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Obrechkoff methods having additional parameters for general second-order differential equations

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Abstract

A class of two-step implicit methods involving higher-order derivatives of y for initial value problems of the form $y'' = f(t, y, y')$ is developed. The methods involve arbitrary parameters p and q, which are determined so that the methods become absolutely stable when applied to the test equation $y'' + \lambda y' + \mu y = 0$. Numerical results for Bessel's and general second-order differential equations are presented to illustrate that the methods are absolutely stable and are of order $O(h^4)$, $O(h^6)$ and $O(h^8)$.

Keywords: Initial value problems; Additive parameter; Absolutely stable

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1. Introduction

When practical problems in science and technology permits mathematical formulation, the chances are rather good that it leads to one or more differential equations. This is true certainly of the vast category of problems associated with force and motion, propagation of waves, flow of heat, diffusion, static or dynamic electricity, etc.

The analytical methods of solving differential equation are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of the familiar types and one is obliged to resort to numerical methods. The numerical methods have become more popular and important with the fast growing computing facilities of memory size and speed in calculations using recent computers.

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We consider the general second-order differential equations of the form

$$y'' = f(t, y, y') \quad \text{with } y(t_0) = y_0, \ y'(t_0) = y'_0 \tag{1.1}$$

Here, we have developed a class of two-step implicit methods involving higher-order derivatives for the initial value problems (1.1). The use of higher-order methods to approximate the theoretical solution restricts the choices of step size to small values. The idea of adapting higher-order methods has been proposed by several authors. The methods contain two arbitrary parameters p, q, which are the new additional values of the coefficients of y' and y in the given differential equation. They have been chosen so that, they are the roots of the characteristic equation $m^2 + pm + q = 0$ satisfying the conditions $0 < p < 2\sqrt{q}$. This condition gives complex conjugate roots and the methods become absolutely stable when applied to the test equation

$$y'' + \lambda y' + \mu y = 0, \tag{1.2}$$

where λ and μ are real numbers.

It is observed that the numerical solution of the special second-order differential equation

$$y'' = f(t, y), \qquad y(t_0) = y_0, \qquad y'(t_0) = y'_0 \tag{1.3}$$

by the Numerov method of $O(h^4)$ with interval of periodicity $(0, 6)$:

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{12}(y''_{n+1} + 10y''_n + y''_{n-1}) \tag{1.4}$$

becomes unstable for large step sizes. Ananthakrishnaiah [2] had developed two-step E-stable methods of higher orders.

Definition. The numerical method

$$y_{n+1} - 2y_n + y_{n-1} = \sum_{i=1}^{m} h^i[\alpha_{i,0}\, y^{(i)}_{n+1} + 2\alpha_{i,1}\, y^{(i)}_n + \alpha_{i,2}\, y^{(i)}_{n-1}] \tag{1.5}$$

is said to be an *E-stable method* if the characteristic equation possesses real and equal roots of modulus less than unity

The following method is of $O(h^4)$ and E-stable:

$$y_{n+1} - 2y_n + y_{n-1} = h(y'_{n+1} - y'_{n-1}) - \tfrac{1}{4}h^2(y''_{n+1} + 2y''_n + y''_{n-1}). \tag{1.6}$$

The purpose of the article is to derive additional parameter methods for (1.1) which are absolutely stable and involve higher-order derivatives. Our derived methods are useful to solve any linear or nonlinear general second-order initial value problems. The truncation error entirely depends on the choice of q such that $q \geqslant (2\pi n/h)^2$, $n = 1, 2, 3, 4, \ldots$ The truncation error tends to zero as n is very large and p near to zero.

2. Derivation of the methods

We write (1.1) in the form

$$y'' + py' + qy = \Phi(t, y, y'), \tag{2.1}$$

where

$$\Phi(t,y,y') = f(t,y,y') + py' + qy \tag{2.2}$$

with $0 < p < 2\sqrt{q}, p$ and q being real numbers to be determined.

Eq. (2.1) can be approximated as

$$y'' + py' + qy = g(t), \tag{2.3}$$

where $g(t)$ is an approximation to $\Phi(t,y,y')$.

Obviously, the general solution of (2.3) is

$$y(t) = Ae^{\sigma_1 t} + Be^{\sigma_2 t} + \frac{1}{\sigma_2 - \sigma_1}\int_{t_n}^{t} (e^{\sigma_2(t-z)} - e^{\sigma_1(t-z)})g(z)\,dz, \tag{2.4}$$

where $\sigma_1 = u + iv$, $\sigma_2 = u - iv$, $u = -p/2$, $v = \sqrt{(4q - p^2)}/2$ and A, B are arbitrary constants. Differentiating (2.4) with respect to t, we have

$$y'(t) = \sigma_1 Ae^{\sigma_1 t} + \sigma_2 Be^{\sigma_2 t} + \frac{1}{\sigma_2 - \sigma_1}\int_{t_n}^{t} (\sigma_2 e^{\sigma_2(t-z)} - \sigma_1 e^{\sigma_1(t-z)})g(z)\,dz. \tag{2.5}$$

