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Local convergence of an at least sixth-order method in Banach spaces

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Abstract. We present a local convergence analysis of an at least sixth-order family of methods to approximate a locally unique solution of nonlinear equations in a Banach space setting. The semilocal convergence analysis of this method was studied by Amat et al. in (Appl Math Comput 206:164–174, 2008; Appl Numer Math 62:833–841, 2012). This work provides computable convergence ball and computable error bounds. Numerical examples are also provided in this study.

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1. Introduction

In this study, we are concerned with the problem of approximating a solution $x^\star$ of the nonlinear equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where $\mathcal{F}$ is a Fréchet-differentiable operator defined on a subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. Using mathematical modeling, many problems in computational sciences and other disciplines can be expressed as a nonlinear equation (1.1) [2,3,6,9,17–19,22]. Closed-form solutions of these nonlinear equations exist only for few special cases which may not be of much practical value. Therefore, solutions of these nonlinear equations (1.1) are approximated by iterative methods. In particular, the practice of Numerical Functional Analysis for approximating solutions iteratively is essentially connected to Newton-like methods [1–23]. The study about convergence of iterative procedures is normally divided into two categories—semilocal and local convergence analysis. The semilocal convergence analysis is based on the information around an initial point to give criteria ensuring the convergence of iterative procedures. While, the local analysis is based on the information around a solution to find estimates of the radii of convergence

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balls. There exist many studies which deal with the local and the semilocal convergence analysis of Newton-like methods such as [1–22].

Amat, Hernández and Romero in [1,2] studied the semilocal convergence of the at least sixth-order method defined for each $n = 0, 1, 2, 3, \dots$ by

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n), \\ z_n &= y_n - \frac{1}{2}L_n\mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n), \\ x_{n+1} &= z_n - H_n\mathcal{F}'(x_n)^{-1}\mathcal{F}(z_n), \end{aligned}\right\} \tag{1.2}$$

where x_0 is an initial point,

$$L_n = \mathcal{F}'(x_n)^{-1}\mathcal{F}''(x_n)\mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n)$$

and

$$H_n = \mathcal{I} + L_n + \frac{3}{2}L_n^2 - \frac{1}{2}\mathcal{F}'(x_n)^{-1}\mathcal{F}'''(x_n)(\mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n))^2.$$

The semilocal convergence analysis was based on the conditions

$$\begin{aligned} \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\right\| &\le \eta, \\ \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}''(x)\right\| &\le \beta, \\ \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'''(x)\right\| &\le \gamma. \end{aligned}$$

Method (1.2) finds applications (see [1,2]) especially when $\mathcal{F}''(x) = B$, where B is a bilinear constant operator. Notice that this way one avoids the computation of $\mathcal{F}'''(x_n)$ and the method (1.2) reduces to

$$\left.\begin{aligned} y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n), \\ z_n &= y_n - \frac{1}{2}L_n\mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n), \\ x_{n+1} &= z_n - (\mathcal{I} + L_n + \frac{3}{2}L_n^2)\mathcal{F}'(x_n)^{-1}\mathcal{F}(z_n). \end{aligned}\right\} \tag{1.3}$$

The efficiency index as defined by Traub [23] was also studied in [1,2]. In this paper, we study the local convergence of method (1.2).

The rest of the paper is organized as follows. Section 2 presents the local convergence of the method (1.2). The numerical examples are presented in the concluding Sect. 3.

2. Local convergence

In this section, we develop a local convergence analysis of the method (1.2). Let $U(w, R)$ and $\overline{U}(w, R)$ stand, respectively, for the open and closed balls in $\mathbf{X}$ centered at $w \in \mathbf{X}$ and of radius $R > 0$.

