Asian-European Journal of Mathematics
Vol. 10, No. 4 (2017) 1750086 (9 pages)

DOI: 10.1142/S1793557117500863

Local convergence of deformed Euler–Halley-type methods in Banach space under weak conditions

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Communicated by C.-H. Chu
Received November 20, 2014
Revised March 23, 2015
Published

We present a unified local convergence analysis for deformed Euler–Halley-type methods in order to approximate a solution of a nonlinear equation in a Banach space setting. Our methods include the Euler, Halley and other high order methods. The convergence ball and error estimates are given for these methods under hypotheses up to the first Fréchet derivative in contrast to earlier studies using hypotheses up to the second Fréchet derivative. Numerical examples are also provided in this study.

Keywords: Chebyshev–Halley-type methods; Banach space; convergence ball; local convergence.

AMS Subject Classification: 65D10, 65D99, 65G99, 47H17, 49M15

1. Introduction

In this study, we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought in a form like (1.1) using mathematical modeling [2–4, 6, 10, 14, 16]. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. In particular, the practice of Numerical Functional Analysis for finding solution x^* of Eq. (1.1) is essentially connected to variants of Newton's method. This method converges quadratically to x^* if the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Euler–Halley-type methods [1, 4, 6, 7–16] require the evaluation of the second Fréchet-derivative, which is very expensive in general. However, there are integral equations, where the second Fréchet-derivative is diagonal by blocks and inexpensive [10–13] or for quadratic equations the second Fréchet-derivative is constant [2, 13]. Moreover, in some applications involving stiff systems [3, 6, 9], high order methods are useful. That is why in a unified way we study the local convergence of deformed Euler–Halley-type methods (DEHTMs) defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n + Q_n(y_n - x_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point and $H_n = \frac{1}{\lambda}F'(x_n)^{-1}(F'(x_n + \lambda(y_n - x_n)) - F'(x_n))$, $Q_n = -\frac{1}{2}(I + \alpha H_n)$, $\alpha \in \mathbb{R} - \{0\}$ and $\lambda \in (0, 1]$. DEHTM was introduced in [15]. This method is motivated by earlier Euler–Halley methods such as [8–11, 13]

$$x_{\mu,n+1} = x_{\mu,n} - \left[I + \frac{1}{2}L_F(x_{\mu,n})(I - \mu L_F(x_{\mu,n}))\right] F'(x_{\mu,n})^{-1}F(x_{\mu,n}), \tag{1.3}$$

where x_0 is an initial point, $\mu \in [0, 1]$ and

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x).$$

Notice that if $\mu = 0$, $\mu = \frac{1}{2}$, $\mu = 1$, method (1.3) reduces to the Chebyshev, Halley and super-Halley methods, respectively. The usual conditions for the semi-local convergence of these methods are $(\mathcal{C})$ [8–15]: There exist constants β, β_1, β_2 such that:

$(\mathcal{C}_1)$ There exist $\Gamma_0 = F'(x_0)^{-1}$ and $\|\Gamma_0\| \le \beta$;

$(\mathcal{C}_2)$

$$\|\Gamma_0 F(x_0)\| \le \eta;$$

$(\mathcal{C}_3)$

$$\|\Gamma_0 F''(x_0)\| \le \beta_1 \quad \text{for each } x \in D;$$

$(\mathcal{C}_4)$

$$\|\Gamma_0[F''(x) - F''(y)]\| \le \beta_2\|x - y\|^p \quad \text{for each } x, y \in D,\ p \in (0, 1].$$

The local convergence conditions are similar but x_0 is x^* in $(\mathcal{C}_1)$–$(\mathcal{C}_4)$. There is a plethora of local and semi-local convergence results based on the $(\mathcal{C})$ conditions [1–16]. As a motivational example, let us define function f on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$f''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x,$$

$$f'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Notice that condition $(\mathcal{C}_4)$ is not satisfied. Hence, the results depending on $(\mathcal{C}_4)$ cannot apply in this case. In this study, we expand the applicability of DEHTM using hypotheses only up to the first Fréchet-derivative (see (2.8), (2.9) and Example 3.3).

The rest of the paper is organized as follows: In Sec. 2, we present the local convergence of these methods. The numerical examples are given in Sec. 3.

2. Local Convergence

In this section, we present the local convergence of DEHTM. Let $L > 0$, $L_0 > 0$, $\alpha \in \mathbb{R} - \{0\}$, $\lambda \in (0, 1]$ be given parameters. It is convenient for the local convergence analysis that follows to introduce some functions and parameters. Define functions on the interval $[0, \frac{1}{L_0})$ by

$$g_1(t) = \frac{Lt}{2(1 - L_0 t)},$$

$$g_2(t) = \frac{L(1 + g_1(t))}{1 - L_0 t},$$

$$g_3(t) = |\alpha| g_2(t) t,$$

$$g_4(t) = \frac{g_2(t) t}{2(1 - g_3(t))},$$

$$g_5(t) = g_1(t) + g_4(t)(1 + g_1(t))$$

and parameter r_A by

$$r_A = \frac{2}{2L_0 + L} < \frac{1}{L_0}. \tag{2.1}$$

We have by the definition of functions g_1, g_2 and (2.1) that

$$0 \leq g_1(t) \leq 1$$

and

$$g_2(t) \geq 0 \quad \text{for each } t \in [0, r_A].$$

Using the definition of functions g_1, g_2 and g_3 we have that function $\bar{g}_3(t) = g_3(t) - 1$ is such that $\bar{g}_3(0) = -1$ and for $s \to \frac{1}{L0}^-$