Eliminating A and B by substituting $t = t_{n+1}, t_n$ and t_{n-1} in (2.4) and (2.5) we obtain the relations

$$y(t_{n+1}) - (e^{\sigma_1 h} + e^{\sigma_2 h})y(t_n) + e^{-ph}y(t_{n-1}) = \frac{1}{\sigma_2 - \sigma_1}\int_{t_n}^{t_{n+1}} (e^{\sigma_2(t_{n+1}-z)} - e^{\sigma_1(t_{n+1}-z)})(g(z) + e^{-ph}g(2t_n - z))\,dz, \tag{2.6}$$

$$y'(t_{n+1}) - (e^{\sigma_1 h} + e^{\sigma_2 h})y'(t_n) + e^{-ph}y'(t_{n-1}) = \frac{1}{\sigma_2 - \sigma_1}\int_{t_n}^{t_{n+1}} (\sigma_2 e^{\sigma_2(t_{n+1}-z)} - \sigma_1 e^{\sigma_1(t_{n+1}-z)})(g(z) + e^{-ph}g(2t_n - z))\,dz. \tag{2.7}$$

2.1. *Implicit multistep methods*

Let us take a natural number k and replace $g(t)$ with the Newton backward difference interpolation polynomial at the points $t_{n+1}, t_n, t_{n-1}, \dots, t_{n-k+1}$ and replacing $\nabla^j g_{n+1}$ by $\nabla^j \Phi_{n+1}$ we get (see Eq. (2.3))

$$g(t) = \phi_{n+1} + \frac{(t - t_{n+1})}{h}\nabla\phi_{n+1} + \frac{(t - t_{n+1})(t - t_n)}{2!h^2}\nabla^2\phi_{n+1} + \cdots + \frac{(t - t_{n+1})(t - t_n)\cdots(t - t_{n-k+2})}{k!h^k}\nabla^k\phi_{n+1} + \frac{(t - t_{n+1})(t - t_n)\cdots(t - t_{n-k+1})}{(k+1)!}\phi^{(k+1)}(\xi), \quad t_n < \xi < t_{n+1}. \tag{2.8}$$

Analogously, if in formula (2.6) we replace the expression $g(t)$ by the Newton polynomial (2.8) and neglect the error term, we get

$$y_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y_n + e^{-ph}y_{n-1} = \frac{h}{\sigma_2 - \sigma_1}\int_0^1 [e^{\sigma_2(1-s)h} - e^{\sigma_1(1-s)h}] \sum_{m=0}^{k} (-1)^m \left[\binom{-s}{m} + \binom{s}{m} e^{-ph}\right] \nabla^m \phi_{n+1}\, ds \tag{2.9}$$

where $(t - t_n)/h = s$.

Proceeding as above and taking into account also the second term (2.7) in (2.8) we obtain the formula

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1} = \frac{h}{\sigma_2 - \sigma_1}\int_0^1 [\sigma_2 e^{\sigma_2(1-s)h} - \sigma_1 e^{\sigma_1(1-s)h}] \sum_{m=0}^{k} (-1)^m \left[\binom{-s}{m} + \binom{s}{m} e^{-ph}\right] \nabla^m \phi_{n+1}\, ds. \tag{2.10}$$

Alternatively, the above equations, (2.9) and (2.10), may be written as

$$y_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y_n + e^{-ph}y_{n-1} = h^2 \sum_{m=0}^{k} \gamma_m \nabla^m \phi_{n+1}, \tag{2.11a}$$

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1} = h \sum_{m=0}^{k} \gamma^*_m \nabla^m \phi_{n+1}, \tag{2.11b}$$

where

$$\gamma_m = \frac{(-1)^m}{(\sigma_2 h - \sigma_1 h)}\int_0^1 (e^{\sigma_2(1-s)h} - e^{\sigma_1(1-s)h})\left[\binom{-s}{m} + e^{-ph}\binom{s}{m}\right] ds$$

$$\gamma^*_m = \frac{(-1)^m}{(\sigma_2 h - \sigma_1 h)}\int_0^1 (\sigma_2 h e^{\sigma_2(1-s)h} - \sigma_1 h e^{\sigma_1(1-s)h})\left[\binom{-s}{m} + e^{-ph}\binom{s}{m}\right] ds.$$

This relation for $k = 2$ takes form of Eqs. (2.11a) and (2.11b) as follows:

$$y_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y_n + e^{-ph}y_{n-1} = h^2(\beta_{1,0}\phi_{n+1} + \beta_{1,1}\phi_n + \beta_{1,2}\phi_{n-1}), \tag{2.12a}$$

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1} = h(\beta'_{1,0}\phi_{n+1} + \beta'_{1,1}\phi_n + \beta'_{1,2}\phi_{n-1}), \tag{2.12b}$$

where $\phi_n = py'_n + qy_n + f(t_n, y_n, y'_n)$.