Let $l_0 > 0$, $l > 0$, $\mathcal{M}_1 > 0$, $\mathcal{M}_2 > 0$ and $\mathcal{M}_3 \ge 0$ be given parameters. It is convenient for the local convergence analysis of method (1.2) to define on the interval $[0, 1/l_0)$, functions g_1, g_2, g_3 and h by

$$g_1(r) = \frac{lr}{2(1 - l_0r)},$$

$$g_2(r) = \frac{r}{2(1-l_0r)}\Big(l + \frac{\mathcal{M}_1^2\mathcal{M}_2}{(1-l_0r)^2}\Big),$$
$$h(r) = 1 + \frac{\mathcal{M}_1\mathcal{M}_2 r}{(1-l_0r)^2} + \frac{3}{2}\frac{\mathcal{M}_1^2\mathcal{M}_2^2r^2}{(1-l_0r)^4} + \frac{1}{2}\frac{\mathcal{M}_3\mathcal{M}_1^2r^2}{(1-l_0r)^3}$$

and

$$g_3(r) = \Big(1 + \frac{\mathcal{M}_1 h(r)}{1-l_0r}\Big)g_2(r)r.$$

Moreover, define polynomial g on the interval $[0, 1/l_0]$ by

$$g(r) = g_3(r) - 1. \tag{2.1}$$

We have that $g(0) = -1 < 0$ and $g(r) \longrightarrow +\infty$ as $t \longrightarrow \frac{1}{l_0}^-$. Then, it follows from the intermediate mean value theorem that polynomial g has roots in the interval $(0, 1/l_0)$. Denote by r_0 the smallest root of polynomial g on the interval $(1, 1/l_0)$. It follows from the definition of the functions g_1, g_2, g_3, h, polynomial g and point r_0 that for each $r \in (0, r_0)$

$$0 < g_1(r) < 1,$$
$$0 < g_2(r) < 1,$$
$$1 < h(r)$$

and

$$0 < g_3(r) < 1.$$

Next, using the preceding notations and definitions, we can show the following local convergence result for the method (1.2).

Theorem 2.1. *Let $\mathcal{F} : \mathbf{D} \subseteq \mathbf{X} \longrightarrow \mathbf{Y}$ be a thrice Fréchet-differentiable operator. Suppose that there exist $x^\star \in \mathbf{D}$, $l_0 > 0$, $l > 0$, $\mathcal{M}_1 > 0$ and $\mathcal{M}_3 \geq 0$ such that for each $x \in \mathbf{D}$*

$$\mathcal{F}(x^\star) = 0, \quad \mathcal{F}'(x^\star)^{-1} \in \mathsf{L}(\mathbf{Y}, \mathbf{X}), \tag{2.2}$$
$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x^\star))\right\| \leq l_0 \left\|x - x^\star\right\|, \tag{2.3}$$

Let $D_0 = D \cap U(x^*, \frac{1}{l_0})$ and for each $x \in D_0$

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}(x) - \mathcal{F}(x^\star) - \mathcal{F}'(x)(x - x^\star))\right\| \leq \frac{l}{2}\left\|x - x^\star\right\|, \tag{2.4}$$
$$\left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x)\right\| \leq \mathcal{M}_1, \tag{2.5}$$
$$\left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}''(x)\right\| \leq \mathcal{M}_2, \tag{2.6}$$
$$\left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}'''(x)\right\| \leq \mathcal{M}_3, \tag{2.7}$$

and

$$\overline{U}(x^\star, r_0) \subseteq \mathbf{X}. \tag{2.8}$$

Then, sequence $\{x_n\}$ generated by the method (1.2) for $x_0 \in U(x^\star, r_0)$ is well defined, remains in $U(x^\star, r_0)$ for each $n = 0, 1, 2, 3, \ldots$ and converges to the solution $x^\star$ of the equation $\mathcal{F}(x) = 0$. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$

$$\left\|y_n - x^\star\right\| \leq g_1(\left\|x_n - x^\star\right\|)\left\|x_n - x^\star\right\|, \tag{2.9}$$

$$\|z_n - x^\star\| \le g_2(\|x_n - x^\star\|)\,\|x_n - x^\star\| \tag{2.10}$$

and

$$\|x_{n+1} - x^\star\| \le g_3(\|x_n - x^\star\|)\,\|x_n - x^\star\|. \tag{2.11}$$

Proof. By hypothesis $x_0 \in U(x^\star, r_0)$. Using (2.3), we get that

$$\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x^\star))\right\| \le l_0\,\|x_0 - x^\star\| < l_0 r_0 < 1. \tag{2.12}$$

