

# Iterative Regularization Methods for Ill-Posed Hammerstein Type Operator Equation with Monotone Nonlinear Part

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## Abstract

We considered a procedure for solving an ill-posed Hammerstein type operator equation  $KF(x) = y$ , by solving the linear equation  $Kz = y$  first for  $z$  and then solving the nonlinear equation  $F(x) = z$ . Convergence analysis is carried out by means of suitably constructed majorizing sequences. The derived error estimate using an adaptive method proposed by Perverzev and Schock (2005) in relation to the noise level and a stopping rule based on the majorizing sequences are shown to be of optimal order with respect to certain assumptions on  $F(\hat{x})$ , where  $\hat{x}$  is the solution of  $KF(x) = y$ .

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## 1 Introduction

In this paper we consider the problem of approximately solving a nonlinear ill-posed operator equation of the Hammerstein type with a monotone nonlinear

part. Recall that a Hammerstein type operator (see [6, 7, 8, 10]) is an operator of the form  $KF$ , where  $F : D(F) \subset X \mapsto Z$  is nonlinear and  $K : Z \mapsto Y$  is a bounded linear operator and  $X, Y, Z$  are taken to be Hilbert spaces. We are interested in the case when  $Z = X$  and  $F$  is a monotone operator (cf. [15]). i.e.,  $F : D(F) \subset X \mapsto X$  satisfies

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in D(F).$$

So we consider an equation of form

$$KF(x) = y \tag{1}$$

where  $F : D(F) \subset X \mapsto X$  is monotone and  $K : X \mapsto Y$  is linear. It is assumed that (1) has a solution  $\hat{x} \in D(F)$ .

We assume throughout that  $y^\delta \in Y$  are the available noisy data with

$$\|y - y^\delta\| \leq \delta. \tag{2}$$

Observe that (cf. [10]) the solution  $\hat{x}$  of (1) can be obtained by first solving the linear equation

$$Kz = y \tag{3}$$

for  $z$  and then solving the nonlinear equation

$$F(x) = z. \tag{4}$$

For the treatment of nonlinear ill-posed problems the standard regularization method is the method of Tikhonov regularization. But if the nonlinear operator is monotone then a simpler regularization strategy available is the Lavrentiev regularization. Note that  $KF$  need not be monotone even if  $F$  is monotone. So in the straight forward approach one has to consider Tikhonov regularization method for approximately solving (1).

What we show in this paper is that for the special case when  $K$  is linear and  $F$  is monotone, by splitting the equation (1) into (3) and (4), one can simplify the procedure by specifying a regularization strategy for linear part (3) and an iterative method for nonlinear part (4). More precisely, for fixed  $\alpha > 0$ ,  $\delta > 0$  we consider the regularized solution of (3) with  $y^\delta$  in place of  $y$  as

$$z_\alpha^\delta = (K + \alpha I)^{-1} y^\delta \tag{5}$$

if the operator  $K$  in (3) is positive self adjoint and  $X = Y$ , otherwise we consider

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1} K^* y^\delta. \tag{6}$$

Note that (5) is the simplified or Lavrentiev regularization (see [5]) of the equation (3) and (6) is the Tikhonov regularization (see [2, 3, 4, 9, 12, 14]) of

(3). The regularization parameter is chosen according to an adaptive method proposed by Pereverzev and Schock in [11]. Also one can see that the iterative method we considered in section 3 and section 4 for the nonlinear equation (4) do not involve any regularization parameter explicitly.

In [10], it is assumed that the bounded inverse of  $F'(x_0)$  exist and considered the sequence

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_0)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta), \tag{7}$$

with  $x_{0,\alpha}^\delta = x_0$  and proved that  $(x_{n,\alpha}^\delta)$  converges linearly to the solution  $x_\alpha^\delta$  of

$$F(x) = z_\alpha^\delta. \tag{8}$$

Later in [8], George and Kunhanadan considered the sequence  $(x_{n,\alpha}^\delta)$  defined iteratively as

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_{n,\alpha}^\delta)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta), \tag{9}$$

with  $x_{0,\alpha}^\delta = x_0$  and proved that  $(x_{n,\alpha}^\delta)$  converges quadratically to the solution  $x_\alpha^\delta$  of (8) under the assumption that the bounded inverse of  $F'(x)$  exist in a neighborhood of  $x_0$ .

