Numer. Math. Theor. Meth. Appl. Vol. **11**, No. 3, pp. 506-517
doi: 10.4208/nmtma.2017-OA-0055 August 2018

Enlarging the Ball Convergence for the Modified Newton Method to Solve Equations with Solutions of Multiplicity under Weak Conditions

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Received 25 April 2017; Accepted (in revised version) 26 November 2017

Abstract. The objective of this paper is to enlarge the ball of convergence and improve the error bounds of the modified Newton method for solving equations with solutions of multiplicity under weak conditions.

AMS subject classifications: 65D10, 65G99, 49M15

Key words: Newton's method, multiple solutions, Ball convergence, derivative, divided difference.

1. Introduction

Many problems in applied sciences and also in engineering can be written in the form like

$$F(x)=0 \tag{1.1}$$

using mathematical modeling, where $F:\Omega\subseteq\mathscr{B}_1\longrightarrow\mathscr{B}_2$ is sufficiently many times differentiable and $\Omega,\mathscr{B}_1,\mathscr{B}_2$ are convex subsets in $\mathbb{R}$. In the present study, we pay attention to the case of a solution p of multiplicity $m>1$, namely, $F(p)=0, F^{(i)}(p)=0$ for $i=1,2,\cdots,m-1$, and $F^{(m)}(p)\neq 0$. The determination of solutions of multiplicity m is of great interest. In the study of electron trajectories, when the electron reaches a plate of zero speed, the function distance from the electron to the plate has a solution of multiplicity two. Multiplicity of solution appears in connection to Van Der Waals equation of state and other phenomena. The convergence order of iterative methods decreases if the equation has solutions of multiplicity m. Modifications in the iterative function are made to improve the order of convergence. The modified Newton's method (MN) defined for each $n=0,1,2,\cdots$ by

$$x_{n+1}=x_n-mF'(x_n)^{-1}F(x_n), \tag{1.2}$$

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where $x_0 \in \Omega$ is an initial point is an alternative to Newton's method in the case of solutions with multiplicity m that converges with second order of convergence. A method with third order of convergence is defined by

$$x_{n+1} = x_n - \left(\frac{m+1}{2m} F'(x_n) - \frac{F''(x_n)F(x_n)}{2F'(x_n)} \right)^{-1} F(x_n). \tag{1.3}$$

Method (1.3) is an extension of the classical Halley's method of the third order. Another cubically convergence method was given by Traub [15]:

$$x_{n+1} = x_n - \frac{m(3-m)}{2} \frac{F(x_n)}{F'(x_n)} - \frac{m^2}{2} \frac{F(x_n)^2 F''(x_n)}{F'(x_n)^3}. \tag{1.4}$$

Method (1.4) is an extension of the Chebyshev's method of the third order. Other iterative methods of high convergence order can be found in [5, 6, 9, 12, 15] and the references therein.

Let $B(p, \lambda) := \{x \in B_1 : |x - p| < \lambda\}$ denote an open ball and $\bar{B}(p, \lambda)$ denote its closure. It is said that $B(p, \lambda) \subseteq \Omega$ is a convergence ball for an iterative method, if the sequence generated by this iterative method converges to p, provided that the initial point $x_0 \in B(p, \lambda)$. But how close x_0 should be to p so that convergence can take place? Extending the ball of convergence is very important, since it shows the difficulty, we confront to pick initial points. It is desirable to be able to compute the largest convergence ball. This is usually depending on the iterative method and the conditions imposed on the function F and its derivatives. We can unify these conditions by expressing them as:

$$\left\| (F^{(m)}(p))^{-1} (F^{(m)}(x) - F^{(m)}(y) \right\| \le \psi(\|x - y\|) \tag{1.5}$$

for all $x, y \in \Omega$, where $\psi : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ is a continuous and nondecreasing function satisfying $\psi(0) = 0$. If we specialize function ψ, for $m \ge 1$ and

$$\psi(t) = \mu t^q, \;\; \mu > 0, \;\; q \in (0, 1], \tag{1.6}$$

then, we obtain the conditions under which the preceding methods were studied in [4, 5, 12, 13, 16, 17]. However, there are cases where even (1.6) does not hold (see Example 4.1). Moreover, the smaller function ψ is chosen, the larger the radius of convergence becomes. The technique, we present next can be used for all preceding methods as well as in methods where $m = 1$. However, in the present study, we only use it for MN. This way, in particular, we extend the results in [4, 5, 12, 13, 16, 17]. In view of (1.5) there always exists a function $\varphi_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing, satisfying

$$\left\| (F^{(m)}(p))^{-1} (F^{(m)}(x) - F^{(m)}(p) \right\| \le \varphi_0(\|x - p\|) \tag{1.7}$$