The coefficients in (2.12) can be determined by expanding both sides of (2.12) into the Taylor series about the point $t = t_n$, and equating the like powers of h. From the system of linear equations thus obtained, we get the coefficients in Eqs. (2.12a) and (2.12b) as follows:

$$\beta_{1,0} = \frac{1}{2q^3h^4}[(2p^2 - 2q - pqh)F - 2pqh(1 - e^{-ph}) + 2q^2h^2], \tag{2.13a}$$

$$\beta_{1,1} = -\frac{2}{2q^3h^4}[(2p^2 - 2q - q^2h^2)F - 2pqh(1 - e^{-ph}) + q^2h^2(1 + e^{-ph})], \tag{2.13b}$$

$$\beta_{1,2} = \frac{1}{2q^3h^4}[(2p^2 - 2q - pqh)F - 2pqh(1 - e^{-ph}) + 2q^2h^2e^{-ph}] \tag{2.13c}$$

and

$$\beta'_{1,0} = \frac{1}{2q^2h^3}[(qh - 2p)F + 2qh(1 - e^{-ph})], \tag{2.14a}$$

$$\beta'_{1,1} = \frac{2}{2q^2h^3}[2pF - 2qh(1 - e^{-ph})], \tag{2.14b}$$

$$\beta'_{1,2} = -\frac{1}{2q^2h^3}[(qh + 2p)F - 2qh(1 - e^{-ph})], \tag{2.14c}$$

where $F = [1 - (e^{\sigma_1 h} + e^{\sigma_2 h}) + e^{-ph})]$.

3. Two-step Obrechkoff methods

We have constructed two-step Obrechkoff methods of $O(h^4), O(h^6)$ and $O(h^8)$ for $k = 2, 3, 4$ in the following formula:

$$y_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y_n + e^{-ph}y_{n-1} = \sum_{i=1}^{k} h^{2i}[\beta_{i,0}\phi_{n+1}^{(2i-2)} + \beta_{i,1}\phi_n^{(2i-2)} + \beta_{i,2}\phi_{n-1}^{(2i-2)}], \tag{3.1a}$$

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1} = \sum_{i=1}^{k} h^{2i-1}[\beta'_{i,0}\phi_{n+1}^{(2i-2)} + \beta'_{i,1}\phi_n^{(2i-2)} + \beta'_{i,2}\phi_{n-1}^{(2i-2)}], \tag{3.1b}$$

where $\phi_n^{(2i-2)} = y_n^{(2i)} + py_n^{(2i-1)} + qy_n^{(2i-2)}$, $i = 1, 2, 3, 4$.

The method is of the form (3.1) which involves higher-order derivatives of the solution $y(t)$ and may be called as Obrechkoff method. The method (3.1) is called explicit if $\beta_{i,0} = \beta'_{i,0} = 0$ for $i = 1(1)k$; otherwise implicit. The coefficients $\beta_{1,0}, \beta_{1,1}, \beta_{1,2}, \beta'_{1,0}, \beta'_{1,1}$, and $\beta'_{1,2}$ are given in (2.13) and (2.14).

3.1. Two-step Obrechkoff method of O(h⁴)

When $k = 2$, Eqs. (3.1a) and (3.1b) may be written as two-step method of order four as follows:

$$y_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y_n + e^{-ph}y_{n-1} = h^2(\beta_{1,0}\phi_{n+1} + \beta_{1,1}\phi_n + \beta_{1,2}\phi_{n-1}) + h^4(\beta_{2,0}\phi''_{n+1} + \beta_{2,1}\phi''_n + \beta_{2,2}\phi''_{n-1}), \tag{3.2a}$$

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1} = h(\beta'_{1,0}\phi_{n+1} + \beta'_{1,1}\phi_n + \beta'_{1,2}\phi_{n-1}) + h^3(\beta'_{2,0}\phi''_{n+1} + \beta'_{2,1}\phi''_n + \beta'_{2,2}\phi''_{n-1}), \tag{3.2b}$$

where

$$\beta_{2,0} = \frac{1}{24q^2h^3}[qh(1+e^{-ph}) - 2(1-e^{-ph})(2p-qh) + \beta_{1,0}(12p^2h + 24p - 6pqh^2 - 3q^2h^3 - 24qh) + \beta_{1,2}(12p^2h - 24p - 6pqh^2 + q^2h^3)], \tag{3.3a}$$

$$\beta_{2,1} = \frac{2}{24q^2h^3}[4p(1-e^{-ph}) - qh(1+e^{-ph}) - \beta_{1,0}(12p^2h + 24p - q^2h^3 - 12qh) - \beta_{1,2}(12p^2h - 24p - q^2h^3 - 12qh)], \tag{3.3b}$$

$$\beta_{2,2} = \frac{1}{24q^2h^3}[qh(1+e^{-ph}) - 2(1-e^{-ph})(2p+qh) + \beta_{1,0}(12p^2h + 24p + 6pqh^2 + q^2h^3) + \beta_{1,2}(12p^2h - 24p + 6pqh^2 - 3q^2h^3 - 24qh)] \tag{3.3c}$$