$$\bar{g}_3(s) = \frac{|\alpha| L(1 + g_1(s))}{1 - L_0 s} - 1 \to \infty.$$

It follows from the intermediate value theorem that function $\bar{g}_3$ has zeros in the interval $(0, \frac{1}{L0})$. Denote by r_3 the smallest such zero. Then, we have that

$$0 \le g_3(t) \le 1 \quad \text{for each } t \in [0, r_3].$$

It also follows

$$0 \le g_4(t) \quad \text{for each } t \in [0, r_3).$$

Moreover, define function $\bar{g}_5$ by $\bar{g}_5(t) = g_5(t) - 1$. Then, we have

$$\bar{g}_5(0) = g_5(0) - 1 = -1 < 0$$

and for $s \to \frac{1}{L0}^-$

$$\bar{g}_5(s) \to \infty.$$

It follows that function $\bar{g}_5$ has zeros in the interval $(0, \frac{1}{L0})$. Denote by r the smallest such zero. Then, we have that

$$0 \le g_1(t) < 1, \tag{2.2}$$

$$g_2(t) \ge 0, \tag{2.3}$$

$$0 \le g_3(t) < 1, \tag{2.4}$$

$$g_4(t) \ge 0, \tag{2.5}$$

and

$$0 \le g_5(t) < 1 \quad \text{for each } t \in [0, r). \tag{2.6}$$

In the rest of this study, $U(w, q)$ and $\overline{U}(w, q)$ stand, respectively, for the open and closed ball in X with center $w \in X$ and of radius $q > 0$. Next, we present the local convergence result for DEHTM.

Theorem 2.1. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $\alpha \in \mathbb{R} - \{0\}$, $\lambda \in (0, 1]$, such that for each $x, y \in D$*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X), \tag{2.7}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0 \|x - x^*\|, \tag{2.8}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L \|x - y\|, \tag{2.9}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.10}$$

where r is defined above Theorem 2.1. *Then, sequence $\{x_n\}$ generated by DEHTM for $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and* *converges to* x^*. *Moreover, the following estimates hold for each* $n = 0, 1, 2, \dots$

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r, \tag{2.11}$$

$$\|H_n\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\|, \tag{2.12}$$

$$\|\alpha H_n\| \leq g_3(\|x_n - x^*\|) < 1, \tag{2.13}$$

$$\|Q_n\| \leq g_4(\|x_n - x^*\|) \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \leq g_5(\|x_n - x^*\|)\|x_n - x^*\|, \tag{2.15}$$

where the "g" functions are given above Theorem 2.1.

Proof. By hypothesis $x_0 \in U(x^*, r)$. Using (2.8), the definition of function g_1 and the definition of radius r, we get that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.16}$$

It follows from (2.16) and the Banach Lemma on invertible operators [3, 6, 14] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r}. \tag{2.17}$$

Hence, y_0, Q_0, H_0 are well defined. Using the first substep of DEHTM we get that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= [F'(x_0)^{-1}F'(x^*)]\left[\int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta\right]. \end{aligned} \tag{2.18}$$

Then, in view of (2.17), (2.9), the definition of function g_1, radius r and (2.2), we get that

$$\begin{aligned} \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \left\|\int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)]d\theta\right\| \|x_0 - x^*\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r, \end{aligned}$$

which shows (2.11) for $n = 0$.

Notice that

$$\begin{aligned} \|x_0 + \lambda(y_0 - x_0) - x^*\| &\leq |\lambda|\|x_0 - x^*\| + |1 - \lambda|\|y_0 - x^*\| \\ &\leq (|\lambda| + |1 - \lambda|)r < r, \end{aligned}$$

which shows that $x_0 + \lambda(y_0 - x_0) \in U(x^*, r)$. We need upper bound on $\|H_0\|$ and $\|Q_0\|$. We have by the definition of H_0, g_2, (2.3), (2.17) and (2.9) that

$$\begin{aligned}\|H_0\| &\le \frac{1}{|\lambda|}\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}[F'(x_0+\lambda(y_0-x_0)) - F'(x_0)]\| \\ &\le \frac{1}{|\lambda|}\frac{L|\lambda|\|y_0-x_0\|}{1-L_0\|x_0-x^*\|} \le \frac{L(\|y_0-x^*\|+\|x_0-x^*\|)}{1-L_0\|x_0-x^*\|} \\ &\le \frac{L(1+g_1(\|x_0-x^*\|)\|x_0-x^*\|)}{1-L_0\|x_0-x^*\|} = g_2(\|x_0-x^*\|)\|x_0-x^*\|,\end{aligned}$$

which shows (2.12) for $n = 0$. Then, we have that

$$\begin{aligned}\|\alpha H_0\| &\le |\alpha|\|H_0\| \le |\alpha| g_2(\|x_0-x^*\|)\|x_0-x^*\| \\ &= g_3(\|x_0-x^*\|) < g_3(r) < 1, \quad \text{by (2.4)}.\end{aligned} \tag{2.19}$$