for all $x \in \Omega$ and $\varphi_0(0) = 0$. We can always choose $\varphi_0(t) = \psi(t)$ for all $t \ge 0$. However, in general

$$\varphi_0(t) \le \psi(t), \quad t \ge 0 \tag{1.8}$$

holds and ψ/φ_0 can be arbitrarily large [2]. Denote by r_0 the smallest positive solution of equation $\varphi_0(t) = 1$. Set $\Omega_0 := \Omega \cap B(p, r_0)$. We have again by (1.5) that there exists function $\varphi : [0, r_0) \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing, such that for each $x, y \in \Omega_0$

$$\left\|(F^{(m)}(p))^{-1}(F^{(m)}(x) - F^{(m)}(y)\right\| \leq \varphi(\|x - y\|), \tag{1.9}$$

and $\varphi(0) = 0$. Clearly, we have

$$\varphi(t) \leq \psi(t) \quad \text{for all } t \in [0, r_0), \tag{1.10}$$

since $\Omega_0 \subseteq \Omega$. It turns out that more precise estimate (1.7)(see (1.8)) than (1.5) can be used to estimate upper bounds on the inverses of the functions involved (see (3.11) or (3.18a)). Moreover, for the upper bounds on the numerators (see (3.12) or (3.18b)) we can use (1.9) tighter than (1.5) (see (1.10)). Using this technique, we obtain (3.13) or (3.18c) which are tighter than the corresponding ones using only ψ (or its special case (1.6)). This way we obtain a larger radius of convergence leading to a wider choice of initial guesses and at least as tight error bounds on the distances $|x_n - p|$ resulting in the computation of at least as few iterates to obtain a desired error tolerance (see also the numerical examples and Remark 3.1). It is worth noticing that the preceding advantages are obtained under the same computational cost as in earlier studies, since in practice the computation of function ψ (or ψ in (1.6)) requires the computation of functions φ_0 and φ as special cases.

The rest of the paper is structured as follows. Section 2 contains some auxiliary results on divided differences and derivatives. The ball convergence of MN is given in Section 3. The numerical examples are presented in the concluding Section 4.

2. Auxiliary results

We need the definition of divided differences, and their standard properties which can also be found in [4, 13, 16, 17].

Definition 2.1. ([4]) *The divided differences $F[y_0, y_1, \cdots, y_k]$, on $k+1$ distinct points $y_0, y_1, \cdots y_k$ of a function $f(x)$ are defined by*

$$F[y_0] = F(y_0) \tag{2.1a}$$

$$F[y_0, y_1] = \frac{F[y_0] - F[y_1]}{y_0 - y_1}, \qquad \cdots, \tag{2.1b}$$

$$F[y_0, y_1, \cdots, y_k] = \frac{F[y_0, y_1, \ldots, y_{k-1}] - F[y_0, y_1, \cdots, y_k]}{y_0 - y_k}. \tag{2.1c}$$

If the function F is sufficiently differentiable, then its divided differences $F[y_0, y_1, \cdots, y_k]$ can be defined if some of the arguments y_i coincide. For instance, if $F(x)$ has $k-th$ derivative at y_0, then it makes sense to define

$$F[\underbrace{y_0, y_1, \ldots, y_k}_{k+1}] = \frac{F^{(k)}(y_0)}{k!}. \tag{2.2}$$

Lemma 2.1. ([4]) *The divided differences $F[y_0, y_1, \cdots, y_k]$ are symmetric functions of their arguments,i.e., they are invariant to permutations of the $y_0, y_1, \cdots, y_k$.*

Lemma 2.2. ([16]) *If the function F has $(k+1)-th$ derivative, and p is a zero of multiplicity m, then for every argument x, the following formulae hold*

$$F(x) = F[y_0] + \sum_{i=1}^{k} F[y_0, y_1, \cdots, y_k] \prod_{j=0}^{i-1}(x - y_j) + F[y_0, y_1, \cdots, y_k, x] \prod_{i=0}^{k}(x - y_i). \quad (2.3)$$

Lemma 2.3. ([17]) *If the function F has $(m+1)-th$ derivative, and p is a zero of multiplicity m, then for every argument x, the following formulae hold*

$$F(x) = F[\underbrace{p, p, \cdots, p}_{m}, x](x - p)^m, \quad (2.4a)$$

$$F'(x) = F[\underbrace{p, p, \cdots, p}_{m}, x, x](x - p)^m + mF[\underbrace{p, p, \cdots, p}_{m}, x](x - p)^{m-1}. \quad (2.4b)$$

We need the following lemma on Genocchi's integral expression formula for divided differences:

Lemma 2.4. ([13]) *If the function F has continuous $k-th$ derivative, then the following formula holds for any points $y_0, y_1, \cdots, y_k$*

$$F[y_0, y_1, \cdots, y_k] = \int_0^1 \cdots \int_0^1 F^{(k)}(y_0 + \sum_{i=1}^{k}(y_i - y_{i-1}) \prod_{j=1}^{i} \theta_j) \prod_{i=1}^{k}(\theta_i^{k-i} d\theta_i). \quad (2.5)$$

We shall also use the following Taylor expansion with integral form remainder.