and

$$\beta'_{2,0} = \frac{1}{24q^2h^3}[4qh(1-e^{-ph}) - 6(1+e^{-ph})(2p-qh) + \beta'_{1,0}(12p^2h + 24p - 6pqh^2 - 3q^2h^3 - 24qh) + \beta'_{1,2}(12p^2h - 24p - 6pqh^2 + q^2h^3)], \tag{3.4a}$$

$$\beta'_{2,1} = \frac{2}{24q^2h^3}[12p(1+e^{-ph}) - 4qh(1-e^{-ph}) + \beta'_{1,0}(12p^2h + 24p - q^2h^3 - 12qh) - \beta'_{1,2}(12p^2h - 24p - q^2h^3 - 12qh)], \tag{3.4b}$$

$$\beta'_{2,2} = \frac{1}{24q^2h^3}[4qh(1-e^{-ph}) - 6(1+e^{-ph})(2p+qh) + \beta'_{1,0}(12p^2h + 24p + 6pqh^2 + q^2h^3) + \beta'_{1,2}(12p^2h - 24p + 6pqh^2 - 3q^2h^3 - 24qh)]. \tag{3.4c}$$

3.2. Two-step Obrechkoff method of $O(h^6)$

In (3.1a) and (3.1b) when $k = 3$ the sixth-order method is written as

$$y_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y_n + e^{-ph}y_{n-1} = h^2(\beta_{1,0}\phi_{n+1} + \beta_{1,1}\phi_n + \beta_{1,2}\phi_{n-1}) + h^4(\beta_{2,0}\phi''_{n+1} + \beta_{2,1}\phi''_n + \beta_{2,2}\phi''_{n-1}) + h^6(\beta_{3,0}\phi^{(iv)}_{n+1} + \beta_{3,1}\phi^{(iv)}_n + \beta_{3,2}\phi^{(iv)}_{n-1}), \tag{3.5a}$$

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1} = h(\beta'_{1,0}\phi_{n+1} + \beta'_{1,1}\phi_n + \beta'_{1,2}\phi_{n-1})$$
$$+ h^3(\beta'_{2,0}\phi''_{n+1} + \beta'_{2,1}\phi''_n + \beta'_{2,2}\phi''_{n-1}) + h^5(\beta'_{3,0}\phi^{(iv)}_{n+1} + \beta'_{3,1}\phi^{(iv)}_n + \beta'_{3,2}\phi^{(iv)}_{n-1}), \qquad (3.5b)$$

where

$$\beta_{3,0} = \frac{1}{720q^2h^3}[qh(1+e^{-ph}) - 3(1-e^{-ph})(2p-qh)$$
$$+ \beta_{1,0}(30p^2h + 120p - 15pqh^2 - 4q^2h^3 - 90qh)$$
$$+ \beta_{1,2}(30p^2h - 120p - 15pqh^2 + 2q^2h^3 + 30qh)$$
$$+ \beta_{2,0}(360p^2h + 720p - 180pqh^2 - 90q^2h^3 - 720qh)$$
$$+ \beta_{2,2}(360p^2h - 720p - 180pqh^2 + 30q^2h^3)], \qquad (3.6a)$$

$$\beta_{3,1} = \frac{2}{720q^2h^3}[6p(1-e^{-ph}) - qh(1+e^{-ph})$$
$$- \beta_{1,0}(30p^2h + 120p - q^2h^3 - 30qh)$$
$$- \beta_{1,2}(30p^2h - 120p - q^2h^3 - 30qh)$$
$$- \beta_{2,0}(360p^2h + 720p - 30q^2h^3 - 360qh)$$
$$- \beta_{2,2}(360p^2h - 720p - 30q^2h^3 - 360qh)], \qquad (3.6b)$$

$$\beta_{3,2} = \frac{1}{720q^2h^3}[qh(1+e^{-ph}) - 3(1-e^{-ph})(2p+qh)$$
$$+ \beta_{1,0}(30p^2h + 120p + 15pqh^2 + 2q^2h^3 + 30qh)$$
$$+ \beta_{1,2}(30p^2h - 120p + 15pqh^2 - 4q^2h^3 - 90qh)$$
$$+ \beta_{2,0}(360p^2h + 720p + 180pqh^2 + 30q^2h^3)$$
$$+ \beta_{2,2}(360p^2h - 720p + 180pqh^2 - 90q^2h^3 - 720qh)], \qquad (3.6c)$$

and

$$\beta'_{3,0} = \frac{1}{720q^2h^3}[6qh(1-e^{-ph}) - 15(1+e^{-ph})(2p-qh)$$
$$+ \beta'_{1,0}(30p^2h + 120p - 15pqh^2 - 4q^2h^3 - 90qh)$$
$$+ \beta'_{1,2}(30p^2h - 120p - 15pqh^2 + 2q^2h^3 + 30qh)$$
$$+ \beta'_{2,0}(360p^2h + 720p - 180pqh^2 - 90q^2h^3 - 720qh)$$
$$+ \beta'_{2,2}(360p^2h - 720p - 180pqh^2 + 30q^2h^3)], \qquad (3.7a)$$

