Research Article

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Modified Minimal Error Method for Nonlinear Ill-Posed Problems

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Abstract: An error estimate for the minimal error method for nonlinear ill-posed problems under general a Hölder-type source condition is not known. We consider a modified minimal error method for nonlinear ill-posed problems. Using a Hölder-type source condition, we obtain an optimal order error estimate. We also consider the modified minimal error method with noisy data and provide an error estimate.

Keywords: Nonlinear Ill-Posed Problem, Minimal Error Method, Regularization Method, Discrepancy Principle

MSC 2010: 65J15, 65J20, 47H17

1 Introduction

In this paper, we deal with the nonlinear ill-posed operator equation

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subseteq X \to Y$ is a nonlinear Fréchet differentiable operator. Here $D(F)$ denotes the domain of F and X, Y are Hilbert spaces with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$, respectively, which can be always identified from the context in which they appear. It is assumed that the operator equation (1.1) has a solution $\hat{x}$ for the exact data y. The operator equation (1.1) is ill-posed in the sense that the solution $\hat{x}$ does not depend continuously on the right-hand side data y. Furthermore, it is assumed that we have only approximate data $y^\delta \in Y$ with

$$\|y - y^\delta\| \le \delta.$$

To approximate the solution $\hat{x}$, iterative methods and iterative regularization methods are studied in [1, 2, 4, 5, 8–10, 12–16, 19]. Let $B(x, \rho)$ and $\overline{B}(x, \rho)$ stand, respectively, for the open ball and the closed ball in X, with center $x \in X$ and of radius $\rho > 0$. In [14], Neubauer and Scherzer considered the minimal error method defined for $k = 1, 2, \dots$ by

$$x_{k+1} = x_k + \alpha_k s_k,$$

where x_0 is the initial guess, $s_k = -F'(x_k)^*(F(x_k) - y)$ is the search direction taken as the negative gradient of the minimization function involved and

$$\alpha_k = \frac{\|F(x_k) - y\|^2}{\|s_k\|^2}$$

is the descent. Convergence analysis in [14] was based on the following assumption.

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Assumption A. We assume the following:
(A1) F has a Lipschitz continuous Fréchet derivative $F'(\cdot)$ in a neighborhood of x_0.
(A2) We have $F'(x) = R_x F'(\hat{x})$, $x \in B(x_0, \rho)$, where $\{R_x : x \in B(x_0, \rho)\}$ is a family of bounded linear operators $R_x : Y \to Y$ with
$$\|R_x - I\| \le C\|x - \hat{x}\|,$$
where C is a positive constant.
(A3) We have $x_0 - \hat{x} = (F'(\hat{x})^* F'(\hat{x}))^{\frac{1}{2}} v$ for some $v \in X$.

Recently, the authors in [8] studied a modified minimal error method in which, we have taken
$$s_k = -F'(x_0)^*(F(x_k) - y) \quad \text{and} \quad \alpha_k = \frac{\|F(x_k) - y\|^2}{\|s_k\|^2}$$
and the convergence analysis in [8] was based on the following assumptions:

Assumption B. We assume the following:
(B0) $\|F'(x)\| \le m$ for some $m > 0$ and for all $x \in D(F)$.
(B1) We have $F'(\hat{x}) = F'(x_0)G(\hat{x}, x_0)$, where $G(\hat{x}, x_0)$ is a bounded linear operator from $X \to X$ with
$$\|G(\hat{x}, x_0) - I\| \le C_0 \rho,$$
where C_0 is a positive constant and $\rho \ge \|x_0 - \hat{x}\|$.
(B2) We have $F'(x) = R(x, y)F'(y)$, $x, y \in B(x_0, \rho)$, where $\{R(x, y) : x, y \in B(x_0, \rho)\}$ is a family of bounded linear operators $R(x, y) : Y \to Y$ with
$$\|R(x, y) - I\| \le C_1\|x - y\|$$
for some positive constant C_1.
(B3) We have $x_0 - \hat{x} = (F'(x_0)^* F'(x_0))^{\frac{1}{2}} v$ for some $v \in X$.

Remark 1.1. It is known that [11], condition (B2) is more restrictive than (A2). So, we give an examples from [10, 14] satisfying (B2) (also see [10] for more examples satisfying (B2)).