Recall that a sequence  $(x_n)$  is  $X$  with  $\lim x_n = x^*$  is said to converge quadratically, if there exists positive number  $M$ , not necessarily less than 1, such that for all  $n$  sufficiently large

$$\|x_{n+1} - x^*\| \leq M\|x_n - x^*\|^2. \tag{10}$$

If the sequence  $(x_n)$  has the property that

$$\|x_{n+1} - x^*\| \leq q\|x_n - x^*\|, \quad 0 < q < 1$$

then  $(x_n)$  is said to be linearly convergent. For an extensive discussion of convergence rate, see Ortega and Rheinboldt [13].

Note that the ill-posedness of equation (1) in [10] and in [8] is due to the ill-posedness of the linear equation (3). In the present paper we assume that (1) is ill-posed in both the linear part (3) and the nonlinear part (4). Using the monotonicity of  $F$ , we carry out the convergence analysis by means of suitably constructed majorizing sequences, deviating from the methods used in [10] and [8]. An advantage of this approach is that the majorizing sequence gives an a priori error estimate which can be used to determine the number of iterations needed to achieve a prescribed solution accuracy before actual computation takes place.

Organization of this paper is as follows. We collected some preparatory results in section 2. Convergence analysis of an iterated sequence converging quadratically is given in section 3 and in section 4 we consider another sequence which converges linearly. In section 5 we give error analysis and derive optimal order error bounds. Finally in section 6 we consider an algorithm for implementing method considered in this paper.

## 2 Preparatory Results

Throughout this paper we assume that the operator  $F$  satisfies the following assumptions.

**Assumption 2.1** *There exists  $r > 0$  such that  $B_r(\hat{x}) \subseteq D(F)$  and  $F$  is Fréchet differentiable at all  $x \in B_r(\hat{x})$ .*

**Assumption 2.2** *There exists a constant  $k_0 > 0$  such that for every  $x, u \in B_r(\hat{x})$  and  $v \in X$ , there exists an element  $\Phi(x, u, v) \in X$  satisfying*

$$[F'(x) - F'(u)]v = F'(u)\Phi(x, u, v), \|\Phi(x, u, v)\| \leq k_0\|v\|\|x - u\|$$

for all  $x, u \in B_r(\hat{x})$  and  $v \in X$ .

The next assumption on source condition is based on a source function  $\varphi$  and a property of the source function  $\varphi$ . We will be using this assumption to obtain an error estimate for  $\|F(\hat{x}) - z_\alpha^\delta\|$ .

**Assumption 2.3** *There exists a continuous, strictly monotonically increasing function  $\varphi : (0, a] \rightarrow (0, \infty)$  with  $a \geq \|K^*K\|$  satisfying;*

- $\lim_{\lambda \rightarrow 0} \varphi(\lambda) = 0$
- $$\sup_{\lambda \geq 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq c_\varphi\varphi(\alpha), \quad \forall \alpha \in (0, a].$$
- *there exists  $v \in X$  such that*

$$F(\hat{x}) = \varphi(K^*K)v \tag{11}$$

Let

$$z_\alpha := (K^*K + \alpha I)^{-1}K^*y.$$

Hereafter we consider  $z_\alpha^\delta$  as in (6). We observe that

$$\begin{aligned} \|F(\hat{x}) - z_\alpha^\delta\| &\leq \|F(\hat{x}) - z_\alpha\| + \|z_\alpha - z_\alpha^\delta\| \\ &\leq \|F(\hat{x}) - z_\alpha\| + \frac{\delta}{\sqrt{\alpha}}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} F(\hat{x}) - z_\alpha &= F(\hat{x}) - (K^*K + \alpha I)^{-1}K^*KF(\hat{x}) \\ &= [I - (K^*K + \alpha I)^{-1}K^*K]F(\hat{x}) \\ &= \alpha(K^*K + \alpha I)^{-1}F(\hat{x}). \end{aligned} \tag{13}$$

So by Assumption 2.3,

$$\|F(\hat{x}) - z_\alpha\| \leq \|v\|c_\varphi\varphi(\alpha). \tag{14}$$

Thus we have the following theorem.