It follows from (2.12) and the Banach lemma on invertible operators [3,6,16] that

$$\mathcal{F}'(x_0)^{-1} \in \mathsf{L}(\mathbf{Y}, \mathbf{X}),$$
$$\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \le \frac{1}{1 - l_0\,\|x_0 - x^\star\|}. \tag{2.13}$$

Then, in view of the first substep in (1.2) for $n = 0$, we have the identity

$$\begin{aligned} y_0 - x^\star &= x_0 - x^\star - \mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0) \\ &= -[\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)][\mathcal{F}'(x^\star)^{-1}(\mathcal{F}(x_0) - \mathcal{F}(x^\star) - \mathcal{F}'(x_0)(x_0 - x^\star))]. \end{aligned} \tag{2.14}$$

Using (2.4), (2.13), (2.14) and the properties of the function g_1, we get in turn that

$$\begin{aligned} \|y_0 - x^\star\| &\le \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}(x_0) - \mathcal{F}(x^\star) - \mathcal{F}'(x_0)(x_0 - x^\star))\right\| \\ &\le \frac{l\,\|x_0 - x^\star\|^2}{2(1 - l_0\,\|x_0 - x^\star\|)} = g_1(\|x_0 - x^\star\|)\,\|x_0 - x^\star\| < \|x_0 - x^\star\| < r_0. \end{aligned} \tag{2.15}$$

Hence, $y_0 \in U(x^\star, r_0)$ and (2.9) holds for $n = 0$. Using the definition of operator L_0, (2.5), (2.6), (2.13) and the properties of function g_2, we get that

$$\begin{aligned} \|L_0\| &\le \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}''(x_0)\right\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \\ &\quad \left\|\int_0^1 \mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x^\star + t(x_0 - x^\star))(x_0 - x^\star)\mathrm{dt}\right\| \\ &\le \frac{\mathcal{M}_1\mathcal{M}_2\,\|x_0 - x^\star\|}{(1 - l_0\,\|x_0 - x^\star\|)^2} \end{aligned} \tag{2.16}$$

so, by the second substep in (1.2), (2.20) and (2.16), we have

$$\begin{aligned} \|z_0 - x^\star\| &\le \|y_0 - x^\star\| + \frac{1}{2}\frac{\mathcal{M}_1^2\mathcal{M}_2\,\|x_0 - x^\star\|^2}{(1 - l_0\,\|x_0 - x^\star\|)^3} \\ &\le \frac{1}{2}\Big(l + \frac{\mathcal{M}_1^2\mathcal{M}_2}{(1 - l_0\,\|x_0 - x^\star\|)^2}\Big)\frac{\|x_0 - x^\star\|^2}{1 - l_0\,\|x_0 - x^\star\|} \\ &= g_2(\|x_0 - x^\star\|)\,\|x_0 - x^\star\| < \|x_0 - x^\star\| < r_0. \end{aligned} \tag{2.17}$$

Hence, we deduce that $z_0 \in U(x^\star, r_0)$ and (2.10) holds for $n = 0$. Using the definition of operator H_0, (2.5), (2.6), (2.7), (2.13) and h we get in turn that

$$\begin{aligned} \|H_0\| &\le 1 + \|L(x_0)\| + \frac{3}{2}\,\|L(x_0)\|^2 \\ &\quad + \frac{1}{2}\left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\| \left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}''(x_0)\right\| \left\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x^\star)\right\|^2 \end{aligned}$$

$$\left\|\int_0^1 \mathcal{F}'(x^\star)^{-1}[\mathcal{F}'(x^\star+t(x_0-x^\star))(x_0-x^\star)]\mathrm{dt}\right\|^2$$
$$\leq 1+\frac{\mathcal{M}_1\mathcal{M}_2\|x_0-x^\star\|}{(1-l_0\|x_0-x^\star\|)^2}+\frac{3}{2}\frac{\mathcal{M}_1^2\mathcal{M}_2^2\|x_0-x^\star\|^2}{(1-l_0\|x_0-x^\star\|)^4}+\frac{1}{2}\frac{\mathcal{M}_3\mathcal{M}_1^2\|x_0-x^\star\|^2}{(1-l_0\|x_0-x^\star\|)^3}$$
$$= h(\|x_0-x^\star\|) \tag{2.18}$$

so, by (2.22), the last substep in (1.2) and the definition of functions g_3, we get that