$$\beta'_{3,1} = \frac{2}{720q^2h^3}[30p(1+e^{-ph}) - 6qh(1-e^{-ph})$$
$$- \beta'_{1,0}(30p^2h + 120p - q^2h^3 - 30qh)$$
$$- \beta'_{1,2}(30p^2h - 120p - q^2h^3 - 30qh)$$
$$- \beta'_{2,0}(360p^2h + 720p - 30q^2h^3 - 360qh)$$
$$- \beta'_{2,2}(360p^2h - 720p - 30q^2h^3 - 360qh)], \tag{3.7b}$$

$$\beta'_{3,2} = \frac{1}{720q^2h^3}[6qh(1-e^{-ph}) - 15(1+e^{-ph})(2p+qh)$$
$$+ \beta'_{1,0}(30p^2h + 120p + 15pqh^2 + 2q^2h^3 + 30qh)$$
$$+ \beta'_{1,2}(30p^2h - 120p + 15pqh^2 - 4q^2h^3 - 90qh)$$
$$+ \beta'_{2,0}(360p^2h + 720p + 180pqh^2 + 30q^2h^3)$$
$$+ \beta'_{2,2}(360p^2h - 720p + 180pqh^2 - 90q^2h^3 - 720qh)]. \tag{3.7c}$$

3.3. Two-step Obrechkoff method of $O(h^8)$

In the identical way as before, when $k = 4$ in (3.1a) and (3.1b), we obtain the following eighth-order method as

$$y_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y_n + e^{-ph}y_{n-1} = h^2(\beta_{1,0}\phi_{n+1} + \beta_{1,1}\phi_n + \beta_{1,2}\phi_{n-1})$$
$$+ h^4(\beta_{2,0}\phi''_{n+1} + \beta_{2,1}\phi''_n + \beta_{2,2}\phi''_{n-1})$$
$$+ h^6(\beta_{3,0}\phi^{(iv)}_{n+1} + \beta_{3,1}\phi^{(iv)}_n + \beta_{3,2}\phi^{(iv)}_{n-1})$$
$$+ h^8(\beta_{4,0}\phi^{(vi)}_{n+1} + \beta_{4,1}\phi^{(vi)}_n + \beta_{4,2}\phi^{(vi)}_{n-1}), \tag{3.8a}$$

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1} = h(\beta'_{1,0}\phi_{n+1} + \beta'_{1,1}\phi_n + \beta'_{1,2}\phi_{n-1})$$
$$+ h^3(\beta'_{2,0}\phi''_{n+1} + \beta'_{2,1}\phi''_n + \beta'_{2,2}\phi''_{n-1})$$
$$+ h^5(\beta'_{3,0}\phi^{(iv)}_{n+1} + \beta'_{3,1}\phi^{(iv)}_n + \beta'_{3,2}\phi^{(iv)}_{n-1})$$
$$+ h^7(\beta'_{4,0}\phi^{(vi)}_{n+1} + \beta'_{4,1}\phi^{(vi)}_n + \beta'_{4,2}\phi^{(vi)}_{n-1}), \tag{3.8b}$$

where

$$\beta_{4,0} = \frac{1}{40\,320q^2h^3}[qh(1+e^{-ph}) - 4(1-e^{-ph})(2p-qh)$$
$$+ \beta_{1,0}(56p^2h + 336p - 28pqh^2 - 5q^2h^3 - 224qh)$$
$$+ \beta_{1,2}(56p^2h - 336p - 28pqh^2 + 3q^2h^3 + 112qh)$$

$$+ \beta_{2,0}(1680p^2h + 6720p - 840pqh^2 - 224q^2h^3 - 5040qh)$$
$$+ \beta_{2,2}(1680p^2h - 6720p - 840pqh^2 + 112q^2h^3 + 1680qh)$$
$$+ \beta_{3,0}(20\,160p^2h + 40\,320p - 10\,080pqh^2 - 5040q^2h^3 - 40\,320qh)$$
$$+ \beta_{3,2}(20\,160p^2h - 40\,320p - 10\,080pqh^2 + 1680q^2h^3)], \tag{3.9a}$$

$$\beta_{4,1} = \frac{2}{40\,320q^2h^3}[8p(1 - \mathrm{e}^{-ph}) - qh(1 + \mathrm{e}^{-ph})$$
$$- \beta_{1,0}(56p^2h + 336p - q^2h^3 - 56qh)$$
$$- \beta_{1,2}(56p^2h - 336p - q^2h^3 - 56qh)$$
$$- \beta_{2,0}(1680p^2h + 6720p - 56q^2h^3 - 1680qh)$$
$$- \beta_{2,2}(1680p^2h - 6720p - 56q^2h^3 - 1680qh)$$
$$- \beta_{3,0}(20\,160p^2h + 40\,320p - 1680q^2h^3 - 20\,160qh)$$
$$- \beta_{3,2}(20\,160p^2h - 40\,320p - 1680q^2h^3 - 20\,160qh)], \tag{3.9b}$$