**THEOREM 2.4** *Let  $z_\alpha^\delta$  be as in (6) and the Assumption 2.3 holds. Then*

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \max\{\|v\|_{c_\varphi}, 1\}(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}). \tag{15}$$

### 2.1 A priori choice of the parameter

Note that the estimate  $\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}$  in (15) attains minimum for the choice  $\alpha := \alpha_\delta$  which satisfies  $\varphi(\alpha_\delta) = \frac{\delta}{\sqrt{\alpha_\delta}}$ . Let  $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$ . Then we have  $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$ , and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)). \tag{16}$$

So Theorem 2.4 and the above observation lead to the following.

**THEOREM 2.5** *Let  $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}, 0 < \lambda \leq \|K\|^2$  and the assumptions of Theorem 2.4 are satisfied. For  $\delta > 0$ , let  $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$ . Then*

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \mathcal{O}(\psi^{-1}(\delta)).$$

### 2.2 An adaptive choice of the parameter

The error estimate in the above Theorem has optimal order with respect to  $\delta$ . Unfortunately, an a priori parameter choice (16) cannot be used in practice since the smoothness properties of the unknown solution  $\hat{x}$  reflected in the function  $\varphi$  are generally unknown. There exist many parameter choice strategies in the literature (cf. [5], [8], [11], [15], etc.).

In [11], Pereverzev and Schock considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. In this method the regularization parameter  $\alpha_i$  are selected from some finite set  $\{\alpha_i : 0 < \alpha_0 < \alpha_1 < \dots < \alpha_N\}$  and the corresponding regularized solution, say  $z_{\alpha_i}^\delta$  are studied on-line. In this paper also, we consider the adaptive method for selecting the parameter  $\alpha$  in  $z_\alpha^\delta$ .

Let  $i \in \{0, 1, 2, \dots, N\}$  and  $\alpha_i = \mu^{2i}\alpha_0$  where  $\mu > 1$  and  $\alpha_0 = \delta^2$ . Let

$$l := \max\{i : \varphi(\alpha_i) \leq \frac{\delta}{\sqrt{\alpha_i}}\} \tag{17}$$

and

$$k := \max\{i : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \leq \frac{4\delta}{\sqrt{\alpha_j}}, j = 0, 1, 2, \dots, i\}. \tag{18}$$

We will be using the following theorem from [8]

**THEOREM 2.6** *(cf. [8], Theorem 4.2) Let  $l$  be as in (17),  $k$  be as in (18) and  $z_{\alpha_k}^\delta$  be as in (6) with  $\alpha = \alpha_k$ . Then  $l \leq k$  and*

$$\|F(\hat{x}) - z_{\alpha_k}^\delta\| \leq (2 + \frac{4\mu}{\mu - 1})\mu\psi^{-1}(\delta).$$

### 3 Quadratic Convergence

Now consider the nonlinear equation (4) with  $z_{\alpha_k}^\delta$  in place of  $z$ . It can be seen as in [15], Theorem 1.1, that for monotone operator  $F$ , the equation

$$F(x) + (x - x_0) = z_{\alpha_k}^\delta. \tag{19}$$

has a unique solution  $x_{\alpha_k}^\delta$ . It is interesting to note that the camouflaged presence of regularization parameter in  $\alpha_k$ , in (19) relieves us of the labour of Lavrentiev regularization in the nonlinear part.

We propose the following iterative method for computing the solution  $x_{\alpha_k}^\delta$ . For  $n \geq 0$ , let

$$x_{n+1, \alpha_k}^\delta = x_{n, \alpha_k}^\delta - (F'(x_{n, \alpha_k}^\delta) + I)^{-1}(F(x_{n, \alpha_k}^\delta) - z_{\alpha_k}^\delta + (x_{n, \alpha_k}^\delta - x_0)), \tag{20}$$

where  $x_0$  is a starting point of the iteration. The main goal of this section is to provide sufficient conditions for the quadratic convergence of method (20) to  $x_{\alpha_k}^\delta$  and obtain an error estimate for  $\|x_{\alpha_k}^\delta - x_{n, \alpha_k}^\delta\|$ . We use a majorizing sequence for proving our results. Recall (see [1], Definition 1.3.11) that a nonnegative sequence  $(t_n)$  is said to be a majorizing sequence of a sequence  $(x_n)$  in  $X$  if

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n, \forall n \geq 0.$$

During the convergence analysis we will be using the following Lemma on majorization, which is a reformulation of Lemma 1.3.12 in [1].