Lemma 2.5. ([17]) *Suppose that $F(x)$ is differentiable $n-$times in the ball $B(x_0, r), r > 0$, and $F^{(n)}(x)$ is integrable from a to $x \in B(a, r)$. Then,*

$$\begin{aligned} F(x) = {} & F(a) + F'(a)(x - a) + \frac{1}{2}F''(a)(x - a)^2 + \cdots + \frac{1}{n!}F^{(n)}(a)(x - a)^n \\ & + \frac{1}{(n-1)!}\int_0^1 \left[F^{(n)}(a + t(x - a)) - F^{(n)}(a)\right](x - a)^n(1 - t)^{n-1}dt, \end{aligned} \quad (2.6a)$$

$$\begin{aligned} F'(x) = {} & F'(a) + F''(a)(x - a) + \frac{1}{2}F'''(a)(x - a)^2 + \cdots + \frac{1}{(n-1)!}F^{(n)}(a)(x - a)^{n-1} \\ & + \frac{1}{(n-2)!}\int_0^1 \left[F^{(n)}(a + t(x - a)) - F^{(n)}(a)\right](x - a)^{n-1}(1 - t)^{n-2}dt. \end{aligned} \quad (2.6b)$$

3. Ball convergence

The ball convergence uses some auxiliary real functions and parameters. Let $\varphi_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ be a continuous and nondecreasing function satisfying $\varphi_0(0) = 0$.

Define function $\beta : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ by

$$\beta(t) = (m-1)! \left[\int_0^1 \cdots \int_0^1 \varphi_0 \left(t \prod_{i=1}^{m-1} \theta_i \right) + (m-1) \int_0^1 \cdots \int_0^1 \varphi_0 \left(t \prod_{i=1}^{m} \theta_i \right) \right] \prod_{i=1}^{m} \theta_i^{m-i} d\theta_i.$$

Notice that $\beta(0) = 0$ and function β is continuous and nondecreasing on $\mathbb{R}_+ \cup \{0\}$. Suppose that

$$(m-1)! \left[\int_0^1 \cdots \int_0^1 \varphi_0 \left(t \prod_{i=1}^{m-1} \theta_i \right) + (m-1) \int_0^1 \cdots \int_0^1 \varphi_0 \left(t \prod_{i=1}^{m} \theta_i \right) \right] \prod_{i=1}^{m} \theta_i^{m-i} d\theta_i - 1$$
$$\longrightarrow +\infty \text{ as } t \longrightarrow \text{a positive number or } +\infty. \tag{3.1}$$

Condition (3.1) can be replaced by a stronger one given by

$$(m-1)! \left[\int_0^1 \cdots \int_0^1 \varphi_0 \left(t_0 \prod_{i=1}^{m-1} \theta_i \right) + (m-1) \int_0^1 \cdots \int_0^1 \varphi_0 \left(t_0 \prod_{i=1}^{m} \theta_i \right) \right] \prod_{i=1}^{m} \theta_i^{m-i} d\theta_i > 1 \tag{3.2}$$

for some $t_0 > 0$. Let $\beta_1(t) = \beta(t) - 1$. By the definition of function β, we have $\beta_1(0) = -1 < 0$. Using (3.1) we get that there exists $t_1 > 0$ such that $\beta_1(t) > 0$ for each $t \geq t_1$. By applying the intermediate value theorem on function β_1 defined on the interval $[0, t_1]$ we deduce that equation $\beta(t) = 1$ has solutions in $(0, t_1)$. Denote by r_0 the smallest positive solution of equation $\beta(t) = 1$. Let $\varphi : [0, r_0) \longrightarrow \mathbb{R}_+ \cup \{0\}$ be a continuous and nondecreasing function satisfying $\varphi(0) = 0$. Moreover, define function α on $[0, r_0)$ by

$$\alpha(t) = (m-1)! \int_0^1 \cdots \int_0^1 \varphi \left(t \prod_{i=1}^{m-1} \theta_i (1 - \theta_m) \right) \prod_{i=1}^{m} \theta_i^{m-i} d\theta_i d\theta_m.$$