$$\beta_{4,2} = \frac{1}{40\,320q^2h^3}[qh(1 + \mathrm{e}^{-ph}) - 4(1 - \mathrm{e}^{-ph})(2p + qh)$$
$$+ \beta_{1,0}(56p^2h + 336p + 28pqh^2 + 3q^2h^3 + 112qh)$$
$$+ \beta_{1,2}(56p^2h - 336p + 28pqh^2 - 5q^2h^3 - 224qh)$$
$$+ \beta_{2,0}(1680p^2h + 6720p + 840pqh^2 + 112q^2h^3 + 1680qh)$$
$$+ \beta_{2,2}(1680p^2h - 6720p + 840pqh^2 - 224q^2h^3 - 5040qh)$$
$$+ \beta_{3,0}(20\,160p^2h + 40\,320p + 10\,080pqh^2 + 1680q^2h^3)$$
$$+ \beta_{3,2}(20\,160p^2h - 40\,320p + 10\,080pqh^2 - 5040q^2h^3 - 40\,320qh)] \tag{3.9c}$$

and

$$\beta'_{4,0} = \frac{1}{40\,320q^2h^3}[8qh(1 - \mathrm{e}^{-ph}) - 28(1 + \mathrm{e}^{-ph})(2p - qh)$$
$$+ \beta'_{1,0}(56p^2h + 336p - 28pqh^2 - 5q^2h^3 - 224qh)$$
$$+ \beta'_{1,2}(56p^2h - 336p - 28pqh^2 + 3q^2h^3 + 112qh)$$
$$+ \beta'_{2,0}(1680p^2h + 6720p - 840pqh^2 - 224q^2h^3 - 5040qh)$$
$$+ \beta'_{2,2}(1680p^2h - 6720p - 840pqh^2 + 112q^2h^3 + 1680qh)$$
$$+ \beta'_{3,0}(20\,160p^2h + 40\,320p - 10\,080pqh^2 - 5040q^2h^3 - 40\,320qh)$$
$$+ \beta'_{3,2}(20\,160p^2h - 40\,320p - 10\,080pqh^2 + 1680q^2h^3)], \tag{3.10a}$$

$$\begin{aligned}\beta'_{4,1} = \frac{2}{40\,320q^2h^3} [&56p(1+e^{-ph}) - 8qh(1+e^{-ph})\\ &- \beta'_{1,0}(56p^2h + 336p - q^2h^3 - 56qh)\\ &- \beta'_{1,2}(56p^2h - 336p - q^2h^3 - 56qh)\\ &- \beta'_{2,0}(1680p^2h + 6720p - 56q^2h^3 - 1680qh)\\ &- \beta'_{2,2}(1680p^2h - 6720p - 56q^2h^3 - 1680qh)\\ &- \beta'_{3,0}(20\,160p^2h + 40\,320p - 1680q^2h^3 - 20\,160qh)\\ &- \beta'_{3,2}(20\,160p^2h - 40\,320p - 1680q^2h^3 - 20\,160qh)],\end{aligned} \tag{3.10b}$$

$$\begin{aligned}\beta'_{4,2} = \frac{1}{40\,320q^2h^3} [&8qh(1-e^{-ph}) - 28(1+e^{-ph})(2p+qh)\\ &+ \beta'_{1,0}(56p^2h + 336p + 28pqh^2 + 3q^2h^3 + 112qh)\\ &+ \beta'_{1,2}(56p^2h - 336p + 28pqh^2 - 5q^2h^3 - 224qh)\\ &+ \beta'_{2,0}(1680p^2h + 6720p + 840pqh^2 + 112q^2h^3 + 1680qh)\\ &+ \beta'_{2,2}(1680p^2h - 6720p + 840pqh^2 - 224q^2h^3 - 5040qh)\\ &+ \beta'_{3,0}(20\,160p^2h + 40\,320p + 10\,080pqh^2 + 1680q^2h^3)\\ &+ \beta'_{3,2}(20\,160p^2h - 40\,320p + 10\,080pqh^2 - 5040q^2h^3 - 40\,320qh)].\end{aligned} \tag{3.10c}$$

To find the coefficients in the methods (3.2), (3.5) and (3.8), we expand the methods using Taylor's series at the point $t = t_n$ and equating the like powers of h on both the sides. From the system of linear equations thus obtained, we get the coefficients in the above methods.

4. Truncation error and order of the methods

The order of the method (3.1) is m, if the truncation errors of the associated linear difference operator L and L' are defined as

$$L[y(t_n), h] = C_{m+1}h^{m+1}y^{(m+1)}(t_n) + O(h^{m+2})$$

and

$$L'[y(t_n), h] = C'_{m+1}h^{m}y^{(m+1)}(t_n) + O(h^{m+1})$$

where C_{m+1}, C'_{m+1} are constants independent of h.