**LEMMA 3.1** *Let  $(t_n)$  be a majorizing sequence for  $(x_n)$  in  $X$ . If  $\lim_{n \rightarrow \infty} t_n = t^*$ , then  $x^* = \lim_{n \rightarrow \infty} x_n$  exists and*

$$\|x^* - x_n\| \leq t^* - t_n, \forall n \geq 0. \tag{21}$$

The next Lemma on majorizing sequence is used to prove the convergence of the method (20).

**LEMMA 3.2** *Assume there exist nonnegative numbers  $q \in [0, 1)$  and  $\kappa_0, \eta$  nonnegative such that for all  $n \geq 0$ ,*

$$\frac{3\kappa_0}{2}q^n\eta \leq q. \tag{22}$$

*Then the iteration  $(t_n), n \geq 0$ , given by  $t_0 = 0, t_1 = \eta$ ,*

$$t_{n+1} = t_n + \frac{3\kappa_0}{2}(t_n - t_{n-1})^2 \tag{23}$$

*is increasing, bounded above by  $t^{**} := \frac{\eta}{1-q}$ , and converges to some  $t^*$  such that  $0 < t^* \leq \frac{\eta}{1-q}$ . Moreover, for  $n \geq 0$ ;*

$$0 \leq t_{n+1} - t_n \leq q(t_n - t_{n-1}) \leq q^n\eta, \tag{24}$$

and

$$t^* - t_n \leq \frac{q^n}{1 - q} \eta. \tag{25}$$

**Proof.** Since the result holds for  $\eta = 0, \kappa_0 = 0$  or  $q = 0$ , we assume that  $\kappa_0 \neq 0, \eta \neq 0$  and  $q \neq 0$ . Observe that  $t_{i+1} - t_i \geq 0$  for all  $i \geq 0$ . If

$$\frac{3\kappa_0}{2}(t_{i+1} - t_i) \leq q, \tag{26}$$

then the estimate (25) follows from (23). So we shall prove (26) by induction on  $i \geq 0$ .

For  $i = 0$ , (26) holds by (22). Suppose (26) holds for all  $i \leq k$  for some  $k$ . Then by (23) we have

$$\frac{3\kappa_0}{2}(t_{k+2} - t_{k+1}) \leq \left(\frac{3\kappa_0}{2}(t_{k+1} - t_k)\right)^2 \leq q^2 < q.$$

Thus by induction (26) holds for all  $i \geq 0$ . Also, for  $k \geq 0$ ,

$$\begin{aligned} t_{k+1} \leq t_k + q(t_k - t_{k-1}) &\leq \dots \leq \eta + q\eta + \dots + q^k \eta \\ &= \frac{1 - q^{k+1}}{1 - q} \eta < \frac{\eta}{1 - q}. \end{aligned}$$

Hence the sequence  $(t_n), n \geq 0$  is bounded above by  $\frac{\eta}{1 - q}$  and is nondecreasing. So it converges to some  $t^* \leq \frac{\eta}{1 - q}$ . Further,

$$t^* - t_n = \lim_{i \rightarrow \infty} t_{n+i} - t_n \leq \lim_{i \rightarrow \infty} \sum_{j=0}^{i-1} (t_{n+1+j} - t_{n+j}) \leq \frac{q^n}{1 - q} \eta.$$

This completes the proof of the lemma.

To prove the convergence of the sequence  $(x_{n,\alpha_k}^\delta)$  defined in (20) we introduce the following notations:

Let  $R(x) := F'(x) + I$  and

$$G(x) := x - R(x)^{-1}[F(x) - z_{\alpha_k}^\delta + (x - x_0)]. \tag{27}$$

Note that with the above notation  $G(x_{n,\alpha_k}^\delta) = x_{n+1,\alpha_k}^\delta$ .