Furthermore, define functions δ and γ on $[0, r_0)$ by

$$\gamma(t) = \frac{\alpha(t)}{1 - \beta(t)},$$
$$\delta(t) = \gamma(t) - 1.$$

We get that $\delta(0) = -1 < 0$ and $\delta(t) \longrightarrow +\infty$ as $t \longrightarrow r_0^-$. Denote by r the smallest solution of equation $\delta(t) = 0$ in $(0, r_0)$. Then, we have that for each $t \in [0, r)$

$$0 \leq \beta(t) < 1, \tag{3.3a}$$
$$0 \leq \gamma(t) < 1. \tag{3.3b}$$

First, we show the ball convergence of the modified Newton's method under conditions $(\mathscr{A})$:

$(\mathscr{A}_1)$ $F : \Omega \subseteq \mathscr{B}_1 \longrightarrow \mathscr{B}_2$ is continuously $m-$times Fréchet-differentiable.

$(\mathscr{A}_2)$ Function F has a zero p of multiplicity $m, m = 1, 2, \cdots$.

($\mathscr{A}_3$) There exists function $\varphi_0 : \mathbb{R}_+ \cup \{0\} \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing satisfying $\varphi_0(0) = 0$ such that for each $x \in \Omega$

$$\left\|F^{(m)}(p)^{-1}(F^{(m)}(x) - F^{(m)}(p))\right\| \le \varphi_0(\|x-p\|).$$

Let $\Omega_0 = \Omega \cup B(p, r_0)$, where r_0 is defined previously.

($\mathscr{A}_4$) There exists $\varphi : [0, r) \longrightarrow \mathbb{R}_+ \cup \{0\}$ continuous and nondecreasing satisfying $\varphi(0) = 0$ such that for each $x, y \in \Omega_0$

$$\left\|F^{(m)}(p)^{-1}(F^{(m)}(x) - F^{(m)}(y))\right\| \le \varphi(\|x-y\|).$$

($\mathscr{A}_5$) Condition (3.1) holds.

($\mathscr{A}_6$) $\bar{B}(p, r) \subseteq \Omega$.

Theorem 3.1. *Suppose that the conditions ($\mathscr{A}$) hold. Then, for starting point $x_0 \in B(p, r) - \{p\}$, the sequence $\{x_n\}$ generated by MN is well defined in $B(p, r)$, remains in $B(p, r)$ for all $n = 0, 1, 2, \cdots$ and converges to p.*

Proof. We shall use mathematical induction. It is convenient to define functions $g(x)$ and $g_0(x)$ as follows

$$g(x) = F[\underbrace{p, p, \cdots, p}_{m}, x], \quad g_0(x) = F[\underbrace{p, p, \cdots, p}_{m}, x, x]. \tag{3.4}$$

Let $e_n = x_n - p$. Using Lemma 2.3, we can write:

$$F(x_0) = g(x_0)e_0^m, \tag{3.5a}$$
$$F'(x_0) = [g_0(x_0)e_0 + mg(x_0)]e_0^{m-1}. \tag{3.5b}$$

By NM, (3.5a) and (3.5b), we have

$$\begin{aligned} e_1 =& e_0 - \frac{mg_0(x_0)e_0^m}{[g_0(x_0)e_0 + mg(x_0)]e_0^{m-1}} \\ =& e_0 - \frac{mg(x_0)e_0}{g_0(x_0)e_0 + mg(x_0)} \\ =& \frac{(mg(p))^{-1}g_0(x_0)e_0}{(mg(p))^{-1}[g_0(x_0)e_0 + mg_0(x_0)]}e_0. \end{aligned} \tag{3.6}$$

We suppose that $g_0(x_0)e_0 + mg(x_0) \neq 0$ (which will be shown later). In view of the definition of divided differences, we have

$$g_0((x_0)e_0 = F[\underbrace{p, p, \cdots, p}_{m-1}, x_0, x_0] - g(x_0). \tag{3.7}$$

Then, we obtain from (2.2) and (3.7) that

$$\begin{aligned}
&\left|1-(mg(p))^{-1}[h_0(x_0)e_0+mg(x_0)]\right|\\
=&\left|(mg(p))^{-1}[g_0(x_0)e_0+mg(x_0)-mg(p)]\right|\\
=&(m-1)!\left|F^{(m)}(p)^{-1}(F[\underbrace{p,p,\cdots,p}_{m-1},x_0,x_0]-g(p)+(m-1)[g(x_0)-g(p)])\right|. \qquad (3.8)
\end{aligned}$$