Example 1.2. Consider the problem of estimating c in
$$-\Delta u + cu = f \quad \text{in } \Omega, \qquad u = g \quad \text{in } \Omega, \tag{1.2}$$
where Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with smooth boundary or with Ω being a parallelepiped, $f \in L^2(\Omega)$ and $g \in H^{\frac{3}{2}}(\partial\Omega)$. The nonlinear mapping $F : D(F) \subseteq L^2(\Omega) \to L^2(\Omega)$ is defined as the parameter to solution mapping
$$F(c) = u(c),$$
where $u(c)$ is the solution of (1.2). Then F is well defined on (see [10, 17])
$$D(F) := \{c \in L^2 : \|c - \bar{c}\| \le \gamma \text{ for some } \gamma > 0 \text{ and } \bar{c} \ge 0 \text{ a.e.}\}.$$
Then the Fréchet derivative of F and its adjoint are given by (see [10, 14, 17])
$$F'(c)h = -A(c)^{-1}(hu(c)), \quad F'(c)^* w = -u(c)A(c)^{-1}w$$
with $A(c) : H^2 \cap H_0^1 \to L^2$ defined by
$$A(c)u = -\Delta u + cu.$$
If $u(c) \ge \kappa$, $\kappa > 0$, for all $c \in B(c_0, \rho)$, $(\rho \le \gamma)$, then
$$F'(d) = R(d, c)F'(c), \quad c, d \in B(c_0, \rho)$$
with
$$R(d, c)^* w = A(c)\Big[\frac{u(d)}{u(c)} A(d)^{-1} w\Big]$$
and
$$\|R(d, c) - I\| \le C_1\|d - c\|, \quad c, d \in B(c_0, \rho),$$
where C_1 is a positive constant independent of c and d. That is F satisfies condition (B2).

The second author and his collaborators studied iterative methods [6, 7, 20, 21] for solving the ill-posed operator equation (1.1) and obtained the error estimate for $\|x_k^\delta - \hat{x}\|$ (x_k^δ is the iterative solution of the method under consideration) under the assumption

$$x_0 - \hat{x} = (F'(x_0)^* F'(x_0))^\nu v, \quad v \in X. \tag{1.3}$$

For frozen-type regularization methods for ill-posed problems, assumption (1.3) is used (see [6, 11] (also see Seminova [18])), instead of the classical Hölder-type source condition,

$$x_0 - \hat{x} = (F'(\hat{x})^* F'(\hat{x}))^\nu v. \tag{1.4}$$

As far as the authors know, for the minimal error method no error estimate is known under the general Hölder-type source condition (1.3) or (1.4) for $\nu \neq \frac{1}{2}$. In order to obtain an error estimate under the general source condition (1.3). The main goal of this study is to obtain an error estimate for a modified form of minimal error method defined by

$$x_{k+1} = x_k + \alpha_k s_k \quad (k = 0, 1, 2, \dots), \qquad s_k = -F'(x_0)^*(F(x_k) - y), \qquad \alpha_k = \frac{\|F(x_k) - y\|^2}{\|A^q(F(x_k) - y)\|^2}, \tag{1.5}$$

where $A = F'(x_0)^* F'(x_0)$ and $0 < q < \frac{1}{2}$ under the Hölder-type source condition (1.3). Note that for $q = \frac{1}{4}$, we have

$$\alpha_k = \frac{\|F(x_k) - y\|^2}{\langle F'(x_0)(F(x_k) - y), F(x_k) - y\rangle}$$

as a special case. We obtain the error estimate $\|x_k - \hat{x}\| = O(k^{-\nu})$ for $0 < \nu < \frac{1}{2} - q$ under assumption (1.3) (see Theorem 2.3). We also considered the method (1.5) with noisy data y^δ and obtained error estimate.

Remark 1.3. We make the following remarks.
(a) For $q = \frac{1}{2}$, method (1.5) reduced to the modified minimal error method considered in [8], but the proof in the present paper cannot be applied for the method considered in [8].
(b) Note that for q close to zero, ν is close to $\frac{1}{2}$, i.e., we obtain the error estimate $O(k^{-\nu})$ for $0 < \nu < \frac{1}{2}$ (see Theorem 2.3).

The rest of the paper is organized as follows. Convergence analysis of method (1.5) is given in Section 2 and the convergence rate result of method (1.5) with noisy data is given in Section 3.