**THEOREM 3.3** *Let  $q \geq t^*$ . Under the assumption 2.2 and the assumptions in the Lemma 3.2 the sequence  $(x_{n,\alpha_k}^\delta)$  defined in (20) is well defined and  $x_{n,\alpha_k}^\delta \in B_{t^*}(x_0)$  for all  $n \geq 0$ . Further  $(x_{n,\alpha_k}^\delta)$  is a Cauchy sequence in  $B_{t^*}(x_0)$  and hence converges to  $x_{\alpha_k}^\delta \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$  and  $F(x_{\alpha_k}^\delta) = z_{\alpha_k}^\delta + (x_0 - x_{\alpha_k}^\delta)$ .*

*Moreover, the following estimates hold for all  $n \geq 0$ ,*

$$\|x_{n+1,\alpha_k}^\delta - x_{n,\alpha_k}^\delta\| \leq t_{n+1} - t_n, \tag{28}$$

$$\|x_{n,\alpha_k}^\delta - x_{\alpha_k}^\delta\| \leq t^* - t_n \leq \frac{q^n \eta}{1 - q}, \quad (29)$$

and

$$\|x_{n+1,\alpha_k}^\delta - x_{\alpha_k}^\delta\| \leq \frac{k_0}{2} \|x_{n,\alpha_k}^\delta - x_{\alpha_k}^\delta\|^2. \quad (30)$$

**Proof.** First we shall prove that

$$\|x_{n+1,\alpha_k}^\delta - x_{n,\alpha_k}^\delta\| \leq \frac{3\kappa_0}{2} \|x_{n,\alpha_k}^\delta - x_{n-1,\alpha_k}^\delta\|^2. \quad (31)$$

With  $G$  as in (27), we have for  $u, v \in B_{t^*}(x_0)$ ,

$$\begin{aligned} G(u) - G(v) &= u - v - R(u)^{-1}[F(u) - z_{\alpha_k}^\delta + (u - x_0)] \\ &\quad + R(v)^{-1}[F(v) - z_{\alpha_k}^\delta + (v - x_0)] \\ &= u - v - [R(u)^{-1} - R(v)^{-1}](F(v) - z_{\alpha_k}^\delta + (v - x_0)) \\ &\quad - R(u)^{-1}(F(u) - F(v) + (u - v)) \\ &= R(u)^{-1}[F'(u)(u - v) - (F(u) - F(v))] \\ &\quad - R(u)^{-1}[F'(v) - F'(u)]R(v)^{-1}(F(v) - z_{\alpha_k}^\delta + (v - x_0)) \\ &= R(u)^{-1}[F'(u)(u - v) - (F(u) - F(v))] \\ &\quad + R(u)^{-1}[F'(v) - F'(u)](v - G(v)) \\ &= R(u)^{-1}[F'(u)(u - v) + \int_0^1 (F'(u + t(v - u)))(v - u)dt] \\ &\quad + R(u)^{-1}[F'(v) - F'(u)](v - G(v)) \\ &= \int_0^1 R(u)^{-1}[(F'(u + t(v - u)) - F'(u))(v - u)dt] \\ &\quad + R(u)^{-1}[F'(v) - F'(u)](v - G(v)) \end{aligned}$$

The last but one step follows from the Fundamental Theorem of Integral Calculus. So by the Assumption 2.2 and the estimate  $\|R(u)^{-1}F'(u)\| \leq 1$ , we have

$$\|G(u) - G(v)\| \leq \frac{\kappa_0}{2} \|u - v\|^2 + \kappa_0 \|u - v\| \|v - G(v)\|. \quad (32)$$

Now taking  $u = x_{n,\alpha_k}^\delta$  and  $v = x_{n-1,\alpha_k}^\delta$  in (32), we obtain (31).

Next we shall prove that the sequence  $(t_n)$  defined in Lemma 3.2 is a majorizing sequence of the sequence  $(x_{n,\alpha_k}^\delta)$ .

Note that  $\|x_{1,\alpha_k}^\delta - x_0\| = \|R(x_0)^{-1}(F(x_0) - z_{\alpha_k}^\delta)\| \leq \eta = t_1 - t_0$ . Assume that  $\|x_{i+1,\alpha_k}^\delta - x_{i,\alpha_k}^\delta\| \leq t_{i+1} - t_i$  for all  $i \leq k$  for some  $k$ . Then by (31),

$$\|x_{k+2,\alpha_k}^\delta - x_{k+1,\alpha_k}^\delta\| \leq \frac{3\kappa_0}{2} \|x_{k+1,\alpha_k}^\delta - x_{k,\alpha_k}^\delta\|^2 \leq \frac{3\kappa_0}{2} (t_{k+1} - t_k)^2 = t_{k+2} - t_{k+1}.$$



