Further Results on Set Sequential and Set Graceful Graphs

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Abstract

Unless mentioned otherwise, we consider only finite simple graphs and for all notations in Graph theory we follow Harary [4].

Several practical problems in real life situations have motivated the study of labeling the vertices and edges of a graph $G = (V, E)$ which are required to obey a variety of conditions depending on the structure of G such as adjacency. There is an enormous amount of literature built up on several kinds of labelings of graphs over the last three decades or so. An interested reader can refer to Gallian [3].

Acharya [1] has initiated a general study of the labelings of the vertices and edges of a graph using subsets of a set and indicated their potential applications in a variety of other areas of human enquiry.

An assignment f of distinct subsets (nonempty subsets) of a finiteset X to the vertices of a given graph $G = (V, E)$ so that the values of the edges $e = uv$ are obtained as the symmetric differences of the sets assigned to the vertices u and v such that both, the vertex function as well as the edge functions are injective, is called a set indexer of G. A set indexer f is called a **set graceful labeling**, if all the nonempty subsets of X are obtained on the edges. A set indexer f is called a **set sequential labeling** if the sets on the vertices and edges together form the set of all nonempty subsets of X. A graph is called **set graceful (set sequential)** if it admits a set graceful (set sequential) labeling with respect to a set X.

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Let X be a nonempty set, 2^X denote the power set of X and $Y(X)$ denote the set of all nonempty subsets of X, and for any two subsets A and B of X let $A \oplus B$ denote the symmetric difference of A and B, $A \oplus B = (A \cup B) - (A \cap B)$.

Given a (p, q)- graph $G = (V, E)$ and a nonempty set X, by a set assignment to the vertices of G we mean a function $f : V(G) \rightarrow 2^X$, and given such a function f on the vertex set V we define the value $f^{\oplus}(e)$ of an edge $e = uv$ of G to be the symmetric difference of $f(u)$ and $f(v)$.

Let $f(G) = \{f(u) : u \in V\}$ and $f^{\oplus}(G) = \{f^{\oplus}(e) : e \in E\}$.

We call f a **set indexer** of G if both f and $f^{\oplus}(G)$ are injective functions. A graph is called **set indexable** if it admits a set indexer. A set indexer f of G is called a **segregation** of X on G if $f(G)$ and $f^{\oplus}(G)$ are disjoint families of subsets of X and if, further, they form a partition of $Y(X)$ then f is called **set sequential labeling** of G. (Acharya and Hegde [2])

Acharya[1] has defined a set indexer f, **set graceful labeling**, if $f^{\oplus}(G) = Y(X)$ and if G admits such a set indexer then the graph is said to be **set graceful**.

Since all the nonempty subsets have to appear in any set sequential labeling of a (p, q)-graph G, a necessary condition for G to be set sequential is that $p + q + 1 = 2^m$, for some positive integer $m = |X|$. Similarly, a necessary condition for G to be set graceful is that $q + 1 = 2^m$.

From the above necessary conditions one can see that the cycles are not set sequential and cycles of lengths not equal to $2^m - 1$ are not set graceful.

The following results will give stronger necessary conditions for set sequential and set graceful graphs.

Theorem 1 *If a (p, q)-graph G has*

(1) exactly one or two vertices of even degree, or
(2) exactly three vertices of even degree say, v_1, v_2, v_3 and any two of these vertices are adjacent, or
(3) exactly four vertices of even degree say, v_1, v_2, v_3, v_4, such that at least v_1v_2 and v_3v_4 are edges in G,

then G is not set sequential.

Corollary 1.1: The complete tripartite graph $K_{\{a,b,c\}}$, where $a = 1, 2$ and b, c

are odd, is not set sequential.

Theorem 2 *If a (p,q)-graph G has*

(1) exactly two vertices of odd degree, or
(2) exactly four vertices of odd degree say, v_1, v_2, v_3, v_4, such that at least v_1v_2 and v_3v_4 are edges in G,

then G is not set graceful.

Corollary 2.1: No path is set graceful.

Corollary 2.2: No Theta graph is set graceful.

Theorem 3 *If a (p,q)-graph G has a set graceful labeling f with respect to a set X of cardinality m, there exists a partition of the vertex set V into two nonempty sets V_1 and V_2 such that the number of edges joining the vertices of V_1 with those of V_2 is exactly $2^{\{m-1\}}$.*

Theorem 4 *If a (p,q)-graph G has a set sequential labeling f with respect to a set X of cardinality m, there exists a partition of the vertex set V into two nonempty sets V_1 and V_2 such that the number of edges joining the vertices of V_1 with those of V_2 is exactly $2^{(m-1)} - |V_2|$.*

Acharya and Hegde [2] have proposed the following conjecture:

Conjecture 1: For every integer $m > 1$ such that $m' = 2^{(m+3)} - 7$ is a perfect square, the complete graph K_n of order $n = (1/2)(\sqrt{m'-1})$ is set sequential.

The first four values of n for which K_n may possibly be set sequential are 1,2,5 and 90 where the values of m are 1,2,4 and 12 respectively. We disprove the conjecture by proving the following.

Theorem 5 *The nontrivial complete graph K_n is set sequential with respect to a set X of cardinality $m > 1$ if and only if n = 2,5.*

Theorem 6 *The nontrivial complete n-ary tree T'_n is set sequential if and only if $n = 2^\alpha - 1$ and t = 1, where t is the number of levels of T'_n.*

Conjecture 2: The Caterpillars (which are not paths) satisfying the necessary conditions are set sequential.

Theorem 7 *The nontrivial complete n-ary tree G is set graceful if and only if $n = 2^m - 1$ and $t = 1$.*

Theorem 8 *The nontrivial plane triangular grid G_n is set graceful if and only if $n = 2$.*

Theorem 9 *A tree is set sequential if and only if it is set graceful.*

Embeddings of set sequential and set graceful graphs

One can embed any graph G with p vertices as a subgraph of a set sequential and set graceful graphs. It is enough to show that complete graph with p vertices can be embedded as a subgraph of a set sequential (set graceful) graph.

Theorem 10 *Any graph G with p vertices can be embedded as an induced subgraph of a connected set sequential graph.*

Remark 1: As any graph can be embedded as an induced subgraph of a set sequential graph, there is no forbidden subgraph characterization for set sequential graphs.

Theorem 11 *Any tree with n vertices can be embedded as an induced subgraph of a set sequential tree.*

Theorem 12 *Any planar graph can be embedded as an induced subgraph of a set sequential graph.*

Theorem 13 *Any graph G with p vertices can be embedded as an induced subgraph of a connected set graceful graph.*

Remark 2: As every graph can be embedded as an induced subgraph of a set graceful graph, there is no forbidden subgraph characterization for set graceful graphs.

Theorem 14 *Any tree can be embedded as an induced subgraph of a set graceful tree.*

Theorem 15 *Any planar graph can be embedded as an induced subgraph of a connected set graceful planar graph.*

References

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