2 Convergence Analysis of Method (1.5)

To obtain an error estimate for $\|x_k - \hat{x}\|$ under assumption (1.3), we need the result of [9, Lemma 2]. Let $\{v_k\}$ be a sequence in X, and let $\nu > 0$ be some parameter such that

$$\|A^\nu v_k\|^2 - \|A^\nu v_{k+1}\|^2 \geq \varepsilon_k \langle A^{\nu+1} v_k, A^\nu v_k\rangle$$

for $k = 0, 1, 2, \dots$, where A is a positive self-adjoint operator and $\varepsilon_k > 0$. Then

$$\|A^\nu v_k\| \leq [2(\nu + 1)]^\nu \|v_k\|^{\frac{1}{\nu+1}} \left[\sum_{i=0}^{k-1} \varepsilon_i \|v_i\|^{-\frac{1}{\nu+1}}\right]^{-\nu}. \tag{2.1}$$

To apply (2.1) with $v_k = A^{-\nu}(x_k - \hat{x})$, one has to prove that

$$\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 \geq \varepsilon_k \langle A(x_k - \hat{x}), x_k - \hat{x}\rangle$$

for some $\varepsilon_k > 0$ and $\|A^{-\nu}(x_k - \hat{x})\|$ is bounded.

Let

$$B = \|A^{\frac{1}{2}-q}\| < \sqrt{2} \quad \text{and} \quad D = \frac{\sqrt{1 + 4B^2} - (B^2 + 1)}{B^2}.$$

Lemma 2.1. *Let assumption* (B2) *and* (1.3) *hold with* $0 < \nu < \frac{1}{2} - q$ *and let* $0 < C_1\rho < D$. *Let* x_k *be as in* (1.5). *Then* $x_k \in B(x_0, 2\rho)$ *and*

$$\|x_{k+1} - \hat{x}\|^2 + \alpha_k \Gamma \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \le \|x_k - \hat{x}\|^2$$

with

$$\Gamma = 2 - (B^2 C_1^2 \rho^2 + 2(B^2+1)C_1\rho + B^2) \tag{2.2}$$

for all $k = 0, 1, 2, \ldots$. *Moreover,*

$$\sum_{k=0}^{\infty} \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 < \infty.$$

Proof. We shall prove the result using induction. Note that $x_0 \in B(x_0, 2\rho)$ and suppose that $x_k \in B(x_0, 2\rho)$. Then using (1.5), we have

$$\begin{aligned}
\|x_{k+1} - \hat{x}\|^2 - \|x_k - \hat{x}\|^2 &= -2\alpha_k \langle x_k - \hat{x}, F'(x_0)^*(F(x_k) - y)\rangle + \alpha_k^2 \|F'(x_0)^*(F(x_k) - y)\|^2 \\
&= -2\alpha_k \langle x_k - \hat{x}, F'(x_0)^*[F(x_k) - F(\hat{x}) - F'(x_0)(x_k - \hat{x})]\rangle \\
&\quad + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\langle x_k - \hat{x}, F'(x_0)^* F'(x_0)(x_k - \hat{x})\rangle] \\
&= -2\alpha_k \Big\langle F'(x_0)(x_k - \hat{x}), \int_0^1 (F'(\hat{x} + t(x_k - \hat{x})) - F'(x_0))\, dt\, (x_k - \hat{x}) \Big\rangle \\
&\quad + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2].
\end{aligned}$$

So by (B2), we have

$$\begin{aligned}
\|x_{k+1} - \hat{x}\|^2 - \|x_k - \hat{x}\|^2 &= -2\alpha_k \Big\langle F'(x_0)(x_k - \hat{x}), \int_0^1 [R(\hat{x} + t(x_k - \hat{x}), x_0) - I]\, dt\, F'(x_0)(x_k - \hat{x}) \Big\rangle \\
&\quad + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2] \\
&\le 2\alpha_k \int_0^1 \|R(\hat{x} + t(x_k - \hat{x}), x_0) - I\| \|F'(x_0)(x_k - \hat{x})\|^2\, dt \\
&\quad + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2] \\
&\le 2\alpha_k C_1 \|\hat{x} + t(x_k - \hat{x}) - x_0\| \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \\
&\quad + \alpha_k[\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 - 2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2].
\end{aligned} \tag{2.3}$$