Thus by induction  $\|x_{n+1,\alpha_k}^\delta - x_{n,\alpha_k}^\delta\| \leq t_{n+1} - t_n$  for all  $n \geq 0$  and hence  $(t_n), n \geq 0$  is a majorizing sequence of the sequence  $(x_{n,\alpha_k}^\delta)$ . So by Lemma 3.1  $(x_{n,\alpha_k}^\delta), n \geq 0$  is a Cauchy sequence and converges to some  $x_{\alpha_k}^\delta \in \overline{B_{t^*}(x_0)} \subset B_{t^{**}}(x_0)$  and

$$\|x_{\alpha_k}^\delta - x_{n,\alpha_k}^\delta\| \leq t^* - t_n \leq \frac{q^n \eta}{1 - q}.$$

To prove (30), we observe that  $G(x_{\alpha_k}^\delta) = x_{\alpha_k}^\delta$ , so (30) follows from (32), by taking  $u = x_{n,\alpha_k}^\delta$  and  $v = x_{\alpha_k}^\delta$  in (32). Now by letting  $n \rightarrow \infty$  in (19) we obtain  $F(x_{\alpha_k}^\delta) = z_{\alpha_k}^\delta + (x_0 - x_{\alpha_k}^\delta)$ . This completes the proof of the Theorem.

**REMARK 3.4** *Note that (30) implies  $(x_{n,\alpha_k}^\delta)$  converges quadratically to  $x_{\alpha_k}^\delta$ .*

## 4 Linear Convergence

In this section, we consider the sequence  $(\tilde{x}_n^\delta)$  defined iteratively by

$$\tilde{x}_{n+1}^\delta := \tilde{x}_n^\delta - (F'(x_0) + I)^{-1}(F(\tilde{x}_n^\delta) - z_{\alpha_k}^\delta + (\tilde{x}_n^\delta - x_0)), \quad (33)$$

where  $x_0$  is a starting point of the iteration. We prove that the sequence  $(\tilde{x}_n^\delta)$  converge to the unique solution  $x_{\alpha_k}^\delta$  of (19) and obtain an error estimate for  $\|x_{\alpha_k}^\delta - \tilde{x}_n^\delta\|$ . The proof of the following lemma is analogous to the proof of Lemma 3.2.

**LEMMA 4.1** *Assume there exist  $\tilde{r} \in [0, 1)$  and nonnegative numbers  $\kappa_0, \eta, \alpha$  such that*

$$\frac{\kappa_0}{(1 - \tilde{r})} \eta \leq \tilde{r}. \quad (34)$$

*Then the sequence  $(\tilde{t}_n)$  defined by*

$$\tilde{t}_{n+1} = \tilde{t}_n + \frac{\kappa_0}{(1 - \tilde{r})} \eta (\tilde{t}_n - \tilde{t}_{n-1}) \quad (35)$$

*is increasing, bounded above by  $\tilde{t}^{**} := \frac{\eta}{1 - \tilde{r}}$ , and converges to some  $\tilde{t}^*$  such that  $0 < \tilde{t}^* \leq \frac{\eta}{1 - \tilde{r}}$ . Moreover, for  $n \geq 0$ ;*

$$0 \leq \tilde{t}_{n+1} - \tilde{t}_n \leq \tilde{r}(\tilde{t}_n - \tilde{t}_{n-1}) \leq \tilde{r}^n \eta, \quad (36)$$

*and*

$$\tilde{t}^* - \tilde{t}_n \leq \frac{\tilde{r}^n}{1 - \tilde{r}} \eta. \quad (37)$$

Let

$$\tilde{R}(x_0) := F'(x_0) + I$$

and

$$\tilde{G}(x) := x - \tilde{R}(x_0)^{-1}[F(x) - z_{\alpha_k}^\delta + (x - x_0)]. \quad (38)$$

Note that with the above notation,  $\tilde{G}(\tilde{x}_n^\delta) = \tilde{x}_{n+1}^\delta$  and  $\|\tilde{R}(x_0)^{-1}\| \leq 1$ .

