

$$\lambda_i \|K_i^* f\|_X \leq \|\{g_n(f)\}\|_{X_d} \leq \mu_i \|f\|_X.$$

By simple calculations, we get

$$\left(\frac{|\alpha|}{\lambda_1} + \frac{|\beta|}{\lambda_2}\right)^{-1} \|(\alpha K_1 + \beta K_2)^* f\|_X \leq \|\{g_n(f)\}\|_{X_d} \leq \left(\frac{\mu_1 + \mu_2}{2}\right) \|f\|_X.$$

Therefore by Theorem 5, $\{g_n\}$ is an atomic system for $\alpha K_1 + \beta K_2$.

□

We now prove that the notions "atomic systems" and "frames for operators" are equivalent under the crucial assumptions. The proof of the result given below follows from Theorem 2 and Theorem 5.

Theorem 7 *Let X_d be a sequence space satisfying the inequality (2) and let $\{g_n\} \subseteq X^*$ be an X_d-Bessel sequence for X. Let $N(L)$ be complemented and $K \in \mathcal{B}(X^*)$. Then the following statements are equivalent:*

1. *$\{g_n\}$ is an X_d-atomic system for X with respect to K.*
2. *$\{g_n\}$ is an X_d-K-frame for X.*

Corollary 2 [7] *Let $\{f_n\}$ be a sequence in a Hilbert space H and let $K \in \mathcal{B}(H)$. Then the following statements are equivalent:*

1. *$\{f_n\}$ is an atomic system for K.*
2. *$\{f_n\}$ is a K-frame for H.*

Proof The proof follows from Theorem 7 because the assumptions are "redundant" if X is considered to be a Hilbert space with the sequence space $X_d = \ell_2$ in Theorem 7. □

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