Note that, by the definition of α_k, we have

$$\begin{aligned}
\alpha_k \|F'(x_0)^*(F(x_k) - y)\|^2 &= \alpha_k \|A^{\frac{1}{2}-q} A^q (F(x_k) - y)\|^2 \\
&\le \|A^{\frac{1}{2}-q}\|^2 \|F(x_k) - y\|^2 \\
&= B^2 \Big\| \int_0^1 F'(\hat{x} + t(x_k - \hat{x}))\, dt\, (x_k - \hat{x}) \Big\|^2 \\
&= B^2 \Big\| \int_0^1 [R(\hat{x} + t(x_k - \hat{x}), x_0) - I + I]\, dt\, F'(x_0)(x_k - \hat{x}) \Big\|^2 \\
&\le B^2 (C_1 \|\hat{x} + t(x_k - \hat{x}) - x_0\| + 1)^2 \|F'(x_0)(x_k - \hat{x})\|^2 \\
&\le B^2 (C_1\rho + 1)^2 \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2.
\end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4) we have

$$\|x_{k+1} - \hat{x}\|^2 - \|x_k - \hat{x}\|^2 \le -\Gamma \alpha_k \|A^{\frac{1}{2}}(x_k - \hat{x})\|^2.$$

This completes the proof. □

Next, we will prove the boundedness of $\|A^{-\nu}(x_k - \hat{x})\|$. Let $B_1 = \|A^{\frac{1}{2}-\nu-q}\|$, $0 < \nu < \frac{1}{2} - q$ with $0 < q < \frac{1}{2}$.

Lemma 2.2. *Let assumption* (B2) *and* (1.3) *hold with* $0 < \nu < \frac{1}{2} - q$ *and* $0 < C_1\rho < D$. *Let* x_k *be as in* (1.5). *Then* $\|A^{-\nu}(x_k - \hat{x})\|$ *is bounded.*

Proof. By using (1.3), one can prove that $x_k - \hat{x} \in R(A^\nu)$ for all $k = 0, 1, 2, \dots$. So, we can apply $A^{-\nu}$ to $x_{k+1} - \hat{x}$ and $x_k - \hat{x}$. Then we have

$$\begin{aligned}\|A^{-\nu}(x_{k+1} - \hat{x})\|^2 - \|A^{-\nu}(x_k - \hat{x})\|^2 &= 2\langle A^{-\nu}(x_k - \hat{x}), A^{-\nu}(x_{k+1} - x_k)\rangle + \|A^{-\nu}(x_{k+1} - x_k)\|^2\\ &= -2\alpha_k\langle A^{-\nu}(x_k - \hat{x}), A^{-\nu}F'(x_0)^*(F(x_k) - y)\rangle + \alpha_k^2\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|^2\\ &\le 2\alpha_k\|A^{-\nu}(x_k - \hat{x})\|\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|\\ &\quad + \alpha_k^2\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|^2.\end{aligned}$$

This implies $\|A^{-\nu}(x_{k+1} - \hat{x})\|^2 \le (\|A^{-\nu}(x_k - \hat{x})\| + \alpha_k\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|)^2$, i.e.,

$$\|A^{-\nu}(x_{k+1} - \hat{x})\| \le \|A^{-\nu}(x_k - \hat{x})\| + \alpha_k\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|. \tag{2.5}$$

By the definition of α_k, we have

$$\begin{aligned}\alpha_k\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|^2 &= \alpha_k\|A^{\frac{1}{2}-\nu-q}A^q(F(x_k) - y)\|^2\\ &\le \|A^{\frac{1}{2}-\nu-q}\|^2\|F(x_k) - y\|^2\\ &= \|A^{\frac{1}{2}-\nu-q}\|^2\left\|\int_0^1 F'(\hat{x} + t(x_k - \hat{x}))\,dt\,(x_k - \hat{x})\right\|^2.\end{aligned} \tag{2.6}$$