$$\|x_1-x^\star\| \leq \|z_0-x^\star\|+\frac{\|H_0\|\mathcal{M}_1\|z_0-x^\star\|}{1-l_0\|x_0-x^\star\|}$$
$$\leq \frac{1}{2}\left(1+\frac{\mathcal{M}_1 h(\|x_0-x^\star\|)}{1-l_0\|x_0-x^\star\|}\right)\left(l+\frac{\mathcal{M}_1^2\mathcal{M}_2}{(1-l_0\|x_0-x^\star\|)^2}\right)\frac{\|x_0-x^\star\|^2}{1-l_0\|x_0-x^\star\|}$$
$$= g_3(\|x_0-x^\star\|)\|x_0-x^\star\| < \|x_0-x^\star\| < r_0. \tag{2.19}$$

Hence, $x_1 \in U(x^\star, r_0)$ and (2.11) holds for $n=1$. If we simply replace x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates, we arrive at the estimates (2.9)–(2.11) and through these estimates to $x_k, y_k, z_k, x_{k+1} \in U(x^\star, r_0)$. Finally, it follows from the estimate

$$\|x_{k+1}-x^\star\| < \|x_k-x^\star\|$$

that

$$\lim_{k\to\infty} x_k = x^\star.$$

□

Remark 2.2. 1. In view of (2.3) and the estimate

$$\left\|\mathcal{F}'(x^\star)^{-1}\mathcal{F}'(x)\right\| = \left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x)-\mathcal{F}'(x^\star))+\mathcal{I}\right\|$$
$$\leq 1+\left\|\mathcal{F}'(x^\star)^{-1}(\mathcal{F}'(x)-\mathcal{F}'(x^\star)\right\| \leq 1+l_0\|x-x^\star\| \tag{2.20}$$

condition (2.5) can be dropped and can be replaced by

$$\mathcal{M}_1(r) = 1+l_0 r.$$

2. It is worth noticing that the earlier results [1,2] use hypotheses in non-affine invariant form. In this study, we use hypotheses in affine invariant form. In the earlier works, neither the local case is covered nor these studies provide a computable convergence ball or computable error bounds based on Lipschitz or other constants.
3. The results obtained here can be used for operators $\mathcal{F}$ satisfying autonomous differential equations [3,6,12,17] of the form

$$\mathcal{F}'(x) = \mathcal{P}(\mathcal{F}(x)),$$

where $\mathcal{P}$ is a continuous operator. Then, since $\mathcal{F}'(x^\star) = \mathcal{P}(\mathcal{F}(x^\star)) = \mathcal{P}(0)$, we can apply the results without actually knowing $x^\star$. For example, let $\mathcal{F}(x) = e^x - 1$. Then, we can choose: $\mathcal{P}(x) = x+1$.

4. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2,6,12,17].
5. In view of (2.8) the radius r is such that

$$r \le r_{\mathcal{A}} = \frac{1}{l_0 + \dfrac{l}{2}}. \tag{2.21}$$

The parameter $r_{\mathcal{A}}$ was shown by us to be the convergence radius of Newton's method [2,6]

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \quad \text{for each} \quad n = 0, 1, 2, \ldots \tag{2.22}$$

under the conditions (2.4) and (2.5). It follows from (2.20) that the convergence radius r of the three-step method (1.2) cannot be larger than the convergence radius $r_{\mathcal{A}}$ of the second-order Newton's method (2.12). As already noted in [2,4,6] $r_{\mathcal{A}}$ is at least as large as the convergence ball given by Rheinboldt [21]

$$r_{\mathcal{R}} = \frac{2}{3\,l}.$$

In particular, for $l_0 < l$ we have that

$$r_{\mathcal{R}} < r_{\mathcal{A}}$$

and

$$\frac{r_{\mathcal{R}}}{r_{\mathcal{A}}} \longrightarrow \frac{1}{3} \quad \text{as} \quad \frac{l_0}{l} \longrightarrow 0.$$

That is, our convergence ball $r_{\mathcal{A}}$ is at most three times larger than Rheinboldt's. The same value for $r_{\mathcal{R}}$ was also given by Traub [22].