**THEOREM 4.2** *Suppose Assumptions 2.1 and 2.2 hold. Let the assumptions in Lemma 4.1 are satisfied with  $\tilde{t}^* \leq \tilde{r}$ . Then the sequence  $(\tilde{x}_n^\delta)$  defined in (33) is well defined and  $\tilde{x}_n^\delta \in B_{\tilde{t}^*}(x_0)$  for all  $n \geq 0$ . Further  $(\tilde{x}_n^\delta)$  is Cauchy sequence in  $B_{\tilde{t}^*}(x_0)$  and hence converges to  $x_{\alpha_k}^\delta \in \overline{B_{\tilde{t}^*}(x_0)} \subset B_{\tilde{t}^{**}}(x_0)$  and  $F(x_{\alpha_k}^\delta) + (x_{\alpha_k}^\delta - x_0) = z_{\alpha_k}^\delta$ .*

*Moreover, the following estimates hold for all  $n \geq 0$ ,*

$$\|\tilde{x}_{n+1}^\delta - \tilde{x}_n^\delta\| \leq \tilde{t}_{n+1} - \tilde{t}_n, \quad (39)$$

and

$$\|\tilde{x}_n^\delta - x_{\alpha_k}^\delta\| \leq \tilde{t}^* - \tilde{t}_n \leq \frac{\tilde{r}^n \eta}{1 - \tilde{r}}. \quad (40)$$

**Proof.**

Let  $G$  be as in (38). Then for  $u, v \in B_{\tilde{t}^*}(x_0)$ ,

$$\begin{aligned} \tilde{G}(u) - \tilde{G}(v) &= u - v - \tilde{R}(x_0)^{-1}[F(u) - z_{\alpha_k}^\delta + (u - x_0)] \\ &\quad + \tilde{R}(x_0)^{-1}[F(v) - z_{\alpha_k}^\delta + (v - x_0)] \\ &= \tilde{R}(x_0)^{-1}[\tilde{R}(x_0)(u - v) - (F(u) - F(v))] + \tilde{R}(x_0)^{-1}(v - u) \\ &= \tilde{R}(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v)) + (u - v)] \\ &\quad + \tilde{R}(x_0)^{-1}(v - u) \\ &= \tilde{R}(x_0)^{-1}[F'(x_0)(u - v) - (F(u) - F(v))] \end{aligned}$$

Thus by Assumption 2.2 we have

$$\|\tilde{G}(u) - \tilde{G}(v)\| \leq \kappa_0 \tilde{t}^* \|u - v\|. \quad (41)$$

The rest of the proof is analogous to the proof of Theorem 3.3.

**REMARK 4.3** *Now by taking  $u = x_{\alpha_k}^\delta$  and  $v = \tilde{x}_{n-1}$  in (41), we obtain linear convergence of  $\tilde{x}_{n-1}$  to  $x_{\alpha_k}^\delta$ .*

**REMARK 4.4** *For the remainder of the paper we shall consider only the quadratically convergent sequence  $(x_{n,\alpha_k}^\delta)$  defined in (20) for detailed analysis. The results verbatim hold good in the case of linearly convergent sequence  $(\tilde{x}_n^\delta)$  defined in (33).*

## 5 Error Bounds Under Source Conditions

The main objective of this section is to obtain an error estimate for  $\|x_{n,\alpha_k}^\delta - \hat{x}\|$  under the assumption

$$\|x_0 - \hat{x}\| \leq c \frac{1}{\mu^k} \quad (42)$$

for some constant  $c$  and source condition (11) on  $F(\hat{x})$ . Note that  $F(x_{\alpha_k}^\delta) + (x_{\alpha_k}^\delta - x_0) = z_{\alpha_k}^\delta$ . So that  $F(x_{\alpha_k}^\delta) - F(\hat{x}) + (x_{\alpha_k}^\delta - x_0) = z_{\alpha_k}^\delta - F(\hat{x})$ . Therefore by monotonicity of  $F$ , by taking inner product with  $x_{\alpha_k}^\delta - \hat{x}$  we obtain the following:

**THEOREM 5.1** *Under the assumption 2.2,*

$$\|x_{\alpha_k}^\delta - \hat{x}\| \leq \|F(\hat{x}) - z_{\alpha_k}^\delta\| + \|x_0 - \hat{x}\|.$$