Using assumption (B2) in (2.6), we get

$$\begin{aligned}\alpha_k\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\|^2 &= \|A^{\frac{1}{2}-\nu-q}\|^2\left\|\int_0^1 [R(\hat{x} + t(x_k - \hat{x}), x_0) - I + I]\,dt\,F'(x_0)(x_k - \hat{x})\right\|^2\\ &\le \|A^{\frac{1}{2}-\nu-q}\|^2(C_1\|\hat{x} + t(x_k - \hat{x}) - x_0\| + 1)^2\|F'(x_0)(x_k - \hat{x})\|^2\\ &\le B_1^2(C_1\rho + 1)^2\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2,\end{aligned}$$

so

$$\sqrt{\alpha_k}\|A^{-\nu}F'(x_0)^*(F(x_k) - y)\| \le B_1(C_1\rho + 1)\|A^{\frac{1}{2}}(x_k - \hat{x})\|. \tag{2.7}$$

Therefore by (2.7) and (2.5), we have

$$\|A^{-\nu}(x_{k+1} - \hat{x})\| \le \|A^{-\nu}(x_k - \hat{x})\| + \sqrt{\alpha_k}B_1(C_1\rho + 1)\|A^{\frac{1}{2}}(x_k - \hat{x})\|. \tag{2.8}$$

Let $z_k = \|A^{-\nu}(x_k - \hat{x})\|$. Then by (2.8),

$$z_{k+1} \le z_k + B_1(C_1\rho + 1)\sqrt{\alpha_k}\|A^{\frac{1}{2}}(x_k - \hat{x})\|,$$

i.e.,

$$z_k \le z_0 + B_1(C_1\rho + 1)\sum_{i=0}^{k-1}\sqrt{\alpha_i}\|A^{\frac{1}{2}}(x_i - \hat{x})\|.$$

By Lemma 2.1, we have

$$z_k \le z_0 + B_1(C_1\rho + 1)M,$$

where M is such that

$$\sum_{k=0}^{\infty}\alpha_k\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2 \le M^2.$$

Now since $z_0 = \|A^{-\nu}(x_0 - \hat{x})\| = \|A^{-\nu}A^\nu v\| = \|v\|$, we obtain

$$z_k \le \|v\| + B_1(C_1\rho + 1)M. \tag{2.9}$$

This completes the proof. □

Theorem 2.3. *Let assumption* (B2) *and* (1.3) *for* $0 < \nu < \frac{1}{2} - q$ *hold and let* $0 < C_1\rho < D$. *Let* x_k *be as in* (1.5). *Then*

$$\|x_k - \hat{x}\| \le \tilde{C}k^{-\nu},$$

where $\tilde{C} = [2(\nu+1)]^{\nu}\varepsilon^{-\nu}(\|v\| + B_1(C_1\rho + 1)M)$.

Proof. Note that

$$\alpha_k \ge \|A^q\|^{-2},$$

since (B2) and (1.3) for $0 < \nu < \frac{1}{2} - q$ hold and $C_1\rho < D$. Set $\varepsilon_k := \varepsilon = \Gamma\|A^q\|^{-2}$, where Γ is as in (2.2). Now Lemma 2.2 implies

$$\begin{aligned}\|x_k - \hat{x}\|^2 - \|x_{k+1} - \hat{x}\|^2 &\ge \Gamma\alpha_k\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2\\ &\ge \Gamma\|A^q\|^{-2}\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2\\ &= \varepsilon\|A^{\frac{1}{2}}(x_k - \hat{x})\|^2\\ &= \varepsilon\langle F'(x_0)^*F'(x_0)(x_k - \hat{x}), x_k - \hat{x}\rangle\\ &= \varepsilon\langle A(x_k - \hat{x}), x_k - \hat{x}\rangle.\end{aligned}$$

Therefore by (2.1), we have

$$\begin{aligned}\|x_k - \hat{x}\| &\le [2(\nu+1)]^{\nu}\|A^{-\nu}(x_k - \hat{x})\|^{\frac{1}{\nu+1}}\Bigg[\sum_{i=0}^{k-1}\varepsilon_i\|A^{-\nu}(x_i - \hat{x})\|^{\frac{-1}{\nu+1}}\Bigg]^{-\nu}\\ &\le [2(\nu+1)]^{\nu}z_k^{\frac{1}{\nu+1}}\varepsilon^{-\nu}\Bigg[\sum_{i=0}^{k-1}z_i^{-\frac{1}{\nu+1}}\Bigg]^{-\nu}.\end{aligned} \tag{2.10}$$

So by (2.9) and (2.10), we have

$$\|x_k - \hat{x}\| \le [2(\nu+1)]^{\nu}\varepsilon^{-\nu}(\|v\| + B_1(C_1\rho + 1)M)k^{-\nu} \le \tilde{C}k^{-\nu},$$

as desired. □

Remark 2.4. The above result shows that we have obtained the error estimate $\|x_k - \hat{x}\| = O(k^{-\nu})$ for $0 < \nu < \frac{1}{2}$ under the general source condition (1.3) as $q \to 0$.