Expand the method (3.2) using the Taylor's series and we can find the truncation errors are as follows:

$$\mathrm{TE}_1 = \frac{h^5}{5!}[(1 - e^{-ph}) - [5ph(\beta_{1,0} + \beta_{1,2}) + (qh^2 + 20)(\beta_{1,0} - \beta_{1,2})] - [60ph(\beta_{2,0} + \beta_{2,2}) + (20qh^2 + 120)(\beta_{2,0} - \beta_{2,2})]]y_n^{(v)} + O(h^6), \tag{4.1a}$$

$$h\mathrm{TE}'_1 = \frac{h^5}{5!}[5(1 + e^{-ph}) - [5ph(\beta'_{1,0} + \beta'_{1,2}) + (qh^2 + 20)(\beta'_{1,0} - \beta'_{1,2})] - [60ph(\beta'_{2,0} + \beta'_{2,2}) + (20qh^2 + 120)(\beta'_{2,0} - \beta'_{2,2})]]y_n^{(v)} + O(h^6). \tag{4.1b}$$

which is of $O(h^4)$.

Similarly, we get the truncation errors in the method (3.5) as follows:

$$\mathrm{TE}_2 = \frac{h^7}{7!}[(1 - e^{-ph}) - [7ph(\beta_{1,0} + \beta_{1,2}) + (qh^2 + 42)(\beta_{1,0} - \beta_{1,2})] - [210ph(\beta_{2,0} + \beta_{2,2}) + (42qh^2 + 840)(\beta_{2,0} - \beta_{2,2})] - [2520ph(\beta_{3,0} + \beta_{3,2}) + (840qh^2 + 5040)(\beta_{3,0} - \beta_{3,2})]]y_n^{(vii)} + O(h^8), \tag{4.2a}$$

$$h\mathrm{TE}'_2 = \frac{h^7}{7!}[7(1 + e^{-ph}) - [7ph(\beta'_{1,0} + \beta'_{1,2}) + (qh^2 + 42)(\beta'_{1,0} - \beta'_{1,2})] - [210ph(\beta'_{2,0} + \beta'_{2,2}) + (42qh^2 + 840)(\beta'_{2,0} - \beta'_{2,2})] - [2520ph(\beta'_{3,0} + \beta'_{3,2}) + (840qh^2 + 5040)(\beta'_{3,0} - \beta'_{3,2})]]y_n^{(vii)} + O(h^8), \tag{4.2b}$$

which is of $O(h^6)$.

In the identical way as before, we obtain the truncation errors in the method (3.8) are as follows

$$\mathrm{TE}_3 = \frac{h^9}{9!}[(1 - e^{-ph}) - [9ph(\beta_{1,0} + \beta_{1,2}) + (qh^2 + 72)(\beta_{1,0} - \beta_{1,2})] - [504ph(\beta_{2,0} + \beta_{2,2}) + (72qh^2 + 3024)(\beta_{2,0} - \beta_{2,2})] - [15\,120ph(\beta_{3,0} + \beta_{3,2}) + (3024qh^2 + 60\,480)(\beta_{3,0} - \beta_{3,2})] - [181\,440ph(\beta_{4,0} + \beta_{4,2}) + (60\,480qh^2 + 362\,880)(\beta_{4,0} - \beta_{4,2})]]y_n^{(ix)} + O(h^{10}), \tag{4.3a}$$

$$h\mathrm{TE}'_3 = \frac{h^9}{9!}[9(1 + e^{-ph}) - [9ph(\beta'_{1,0} + \beta'_{1,2}) + (qh^2 + 72)(\beta'_{1,0} - \beta'_{1,2})] - [504ph(\beta'_{2,0} + \beta'_{2,2}) + (72qh^2 + 3024)(\beta'_{2,0} - \beta'_{2,2})] - [15\,120ph(\beta'_{3,0} + \beta'_{3,2}) + (3024qh^2 + 60\,480)(\beta'_{3,0} - \beta'_{3,2})] - [181\,440ph(\beta'_{4,0} + \beta'_{4,2}) + (60\,480qh^2 + 362\,880)(\beta'_{4,0} - \beta'_{4,2})]]y_n^{(ix)} + O(h^{10}). \tag{4.3b}$$

which is of $O(h^8)$.

Note: In particular, when $p = 0$ and $q = (2\pi n/h)^2$, $n = 1, 2, 3, \ldots$, the truncation errors in the methods (3.2), (3.5) and (3.8) are as follows:

$$\mathrm{TE}_4 = \frac{2401}{3\,748\,096 n^4} h^6 y^6(t_n) + O(h^8), \tag{4.4}$$

$$\mathrm{TE}_5 = -\frac{117\,649}{7\,256\,313\,856 n^6} h^8 y^8(t_n) + O(h^{10}), \tag{4.5}$$

$$\mathrm{TE}_6 = \frac{5\,764\,801}{14\,048\,223\,625\,216 n^8} h^{10} y^{10}(t_n) + O(h^{12}), \tag{4.6}$$

which are of $O(h^4), O(h^6)$ and $O(h^8)$, respectively.