It follows from (2.19) and the Banach Lemma on invertible operators that $(I + \alpha H_0)^{-1} \in L(Y, X)$ and

$$\|(I+\alpha H_0)^{-1}\| \le \frac{1}{1-g_3(\|x_0-x^*\|)} \le \frac{1}{1-g_3(r)}. \tag{2.20}$$

Using the definition of Q_0, g_4, (2.12) (for $n = 0$), (2.15), we obtain that

$$\|Q_0\| \le \frac{1}{2}\frac{g_2(\|x_0-x^*\|)\|x_0-x^*\|}{1-g_3(\|x_0-x^*\|)} = g_4(\|x_0-x^*\|),$$

which shows (2.15) for $n = 0$. Then, using the last substep of DEHTM for $n = 0$, (2.6), (2.11), (2.15), we get that

$$\begin{aligned}\|x_1-x^*\| &\le \|y_0-x^*\| + \|Q_0\|\|y_0-x_0\| \\ &\le g_1(\|x_0-x^*\|)\|x_0-x^*\| + g_4(\|x_0-x^*\|)(\|y_0-x^*\|+\|x_0-x^*\|) \\ &\le [g_1(\|x_0-x^*\|) + g_4(\|x_0-x^*\|)(1+g_1(\|x_0-x^*\|))]\|x_0-x^*\| \\ &= g_5(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r,\end{aligned}$$

which shows (2.14) for $n = 0$. By simply replacing x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the preceding estimates we arrive at (2.11)–(2.15). Finally, using the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\|$ we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim_{k\to\infty} x_k = x^*$. □

Remark 2.2. (a) Condition (2.8) can be dropped, since this condition follows from (2.9). Notice, however, that

$$L_0 \le L \tag{2.21}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [2–6].

(b) It is worth noticing that it follows from the first term in (2.17) that r is such that

$$r < r_A = \frac{2}{2L_0 + L}. \tag{2.22}$$

The convergence ball of radius r_A was given by us in [3, 4, 6] for Newton's method under conditions (2.7)–(2.9). Estimate (2.22) shows that the convergence ball of higher than two DEHTMs is smaller than the convergence ball of the quadratically convergent Newton's method. The convergence ball given by Rheinboldt [16] for Newton's method is

$$r_R = \frac{2}{3L} < r_A \tag{2.23}$$

if $L_0 < L$ and $\frac{r_R}{r_A} \to \frac{1}{3}$ as $\frac{L_0}{L} \to 0$.

(c) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2–4, 6, 14, 15].

(d) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [2–4, 6, 14, 15]:

$$F'(x) = T(F(x)),$$

where T is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, $F''(x^*) = F'(x^*)T'(F(x^*)) = T(0)T'(0)$, we can apply the results without actually knowing the solution x^*. Let us consider an example $F(x) = e^x - 1$. Then, we can choose $T(x) = x + 1$ and $x^* = 0$.

(e) We can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) \Big/ \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate COC

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) \Big/ \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right),$$

since the bounds given in Theorem 2.1 may be very pessimistic.

(f) The restriction $\lambda \in (0, 1]$ can be dropped, if $\lambda \in \mathbb{R} - \{0\}$ and (2.10) is replaced by

$$U_1 = \bar{U}(x^*, (|\lambda| + |1 - \lambda|)r) \subseteq D. \tag{2.24}$$

Indeed, we will then have

$$\begin{aligned}\|x_n + \lambda(y_n - x_n) - x^*\| &\leq |\lambda|\|x_n - x^*\| + |1 - \lambda|\|y_n - x^*\| \\ &\leq (|\lambda| + |1 - \lambda|)r \Rightarrow x_n + \lambda(y_n - x_n) \in U_1.\end{aligned}$$

3. Numerical Examples

We present numerical examples where we compute the radii of the convergence balls.

Example 3.1. Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define F on D for $v = (x,y,z)^T$ by

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T. \tag{3.1}$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $x^* = (0,0,0)^T$, $F'(x^*) = F'(x^*)^{-1} = \text{diag}\{1,1,1\}$, $L_0 = e - 1 < L = e$, $\alpha = 0.1$. Then we have

$$r = 0.1547 < r_R = 0.2453 < r_A = 0.3249.$$

Example 3.2. Let $X = Y = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D = \overline{U}(0,1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.2}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta \quad \text{for each } \xi \in D.$$

Then, we get that $x^* = 0$, $L_0 = 7.5$, $L = 15, \alpha = 0.1$ and

$$r = 0.0310 < r_R = 0.0444 < r_A = 0.0667.$$

Example 3.3. Returning back to the motivational example at the introduction of this study, we have $L_0 = L = 146.6629073, \alpha = 0.1$. Then we have

$$r = 0.0022 < r_R = 0.0045 \le r_A = 0.0045.$$

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