3 Convergence Rate Result of Method (1.5) with Noisy Data

In this section we study the modified form of minimal error method (1.5) for noisy data y^δ instead of exact data y. We assume that $\|y - y^\delta\| \le \delta$ as stated in the introduction. The minimal error method (1.5) with noisy data takes the form

$$x_{k+1}^\delta = x_k^\delta + \alpha_k^\delta s_k^\delta \quad (k = 0, 1, 2, \dots), \qquad s_k^\delta = -F'(x_0)^*(F(x_k^\delta) - y^\delta), \qquad \alpha_k^\delta = \frac{\|F(x_k^\delta) - y^\delta\|^2}{\|A^q(F(x_k^\delta) - y^\delta)\|^2}. \tag{3.1}$$

As in [8], we assume:

(B4) F satisfies the local property

$$\|F(u) - F(v) - F'(x_0)(u - v)\| \le \eta\|F(u) - F(v)\|$$

for all $u, v \in B(x_0, \rho)$ with $\max\{\frac{1-B^2}{3}, 1 - \frac{B^2}{2} - \frac{\|A^q\|^2}{2m^2}, 0\} < \eta < 1 - \frac{B^2}{2}$.

Throughout this section we assume that $B(x_0, 2\rho) \subset D(F)$.

Due to the instability of (1.5) for the noisy data, it is not possible to use an a priori regularization strategy as a stopping rule. So we need an a posteriori strategy as a stopping rule (i.e., discrepancy principle). In [14], Neubauer and Scherzer noticed that no convergence rate result has been proven for the minimal error method with noisy data. But the authors in [8] proved the convergence rate by proposing a modified discrepancy principle. Using the idea from [8], we can prove a convergence rate result for method (3.1).

3.1 Discrepancy Principle

Proposition 3.1. *Let assumption* (B4) *holds and let* x_k^δ *be as in* (3.1). *Then* $x_k^\delta \in B(x_0, 2\rho) \subset D(F)$ *for all* $k = 0, 1, 2, \dots,$ *and if*

$$\|F(x_k^\delta) - y^\delta\| \geq \tau\delta, \tag{3.2}$$

where

$$\tau > 2\frac{(1+\eta)}{2 - 2\eta - B^2} > 2, \tag{3.3}$$

then, for all $0 \leq k < k_*$ *with* τ *as in* (3.3), *we have*

$$k_*(\tau\delta)^2 \leq \sum_{k=0}^{k_*-1} \|F(x_k^\delta) - y^\delta\|^2 \leq \frac{\tau\|F'(x_0)\|^2}{(2 - 2\eta - B^2)\tau - 2(1+\eta)}\|x_0 - \hat{x}\|^2.$$

Proof. Note that $x_0 \in B(x_0, 2\rho)$. Suppose that $x_k^\delta \in B(x_0, 2\rho)$. Using (3.1), we have

$$\begin{aligned}
\|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 &= -2\alpha_k^\delta\langle x_k^\delta - \hat{x}, F'(x_0)^*(F(x_k^\delta) - y^\delta)\rangle + {\alpha_k^\delta}^2\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 \\
&= 2\alpha_k^\delta\langle F(x_k^\delta) - y^\delta - F'(x_0)(x_k^\delta - \hat{x}), F(x_k^\delta) - y^\delta\rangle \\
&\quad + \alpha_k^\delta[\alpha_k^\delta\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2] \\
&\leq 2\alpha_k^\delta\|F(x_k^\delta) - F(\hat{x}) + y - y^\delta - F'(x_0)(x_k^\delta - \hat{x})\|\|F(x_k^\delta) - y^\delta\| \\
&\quad + \alpha_k^\delta[\alpha_k^\delta\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2].
\end{aligned} \tag{3.4}$$