Combining the estimates in Theorem 2.4 Theorem 3.3, Theorem 5.1, (42) and the relation  $\frac{1}{\mu^k} = \frac{1}{\mu^{k-l}\mu^l} = \frac{1}{\mu^{k-l}} \frac{\delta}{\sqrt{\alpha_l}}$ . Now since  $\sqrt{\alpha_\delta} \leq \sqrt{\alpha_{l+1}} = \mu\sqrt{\alpha_l}$  we have  $\frac{1}{\mu^k} \leq \frac{1}{\mu^{k-l-1}} \frac{\delta}{\sqrt{\alpha_\delta}} = \frac{1}{\mu^{k-l-1}} \psi^{-1}(\delta)$ , so we obtain the following

**THEOREM 5.2** *Let  $x_{\alpha_k}^\delta$  be the unique solution of (19) and  $x_{n,\alpha_k}^\delta$  be as in (20). Let the assumptions in Theorem 2.4, Assumption 2.1, Assumption 2.2 and Assumption 2.3 be satisfied. Then we have*

$$\|x_{n,\alpha_k}^\delta - \hat{x}\| \leq \frac{q^n \eta}{1 - q} + \mathcal{O}(\psi^{-1}(\delta)). \tag{43}$$

### 5.1 Stopping index

Let

$$n_k = \min\{n : q^n \leq \frac{1}{\mu^k}\}. \tag{44}$$

Then we have the following

**THEOREM 5.3** *Let  $x_{\alpha_k}^\delta$  be the unique solution of (19) and  $x_{n,\alpha_k}^\delta$  be as in (20). Let the assumptions in Theorem 2.4, Assumption 2.1, Assumption 2.2 and Assumption 2.3 be satisfied. Let  $n_k$  be as in (44). Then we have*

$$\|x_{n_k,\alpha_k}^\delta - \hat{x}\| = \mathcal{O}(\psi^{-1}(\delta)). \tag{45}$$

## 6 Implementation of adaptive choice rule

The main goal of this section is to provide a starting point for the iteration approximating the unique solution  $x_\alpha^\delta$  of (19) and then to provide an algorithm for the determination of a parameter fulfilling the balancing principle (18). Hereafter we assume without loss of generality that  $k_0 \leq \frac{1}{4\eta}$  (if not replace  $F$  by  $cF$  where  $c \leq \frac{1}{4k_0\eta}$ ).

For  $i, j \in \{0, 1, 2, \dots, N\}$ , we have

$$z_{\alpha_i}^\delta - z_{\alpha_j}^\delta = (\alpha_j - \alpha_i)(K^*K + \alpha_i I)^{-1}(K^*K + \alpha_j I)^{-1}K^*y^\delta.$$

The implementation of our method involves the following steps:

#### Step I

- $i=1$
- Solve for  $w_i : (K^*K + \alpha_i I)w_i = K^*y^\delta$
- Solve for  $z_{i,j} : (K^*K + \alpha_i I)^{-1}z_{i,j} = (\alpha_j - \alpha_i)w_i, j \leq i$
- If  $\|z_{i,j}\| > \frac{4}{\mu^j}$ , then take  $k = i - 1$ .
- Otherwise, repeat with  $i + 1$  in place of  $i$ .

### Step II

- choose  $x_0 \in D(F)$  such that  $\|x_0 - \hat{x}\| < \frac{c}{\mu^k}$  for some constant  $c$ .
- Choose  $\eta > 0$  such that  $\|F(x_0) - z_{\alpha_k}^\delta\| \leq \eta$ .
- Choose  $q \leq \frac{1 - \sqrt{1 - 4k_0\eta}}{2}$

### Step III

- $n = 1$
- If  $q^n \leq \frac{1}{\mu^k}$ , then take  $n_k := n$
- Otherwise, repeat with  $n + 1$  in place of  $n$

### Step IV

- Solve  $x_{j,\alpha_k}^\delta : (F'(x_{j-1,\alpha_k}^\delta) + I)(x_{j,\alpha_k}^\delta - x_{j-1,\alpha_k}^\delta) = F(x_{j-1,\alpha_k}^\delta) - w_k + x_{j-1,\alpha_k}^\delta - x_0$  for  $j = 1, 2, \dots, n_k$ .

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