By Lemma 2.4, we get

$$F[\underbrace{p,p,\ldots,p}_{m-1},x_0,x_0]=\int_0^1\cdots\int_0^1 F^{(m)}\Big(p+e_0\prod_{i=1}^{m-1}\theta_i\Big)\prod_{i=1}^{m}(\theta_i^{m-1}d\theta_i), \qquad (3.9a)$$

$$g(x_0)=\int_0^1\cdots\int_0^1 F^{(m)}\Big(p+e_0\prod_{i=1}^{m}\theta_i\Big)\prod_{i=1}^{m}(\theta_i^{m-1}d\theta_i), \qquad (3.9b)$$

$$g(p)=\int_0^1\cdots\int_0^1 F^{(m)}(p)\prod_{i=1}^{m}(\theta_i^{m-1}d\theta_i). \qquad (3.9c)$$

Substituting (3.9a)–(3.9c) into (3.8), using condition $(\mathscr{A}_3)$, $x_0\in B(p,r)$, and the definition of r, we get

$$\begin{aligned}
&\left|1-(mg(p))^{-1}[g_0(x_0)e_0+mg(x_0)]\right|\\
=&(m-1)!\Bigg|\int_0^1\cdots\int_0^1 F^{(m)}(p)^{-1}\left(F^{(m)}\Big(p+e_0\prod_{i=1}^{m-1}\theta_i\Big)-F^{(m)}(p)\right)\prod_{i=1}^{m}(\theta_i^{m-i}d\theta_i)\\
&+(m-1)F^{(m)}(p)^{-1}\left(F^{(m)}\Big(p+e_0\prod_{i=1}^{m-1}\theta_i\Big)-F^{(m)}(p)\right)\prod_{i=1}^{m}(\theta_i^{m-i}d\theta_i)\Bigg|\\
\le&(m-1)!\Bigg(\int_0^1\cdots\int_0^1\left|F^{(m)}(p)^{-1}\Big(F^{(m)}(p+e_0\prod_{i=1}^{m-1}\theta_i)-F^{(m)}(p)\Big)\right|\prod_{i=1}^{m}(\theta_i^{m-i}d\theta_i)\\
&+(m-1)\int_0^1\cdots\int_0^1\left|F^{(m)}(p)^{-1}\Big(F^{(m)}(p)+e_0\prod_{i=1}^{m-1}\theta_i\Big)-F^{(m)}(p)\Big)\right|\prod_{i=1}^{m}(\theta_i^{m-i}d\theta_i)\\
\le&(m-1)!\Bigg[\int_0^1\cdots\int_0^1\varphi_0\Big(|e_0|\prod_{i=1}^{m-1}\theta_i\Big)\prod_{i=1}^{m}\theta_i^{m-i}d\theta_i\\
&+(m-1)\int_0^1\cdots\int_0^1\varphi_0\Big(|e_0|\prod_{i=1}^{m}\theta_i\Big)\prod_{i=1}^{m}\theta_i^{m-i}d\theta_i\Bigg]\\
\le&\beta(|e_0|)<\beta(r)<1. \qquad (3.10)
\end{aligned}$$

It follows from the Banach perturbation lemma [1,3] and (3.10) that, $g_0(x_0)e_0+mg(x_0)\neq 0$ and

$$\left|(mg(p))^{-1}g_0(x_0)e_0+mg(x_0))^{-1}\right|\le\frac{1}{1-\beta(|e_0|)}<\frac{1}{1-\beta(r)}. \qquad (3.11)$$

Moreover, using (3.7), (3.9a), (3.9b) and ($\mathscr{A}_4$), we have in turn that

$$
\begin{aligned}
&\left|(mg(p))^{-1}g_0(x_0)e_0\right| \\
=&(m-1)!\left|\int_0^1\cdots\int_0^1 F^{(m)}(p)^{-1}\left(F^{(m)}\left(p+e_0\prod_{i=1}^{m-1}\theta_i\right)\right.\right. \\
&\left.\left.-F^{(m)}\left(p+e_0\prod_{i=1}^{m}\theta_i\right)\right)\prod_{i=1}^{m}(\theta_i^{m-i}d\theta_i)\right| \\
=&(m-1)!\int_0^1\cdots\int_0^1 \left|F^{(m)}(p)^{-1}\left(F^{(m)}\left(p+e_0\prod_{i=1}^{m-1}\theta_i\right)\right.\right. \\
&\left.\left.-F^{(m)}\left(p+e_0\prod_{i=1}^{m}\theta_i\right)\right)\right|\prod_{i=1}^{m}(\theta_i^{m-i}d\theta_i) \\
\le&(m-1)!\left|\int_0^1\cdots\int_0^1 \varphi_0\left(|e_0|\prod_{i=1}^{m-1}\theta_i(1-\theta_m)\right)\prod_{i=1}^{m}\theta_i^{m-i}d\theta_i d\theta_m\right| \\
=&\alpha(|e_0|)<\alpha(r)<1.
\end{aligned}
\tag{3.12}
$$