3. Numerical examples

It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger (**C**) conditions used in [1]. Moreover, we can compute the computational order of convergence (COC) by the equation

$$\xi = \frac{\ln \frac{\|x_{n+1} - x^\star\|}{\|x_n - x^\star\|}}{\ln \frac{\|x_n - x^\star\|}{\|x_{n-1} - x^\star\|}} \tag{3.1}$$

or approximate the computational order of convergence by the equation

$$\xi_1 = \frac{\ln \frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}}{\ln \frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}}. \tag{3.2}$$

This way we obtain in practice the order of convergence. For solving $\mathcal{F}(x) = 0$ in $\mathbb{R}^m$, the method (1.2) yields

$$\left.\begin{aligned} \mathcal{F}'(x_n)c_n &= -\mathcal{F}(x_n),\\ \mathcal{F}'(x_n)d_n &= -\mathcal{F}(x_n) - \frac{1}{2}\mathcal{F}''(x_n)\,c_n^2,\\ z_n &= x_n + d_n\\ \mathcal{F}'(x_n)m_n &= -\mathcal{F}(z_n),\\ \mathcal{F}'(x_n)p_n &= -\mathcal{F}''(x_n)c_n m_n,\\ \mathcal{F}'(x_n)g_n &= -F(z_n) - \mathcal{F}''(x_n)c_n m_n - \frac{3}{2}\mathcal{F}''(x_n)c_n p_n - \frac{1}{2}\mathcal{F}'''(x_n)c_n^2 m_n,\\ x_{n+1} &= z_n + g_n. \end{aligned}\right\}$$

We present two numerical examples in this section.

Example 3.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}^3$, $\mathbf{D} = \overline{\mathbf{U}}(0,1)$ and $x^\star = (0,0,0)^T$. We define function $\mathcal{F}$ on $\mathbf{D}$ as

$$\mathcal{F}(x,y,z) = \left(e^x - 1, \frac{e-1}{2}\,y^2 + y, z\right)^T. \tag{3.3}$$

Then, the Fréchet derivative of $\mathcal{F}$ is given by

$$\mathcal{F}'(x,y,z) = \begin{pmatrix} e^x & 0 & 0\\ 0 & (e-1)\,y+1 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that we have:

$$\left.\begin{aligned} \mathcal{F}(x^\star) = 0, \quad \mathcal{F}'(x^\star) = \mathcal{F}'(x^\star)^{-1} = \text{ diag } \{1,1,1\}\\ l = \mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3 = e^{\frac{1}{e-1}}, \quad l_0 = e-1. \end{aligned}\right\}$$

Now, we perform the local convergence analysis as stated in the Sect. 2. For the function $g(r)$, the smallest positive root is

$$r_0 = 0.064170054328851366953756496513961$$

For the functions g_1, g_2, g_3 and h, we obtain the Fig. 1.

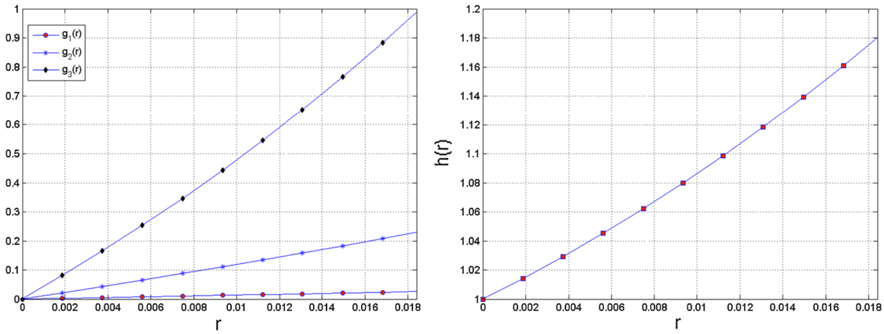

Figure 1. Functions $g_1(r), g_2(r), g_3(r)$ and $h(r)$ for Example 3.1 on the interval $r \in (0, r_0)$