So by (B4) and (3.4), we have

$$\begin{aligned}
\|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 &\leq 2\alpha_k^\delta(\eta\|F(x_k^\delta) - F(\hat{x})\| + \delta)\|F(x_k^\delta) - y^\delta\| \\
&\quad + \alpha_k^\delta[\alpha_k^\delta\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2] \\
&\leq 2\alpha_k^\delta[\eta\|F(x_k^\delta) - y^\delta\| + (1+\eta)\delta]\|F(x_k^\delta) - y^\delta\| \\
&\quad + \alpha_k^\delta[\alpha_k^\delta\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 - 2\|F(x_k^\delta) - y^\delta\|^2] \\
&= \alpha_k^\delta(2\eta - 2)\|F(x_k^\delta) - y^\delta\|^2 + \alpha_k^\delta 2(1+\eta)\delta\|F(x_k^\delta) - y^\delta\| + (\alpha_k^\delta)^2\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2.
\end{aligned}$$

Note that

$$\alpha_k^\delta\|F'(x_0)^*(F(x_k^\delta) - y^\delta)\|^2 = \alpha_k^\delta\|A^{\frac{1}{2}}(F(x_k^\delta) - y^\delta)\|^2 \leq \alpha_k^\delta\|A^{\frac{1}{2}-q}\|^2\|A^q(F(x_k^\delta) - y^\delta)\|^2 \leq B^2\|F(x_k^\delta) - y^\delta\|^2.$$

Therefore we have

$$\|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \leq \alpha_k^\delta[(2\eta + B^2 - 2)\|F(x_k^\delta) - y^\delta\|^2 + 2(1+\eta)\delta\|F(x_k^\delta) - y^\delta\|],$$

so by (3.2),

$$\|x_{k+1}^\delta - \hat{x}\|^2 - \|x_k^\delta - \hat{x}\|^2 \leq \alpha_k^\delta\Big((2\eta + B^2 - 2) + 2\frac{(1+\eta)}{\tau}\Big)\|F(x_k^\delta) - y^\delta\|^2 < 0. \tag{3.5}$$

This implies $\|x_{k+1}^\delta - \hat{x}\| < \|x_k^\delta - \hat{x}\| < \|x_0 - \hat{x}\| < \rho$. Thus we obtain $\|x_{k+1}^\delta - x_0\| \leq \|x_{k+1}^\delta - \hat{x}\| + \|x_0 - \hat{x}\| < 2\rho$, i.e., $x_{k+1}^\delta \in B(x_0, 2\rho) \subset D(F)$ for all $k = 0, 1, 2, \dots$. Now since $\alpha_k^\delta \geq \|A^q\|^{-2}$, we have by (3.5)

$$\|A^q\|^{-2}\Big((2 - 2\eta - B^2) - 2\frac{(1+\eta)}{\tau}\Big)\|F(x_k^\delta) - y^\delta\|^2 \leq \|x_k^\delta - \hat{x}\|^2 - \|x_{k+1}^\delta - \hat{x}\|^2. \tag{3.6}$$

Adding inequality (3.6) for k from 0 through $k_* - 1$, we obtain

$$\|A^q\|^{-2}\Big((2 - 2\eta - B^2) - 2\frac{(1+\eta)}{\tau}\Big)\sum_{k=0}^{k_*-1}\|F(x_k^\delta) - y^\delta\|^2 \leq \|x_0 - \hat{x}\|^2 - \|x_{k_*}^\delta - \hat{x}\|^2. \tag{3.7}$$

This completes the proof. □

Remark 3.2. Note that (3.7) implies that, for $y^\delta \neq y$, there must be a unique index k_* such that (3.2) holds for all $k < k_*$ but is violated at $k = k_*$ (see also [3, p. 282]).

Let $\Omega := \|A^q\|^{-2}((2-2\eta-B^2) - 2\frac{(1+\eta)}{\tau})$ and

$$q = 1 - \Omega\Big(\frac{m}{\tau-1}\Big)^2.$$

Now, we shall prove that $q < 1$ for $\tau > 2$. Note that, to prove $q < 1$, it is enough to prove that

$$\Omega(\frac{m}{\tau-1})^2 = \|A^q\|^{-2}\Big((2-2\eta-B^2) - 2\frac{(1+\eta)}{\tau}\Big)\Big(\frac{m}{\tau-1}\Big)^2 < 1$$

for $\tau > 2$. That is to prove that

$$p(\tau) := \tau^3 - 2\tau^2 + (1 - \|A^q\|^{-2}(2-2\eta-B^2)m^2)\tau + 2(1+\eta)m^2\|A^q\|^{-2} > 0$$

for $\tau > 2$. This follows from the condition $\eta > 1 - \frac{B^2}{2} - \frac{\|A^q\|^2}{2m^2}$.