Furthermore, by (3.6), (3.11), (3.12) and the definition of r, we get that

$$
\begin{aligned}
|e_1|\le{}&|e_0|\frac{\alpha(|e_0|)}{1-\beta(|e_0|)} \\
\le{}&|e_0|\frac{\alpha(r)}{1-\beta(r)}<|e_0|<r.
\end{aligned}
\tag{3.13}
$$

Hence, we deduce $x_1\in B(p,r)$ and $|e_1|\le c|e_0|$ where $c=\alpha(|e_0|)/1-\beta(|e_0|)\in[0,1)$. By simply replacing x_0,x_1, by x_k,x_{k+1}, we arrive at which shows $\lim_{k\longrightarrow+\infty}x_k=p$ and $x_{k+1}\in B(p,r)$. ☐

Concerning the uniqueness of the solution p, we have:

Proposition 3.1. *Suppose that conditions ($\mathscr{A}$) and*

$$\frac{m}{(s_2-s_1)^m}\int_{s_1}^{s_2}\varphi_0(|t-s_1|)|s_2-t|^{m-1}dt<1 \tag{3.14}$$

for all s_1,t,s_2 with $0\le s_1\le t\le s_2\le\bar{r}$ for some $\bar{r}\ge r$ hold. Then, the solution p of equation $F(x)=0$ is unique in $\Omega_1=\Omega\cap\bar{B}(p,\bar{r})$.

Proof. Suppose that $p^*\in\Omega_1$ is a solution of equation $F(x)=0$ with $p\neq p^*$. Without loss of generality, suppose $p<p^*$. We can write

$$F(p^*)-F(p)=\frac{1}{(m-1)!}\int_p^{p^*}F^{(m)}(t)(p^*-t)^{m-1}dt. \tag{3.15}$$

Using ($\mathscr{A}_3$) and (3.14), we obtain in turn that

$$
\begin{aligned}
&\left|1-\left(\frac{(p^*-p)^m}{m}F^{(m)}(p)\right)^{-1}\int_p^{p^*}F^{(m)}(t)(p^*-t)^{m-1}dt\right| \\
=&\left|\left(\frac{(p^*-p)^m}{m}F^{(m)}(p)\right)^{-1}\int_p^{p^*}\left[F^{(m)}(t)-F^{(m)}(p)\right](p^*-t)^{m-1}dt\right| \\
\le&\frac{m}{(p^*-p)^m}\int_p^{p^*}\varphi_0(|t-p|)|p^*-t|^{m-1}dt<1,
\end{aligned}
\tag{3.16}
$$

so $\left(\frac{(p^*-p)^m}{m}F^{(m)}(p)\right)^{-1}\int_p^{p^*}F^{(m)}(t)(p^*-t)^{m-1}dt$ is invertible, i.e., $\int_p^{p^*}F^{(m)}(t)(p^*-t)^{m-1}dt$ is invertible. □

Next, in an analogous way, we shall present a ball convergence result for NM by dropping ($\mathscr{A}_4$) from conditions ($\mathscr{A}$). Consider, again functions $\bar{\alpha},\bar{\beta},\bar{\gamma}$ and $\bar{\delta}$ defined by

$$
\begin{aligned}
\bar{\alpha}(t)&=\int_0^1\varphi_0(t\theta)|1-m\theta|d\theta,\\
\bar{\beta}(t)&=(m-1)\int_0^1\varphi_0(t\theta)(1-\theta)^{m-2}d\theta,\\
\bar{\gamma}(t)&=\frac{\bar{\alpha}(t)}{1-\bar{\beta}(t)},\quad \bar{\delta}(t)=\bar{\gamma}(t)-1
\end{aligned}
$$

with corresponding radii, ρ_0 and ρ. Replace r_0, r by ρ_0 and ρ, respectively and drop ($\mathscr{A}_4$) from the conditions ($\mathscr{A}$). Denote the resulting conditions by ($\mathscr{A}'$). Then, Theorem 3.1 and Proposition 3.1 can be reproduced in this weaker setting.

Theorem 3.2. *Suppose that conditions ($\mathscr{A}'$) hold. Then, the conclusions of Theorem 3.1 hold.*