TABLE 1. Solving (3.4) by the method (1.2) for $\mathbf{x}_0 = 1.0$

n	$\|x_n\|$	$\|x_n - x_{n-1}\|$	$\|\mathcal{F}(x_n)\|_2$	COC
0	$1.000000 \times 10^{+00}$	3.768260×10^{-01}	$5.000000 \times 10^{+00}$	–
1	$1.376826 \times 10^{+00}$	1.159603×10^{-02}	1.925800×10^{-01}	$6.598731 \times 10^{+00}$
2	$1.365230 \times 10^{+00}$	1.505609×10^{-12}	2.486272×10^{-11}	$5.997595 \times 10^{+00}$
3	$1.365230 \times 10^{+00}$	7.619151×10^{-72}	1.258181×10^{-70}	$6.000000 \times 10^{+00}$
4	$1.365230 \times 10^{+00}$	$1.279596 \times 10^{-427}$	$2.113048 \times 10^{-426}$	$6.000000 \times 10^{+00}$
5	$1.365230 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$2.697484 \times 10^{-2008}$	$6.000000 \times 10^{+00}$
6	$1.365230 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$2.697484 \times 10^{-2008}$	$6.000000 \times 10^{+00}$
7	$1.365230 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$2.697484 \times 10^{-2008}$	$6.000000 \times 10^{+00}$
8	$1.365230 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$2.697484 \times 10^{-2008}$	$6.000000 \times 10^{+00}$
9	$1.365230 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$2.697484 \times 10^{-2008}$	$6.000000 \times 10^{+00}$

In the Fig. 1, we observe that for $r \in (0, r_0)$

$$0 < g_1(r) < 1, \quad 0 < g_2(r) < 1, \quad 0 < g_3(r) \quad \text{and} \quad 1 < h(r).$$

Now we evaluate convergence balls (see Remark 2.2)

$$r_A = 0.38269191223238574472986783803208 \quad \text{and}$$
$$r_R = 0.37252846984183135559121069491084.$$

We notice that $r_0 < r_R < r_A$.

Example 3.2. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}$, $\mathbf{D} = \mathbf{U}(-2, 2)$ and $x^\star \approx 1.36523001341409684576080682898$. On the domain $\mathbf{D}$, the function $\mathcal{F}$ is given as

$$\mathcal{F}(x) = x^3 + 4x^2 - 10. \tag{3.4}$$

Notice that we have:

$$\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3 = 2.0, \quad l_0 = 0.8, \quad l = 1.2.$$

To verify estimates (2.9)–(2.11) of the Theorem 2.1, we solve (3.4) by the method (1.2). We implement the method (1.2) with the help of high-performance package ARPREC and solve the Eq. (3.4) to very high precision. Results of numerical work are reported in the Tables 1, 2 and 3.

The Table 1 reports performance of the method (1.2) for the problem (3.4). In the Table 1, we notice that the computational order of convergence of the method is 6. For evaluating COC, we use the Eq. (3.1). In this equation, $x^\star$ is replaced by the 20th-iteration produced by the method (1.2). In the Tables 2 and 3, we observe that the estimates (2.9), (2.10) and (2.11) of the Theorem 2.1 hold.

Table 2. Verification of estimates (2.9) and (2.10) of the Theorem 2.1

n	$\|y_n - x^\star\|$	$g_1(\|x_n - x^\star\|)$ $\|x_n - x^\star\|$	$\|z_n - x^\star\|$	$g_2(\|x_n - x^\star\|)$ $\|x_n - x^\star\|$
0	8.931544×10^{-02}	1.130743×10^{-01}	4.216465×10^{-02}	$3.051819 \times 10^{+00}$
1	6.536686×10^{-05}	8.143617×10^{-05}	6.425571×10^{-07}	1.161754×10^{-03}
2	1.111327×10^{-24}	1.360115×10^{-24}	1.433914×10^{-36}	1.906995×10^{-23}
3	$2.845972 \times 10^{-143}$	$3.483088 \times 10^{-143}$	$1.858259 \times 10^{-214}$	$4.883580 \times 10^{-142}$
4	$8.027184 \times 10^{-855}$	$9.824198 \times 10^{-855}$	$8.802486 \times 10^{-1282}$	$1.377434 \times 10^{-853}$
5	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$
6	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$
7	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$
8	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$
9	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$