Theorem 3.3. *Let assumptions* (B2) *and* (B4) *hold and let* $\rho < \min\{\frac{(\tau-1)^2\delta}{m}, \frac{2}{m\sqrt{\Omega}}\}$. *Let* x_{k+1}^δ *be as in* (3.1). *Then for* $0 \leq k < k_*$,

$$\|x_{k+1}^\delta - \hat{x}\| = \begin{cases} O(q^{\frac{k+1}{2}}) & \text{if } \delta < q^{k+1}, \\ O(\delta^{\frac{1}{2}}) & \text{if } q^{k+1} \leq \delta, \end{cases}$$

where $q := 1 - \frac{\Omega m^2}{(\tau-1)^2}$.

Proof. By the definition of k_*, we have for $k \leq k_*$,

$$\tau\delta < \|F(x_k^\delta) - y^\delta\| \leq \|F(x_k^\delta) - F(\hat{x})\| + \|y - y^\delta\|. \tag{3.8}$$

So,

$$\|F(x_k^\delta) - F(\hat{x})\| > (\tau - 1)\delta. \tag{3.9}$$

Again by (3.8), we have

$$\tau\delta < \|F(x_k^\delta) - y^\delta\| < \Big\|\int_0^1 F'(\hat{x} + \theta(x_k^\delta - \hat{x}))\,d\theta\,(x_k^\delta - \hat{x})\Big\| + \delta \leq m\|x_k^\delta - \hat{x}\| + \delta,$$

i.e.,

$$\delta < \frac{m\|x_k^\delta - \hat{x}\|}{\tau - 1}. \tag{3.10}$$

Thus, by (3.9) and (3.10),

$$\|F(x_k^\delta) - y^\delta\| \geq \|F(x_k^\delta) - F(\hat{x})\| - \delta \geq (\tau-1)\delta - \frac{m\|x_k^\delta - \hat{x}\|}{\tau-1} \geq (\tau-1)\delta - \frac{m\rho}{\tau-1} > 0. \tag{3.11}$$

It follows from (3.11) that

$$\|F(x_k^\delta) - y^\delta\|^2 \geq (\tau-1)^2\delta^2 + \Big(\frac{m\|x_k^\delta - \hat{x}\|}{\tau-1}\Big)^2 - 2\delta m\|x_k^\delta - \hat{x}\|. \tag{3.12}$$

So by (3.12) and (3.6), we have

$$\begin{aligned}\|x_{k+1}^\delta - \hat{x}\|^2 &\leq \Big(1 - \Omega\Big(\frac{m}{\tau-1}\Big)^2\Big)\|x_k^\delta - \hat{x}\|^2 - \Omega(\tau-1)^2\delta^2 + 2\Omega\delta m\|x_k^\delta - \hat{x}\| \\ &\leq \Big(1 - \Omega\Big(\frac{m}{\tau-1}\Big)^2\Big)\|x_k^\delta - \hat{x}\|^2 - \Omega(\tau-1)^2\delta^2 + 2\Omega\delta m\rho \\ &\leq \Big(1 - \Omega\Big(\frac{m}{\tau-1}\Big)^2\Big)\|x_k^\delta - \hat{x}\|^2 + 2\Omega\delta m\rho.\end{aligned}$$

Therefore,

$$\|x_{k+1}^{\delta} - \hat{x}\|^2 \le q\|x_k^{\delta} - \hat{x}\|^2 + L\delta,$$

where $q = 1 - \Omega(\frac{m}{\tau-1})^2$ and $L = 2\Omega m\rho$. Then

$$\|x_{k+1}^{\delta} - \hat{x}\|^2 \le q^{k+1}\|x_0^{\delta} - \hat{x}\|^2 + q^k L\delta + \cdots + qL\delta + L\delta \le q^{k+1}\rho^2 + \frac{L\delta}{1-q}.$$

This completes the proof. □

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