Proof. Using Lemma 2.5 and MN, we get instead of (3.6), (3.10)-(3.13), respectively

$$
e_1=\frac{\int_0^1\left[F^{(m)}(p+\theta e_0)-F^{(m)}(p)\right]\left[(m-1)(1-t)^{m-2}-m(1-t)^{m-1}\right]dte_0}{F^{(m)}(p)+(m-1)\int_0^1[F^{(m)}(p+te_0)-F^{(m)}(p)](1-t)^{m-2}dt},\tag{3.17a}
$$

$$
\begin{aligned}
&\left|1-(F^{(m)}(p))^{-1}\times F^{(m)}(p)+(m-1)\int_0^1\left[F^{(m)}(p+\theta e_0)-F^{(m)}(p)\right](1-t)^{m-2}dt\right|\\
=&(m-1)\left|\int_0^1(F^{(m)}(p))^{-1}\left[F^{(m)}(p+te_0)-F^{(m)}(p)\right](1-t)^{m-2}dt\right|\\
\le&(m-1)\int_0^1\varphi_0(t|e_0|)(1-t)^{m-2}dt=\bar{\beta}(|e_0|)<\bar{\beta}(\rho)<1.
\end{aligned}
\tag{3.17b}
$$

Hence, we have that $F^{(m)}(p)+(m-1)\int_0^1\left[F^{(m)}(p+te_0)-F^{(m)}(p)\right](1-t)^{m-2}dt\neq 0$,

$$\left|\left(F^{(m)}(p)+(m-1)\int_0^1\left[F^{(m)}(p+te_0)-F^{(m)}(p)\right]\,(1-t)^{m-2}dt\right)^{-1}F^{(m)}(p)\right|$$
$$\leq\frac{1}{1-\bar{\beta}(|e_0|)}<\frac{1}{1-\bar{\beta}(\rho)}, \tag{3.18a}$$
$$\left|\int_0^1 F^{(m)}(p)^{-1}\left[F^{(m)}(p+te_0)-F^{(m)}(p)\right]\left[(m-1)(1-t)^{m-2}-m(1-t)^{m-1}\right]dt\right|$$
$$\leq\int_0^1\varphi_0(t|e_0|)|1-mt|dt=\bar{\alpha}(e_0)<\bar{\alpha}(\rho), \tag{3.18b}$$
$$|e_1|\leq\frac{\bar{\alpha}(|e_0|)}{1-\bar{\beta}(|e_0|)}|e_0|\leq\bar{c}\gamma(\rho)<|e_0|, \tag{3.18c}$$
$$|x_{k+1}-p|\leq\bar{c}|x_k-p|<\rho, \tag{3.18d}$$

where $\bar{c}=\bar{\gamma}(|x_0-p|)\in[0,1)$. □

Proposition 3.2. *Suppose that the conditions ($\mathscr{A}'$) and*

$$\frac{m}{(s_2-s_1)^m}\int_{s_1}^{s_2}\varphi_0(|t-s_1|)|s_2-t|^{m-1}dt<1 \tag{3.19}$$

for all s_1,t,s_2 with $0\leq s_1\leq t\leq s_2\leq\bar{\rho}$ for some $\bar{\rho}\geq\rho$ hold. Then, the solution p of equation $F(x)=0$ is unique in $\Omega_2=\Omega\cap\bar{B}(p,\bar{\rho})$.

Proof. Simply replace Ω_1, (3.14), $r,\bar{r}$ by Ω_2, (3.19), $\rho,\bar{\rho}$, respectively, in the proof of Proposition 3.1. □

Remark 3.1. (a) Let functions $\alpha^*,\beta^*,\gamma^*,\delta^*$ be functions corresponding to $\alpha,\beta,\gamma,\delta$ respectively but with φ_0 and φ replaced by ψ. Then, with the old approach we must solve equation

$$\delta^*(t)=\gamma^*(t)-1=0 \tag{3.20}$$

to obtain the solution r^* corresponding to r. In view of (1.8) and (1.10), we have that

$$\alpha(t)\leq\alpha^*(t),\quad \beta(t)\leq\beta^*(t)<1$$

Consequently,

$$\frac{\alpha(t)}{1-\beta(t)}\leq\frac{\alpha^*(t)}{1-\beta^*(t)},\quad \gamma(t)\leq\gamma^*(t)$$

implies that

$$\delta(t)\leq\delta^*(t). \tag{3.21}$$

We have $\delta(0)=-1<0$ and $\delta^*(r)\geq\delta(r)=0$ so

$$r^*\leq r. \tag{3.22}$$

(b) Similarly let $\bar{\alpha}^*, \bar{\beta}^*, \bar{\gamma}^*, \bar{\delta}^*$ be functions corresponding to $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$, respectively with function φ_0 replaced by ψ. Then, in view of (1.8) as in part (a) we have

$$\bar{\delta}(t) \le \bar{\delta}^*(t) \tag{3.23}$$

leading to

$$\rho^* \le \rho, \tag{3.24}$$

where ρ^* is the smallest positive solution of equation

$$\bar{\delta}^*(t) = \bar{\gamma}^*(t) - 1 = 0. \tag{3.25}$$

4. Numerical examples

We present two numerical examples in this section.