Table 3. Verification of estimate (2.11) of the Theorem 2.1

n	$\|x_{n+1} - x^\star\|$	$g_3(\|x_n - x^\star\|)\,\|x_n - x^\star\|$
0	1.159603×10^{-02}	$1.597793 \times 10^{+02}$
1	1.505609×10^{-12}	3.627001×10^{-03}
2	7.619151×10^{-72}	5.720984×10^{-23}
3	$1.279596 \times 10^{-427}$	$1.465074 \times 10^{-141}$
4	$0.000000 \times 10^{+00}$	$4.132303 \times 10^{-853}$
5	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$
6	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$
7	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$
8	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$
9	$0.000000 \times 10^{+00}$	$0.000000 \times 10^{+00}$

References

[1] Amat, S., Hernández, M.A., Romero, N.: A modified Chebyshevs iterative method with at least sixth order of convergence. Appl. Math. Comput. **206**, 164–174 (2008)

[2] Amat, S., Hernández, M.A., Romero, N.: Semilocal convergence of a sixth order iterative method for quadratic equations. Appl. Numer. Math. **62**, 833–841 (2012)

[3] Argyros, I.K.: Computational theory of iterative methods. In: Chui, C.K., Wuytack, L. (eds.) Series: Studies in Computational Mathematics, vol. 15. Elsevier, New York (2007)

[4] Argyros, I.K.: A semilocal convergence analysis for directional Newton methods. Math. Comput. **80**, 327–343 (2011)

[5] Argyros, I.K., Hilout, S.: Weaker conditions for the convergence of Newton's method. J. Complex. **28**, 364–387 (2012)

[6] Argyros, I.K., Hilout, S.: Computational Methods in Nonlinear Analysis. Efficient Algorithms, Fixed Point Theory and Applications. World Scientific, Singapore (2013)

[7] Candela, V., Marquina, A.: Recurrence relations for rational cubic methods II: the Chebyshev method. Computing **45**, 355–367 (1990)

[8] Candela, V., Marquina, A.: Recurrence relations for rational cubic methods I: the Halley method. Computing **44**, 169–184 (1990)

[9] Cătinaş, E.: The inexact, inexact perturbed, and quasi-Newton methods are equivalent models. Math. Comput. **74**, 291–301 (2005)

[10] Chun, C., Stănică, P., Neta, B.: Third-order family of methods in Banach spaces. Comput. Math. Appl. **61**, 1665–1675 (2011)

[11] Ezquerro, J.A., Hernández, M.A.: Recurrence relations for Chebyshev-type methods. Appl. Math. Optim. **41**, 227–236 (2000)

[12] Ezquerro, J.A., Hernández, M.A.: On the R-order of the Halley method. J. Math. Anal. Appl. **303**, 591–601 (2005)

[13] Gutiérrez, J.M., Hernández, M.A.: Third-order iterative methods for operators with bounded second derivative. J. Comput. Math. Appl. **82**, 171–183 (1997)

[14] Gutiérrez, J.M., Magreñán, Á.A., Romero, N.: On the semilocal convergence of Newton–Kantorovich method under center-Lipschitz conditions. Appl. Math. Comput. **221**, 79–88 (2013)

[15] Hernández, M.A., Salanova, M.A.: Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev method. J. Comput. Appl. Math. **126**, 131–143 (2000)

[16] Kantorovich, L.V., Akilov, G.P.: Functional Analysis. Pergamon Press, Oxford (1982)

[17] Kou, J.-S., Li, Y.-T., Wang, X.-H.: A modification of Newton method with third-order convergence. Appl. Math. Comput. **181**, 1106–1111 (2006)

[18] Ortega, L.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York (1970)

[19] Parida, P.K., Gupta, D.K.: Recurrence relations for a Newton-like method in Banach spaces. J. Comput. Appl. Math. **206**, 873–887 (2007)

[20] Potra, F.A., Pták, V.: Nondiscrete Induction and Iterative Processes. In: Research Notes in Mathematics, vol. 103. Pitman, Boston (1984)

[21] Proinov, P.D.: General local convergence theory for a class of iterative processes and its applications to Newton's method. J. Complex. **25**, 38–62 (2009)

[22] Rheinboldt, W.C.: An adaptive continuation process for solving systems of nonlinear equations. Pol. Acad. Sci. Banach Cent. Publ. **3**, 129–142 (1977)

[23] Traub, J.F.: Iterative Methods for the Solution of Equations. AMS Chelsea Publishing, Providence (1982)

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