Example 4.1. Let $\mathscr{B}_1 = \mathscr{B}_2 = \mathbb{R}, \Omega = [0,1], m = 2, p = 0$. Define function F on Ω by

$$F(x) = \frac{4}{15}x^{\frac{5}{2}} + \frac{1}{2}x^2.$$

We have $F'(x) = \frac{2}{3}x^{\frac{3}{2}} + x$, $F''(x) = x^{\frac{1}{2}} + 1$, $F''(0) = 1$. Function F'' cannot satisfy (1.5) with ψ given by (1.6) since $F'''(0)$ does not exist. Hence, the results in [4, 5, 12, 13, 16, 17] cannot apply. However, the new results apply for $\varphi_0(t) = \varphi(t) = t^{\frac{1}{2}}$. Moreover, the convergence radii are: $r = 0.5407$ and $\rho = 1.1480$, so we can choose $\rho = 1$.

Example 4.2. Let $\mathscr{B}_1 = \mathscr{B}_2 = \mathbb{R}, \Omega = [-1,1], m = 2, p = 0$. Define function F on Ω by

$$F(x) = e^x - x - 1.$$

We get $r_0 = 2.0951, \varphi(t) = \psi(t) = et$ and $\varphi_0(t) = (e-1)t$. Notice that

$$\varphi_0(t) < \varphi(t) = \psi(t) \quad \text{for all } t \ge 0.$$

Then, the old results give $r^* = 0.5518191617571$ and $\rho^* = 1.1036$ The new results give: $r = 1.745930120607978$ and $\rho = 1.7459$, so we choose $\rho = r = 1$. Notice that

$$\begin{aligned} &r^* < r, \quad \rho^* < \rho, \\ &\delta(t) < \delta^*(t), \\ &\bar{\delta}(t) < \bar{\delta}^*(t). \end{aligned}$$

Hence, we obtain a larger radius of convergence and a smaller ratio of convergence than the ones given before in [4, 13, 16, 17].

References

[1] S. Amat, M. A. Hernández and N. Romero, *Semilocal convergence of a sixth order iterative method for quadratic equations*, Elsevier Science Publishers B. V., 62 (2012), pp. 833–841.

[2] I. K. Argyros, Computational Theory of Iterative Methods, Series: Studies in Computational Mathematics 15, Elsevier B, 2007.

[3] I. K. Argyros, *On the convergence and application of Newtons method under Hölder continuity assumptions*, Int. J. Comput. Math. , 80 (2003), pp. 767–780.

[4] W. Bi, H. M. Ren and Q. B. Wu, *Convergence of the modified Halley's method for multiple zeros under Höder continous derivatives*, Numer. Algorithms, 58 (2011), pp. 497–512.

[5] C. Chun and B. Neta, *A third order modification of Newton's method for multiple roots*, Appl. Math. Comput., 211 (2009), pp. 474–479.

[6] E. Hansen and M. Patrick, *A family of root finding methods*, Numer. Math., 27 (1977), pp. 257–269.

[7] A. A. Magreñán, *Different anomalies in a Jarratt family of iterative root finding methods*, Appl. Math. Comput., 233 (2014), pp. 29–38.

[8] A. A. Magreñán, *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput., 248 (2014), pp. 29–38.

[9] B. Neta, *New third order nonlinear solvers for multiple roots*, Appl. Math. Comput., 202 (2008), pp. 162–170.

[10] N. Obreshkov, *On the numerical solution of equations (Bulgarian)*, Annuaire Univ. Sofia. Fac. Sci. Phy. Math., 56 (1963), pp. 73–83.

[11] N. Osada, *An optimal mutiple root-finding method of order three*, J. Comput. Appl. Math., 52 (1994), pp. 131–133.

[12] M. S. Petkovic, B. Neta, L. Petkovic and J. Džunič, Multipoint Methods for Solving Nonlinear Equations, Elsevier, 2013.

[13] H. M. Ren and I. K. Argyros, *Convergence radius of the modified Newton method for multiple zeros under Hölder continuity derivative*, Appl. Math. Comput., 217, (2010), pp. 612–621.

[14] E. Schröder, *Uber unendlich viele Algorithmen zur Auflosung der Gieichunger*, Math. Ann., 2 (1870), pp. 317–365.

[15] J. F. Traub, *Iterative methods for the solution of equations*, AM. Math. Mon., 19(89) (1964), pp. 310.

[16] X. Zhou and Y. Song, *Convergence radius of Osada's method for multiple roots under Hölder and Center-Hölder continous conditions*, AIP Conf. Proc., 1389 (2011), pp. 1836–1839.

[17] X. Zhou, X. Chen and Y. Song, *On the convergence radius of the modified Newton method for multiple roots under the center- Hölder condition*, Numer. Algorithms, 65(2) (2014), pp. 221–232.