5. Stability

Applying the above derived methods (3.2), (3.5) and (3.8) to the test equation (1.2), we obtain

$$y_{n+1} - \{\exp(\sigma_1 h) + \exp(\sigma_2 h)\} y_n + \exp(-ph) y_{n-1} = 0, \tag{5.1}$$

where λ and μ are chosen to be equal to p and q, respectively.

Definition. The linear multistep method is said to be *absolutely stable* if the roots of the characteristic equation are in moduli less than one for all values of the step length h.

The characteristic equation of the recurrence equation (5.1) is

$$\xi^2 - \{\exp(\sigma_1 h) + \exp(\sigma_2 h)\} \xi + \exp(-ph) = 0 \tag{5.2}$$

so its roots are $\exp(\sigma_1 h)$ and $\exp(\sigma_2 h)$. Hence, their moduli are equal to $\exp(-ph/2)$. Therefore, our method is absolutely stable for all $p > 0$.

If $p = 0$, the methods reduce to P-stable for periodic initial value problems of the form (1.3) (see [8]).

6. Numerical results

Numerical results are presented for the following initial value problems of Bessel's and general second-order nonlinear differential equations to illustrate the order, accuracy and implementational aspects of the methods (3.2), (3.5) and (3.8).

Problem 1. Consider the Bessel's differential equation

$$t^2 y'' + t y' + (t^2 - 0.25) y = 0 \tag{6.1}$$

with $y(1) = 0.6714$, $y'(1) = 0.0954$.

It is well known that $y(t) = J_{1/2}(t) = \sqrt{(2/\pi t)} \sin t$ is the exact solution of (6.1).

Problem 2. We consider the linear second-order differential equation

$$(1+t)y'' + 2y' - (1+t)y = 0, \quad y(0) = 1, \quad y'(0) = 0 \tag{6.2}$$

with exact solution $y(t) = e^{-t}/(1+t)$.

Problem 3. We consider the nonlinear second-order differential equation

$$y'' + 2yy' - (1-y)y = 0, \quad y(0) = \tfrac{1}{2},\ y'(0) = \tfrac{1}{4} \tag{6.3}$$

with exact solution $y(t) = 1/(1+e^{-t})$.

The above problems are solved using the methods (3.2), (3.5) and (3.8), which are of $O(h^4)$, $O(h^6)$ and $O(h^8)$, respectively. We take different arbitrary parameters of p and q (see Tables 1–3). The numerical solutions are compared with exact solutions and the absolute error values $e = |y_n - y(t_n)|$ are found for $t = 1$ to 10 and the values e at $t = 8$ are presented in Tables 1–3. One can see from the tables that they are absolutely stable and these errors are of $O(h^4)$, $O(h^6)$, $O(h^8)$, respectively.

The problem (6.3) is also solved by method (3.2) of $O(h^4)$ with three different pairs of the values of p and q. The absolute errors of the numerical solutions at $t = 8$ are presented in Table 4. From Table 4, we can clearly see that, as q becomes very large and p tends to zero we get more accurate results, since the truncation error decreases.

Table 1
Numerical results using the method (3.2)

h	Errors at $t = 8.0$, when $p = 0.1$ and $q = (20\pi/h)^2$		
	Problem (6.1) $O(h^4)$	Problem (6.2) $O(h^4)$	Problem (6.3) $O(h^4)$
2^{-1}	0.66695915(−06)	0.46404758(−05)	0.46393722(−08)
2^{-2}	0.44310303(−07)	0.36343158(−06)	0.25105829(−09)
2^{-3}	0.27795443(−08)	0.25523150(−07)	0.14697021(−10)

Table 2
Numerical results using the method (3.5)

h	Errors at $t = 8.0$, when $p = 0.1$ and $q = (20\pi/h)^2$		
	Problem (6.1) $O(h^6)$	Problem (6.2) $O(h^6)$	Problem (6.3) $O(h^6)$
2^{-1}	0.25526375(−09)	0.16996989(−06)	0.32985614(−09)
2^{-2}	0.38778980(−11)	0.53095164(−08)	0.43316462(−11)
2^{-3}	0.58175686(−13)	0.17769253(−09)	0.64170891(−13)

Table 3
Numerical results using the method (3.8)

h	Errors at $t = 8.0$, when $p = 0.1$ and $q = (10\pi/h)^2$		
	Problem (6.1) $O(h^8)$	Problem (6.2) $O(h^8)$	Problem (6.3) $O(h^8)$
2^{-1}	0.20435875(−10)	0.16778472(−08)	0.30300873(−10)
2^{-2}	0.76605389(−13)	0.82991392(−11)	0.10957901(−12)
2^{-3}	0.33306691(−15)	0.56843419(−13)	0.44408921(−15)

Table 4
Numerical results using the method (3.2) for different values of p and q

h	Errors at $t = 8.0$ for the problem (6.3)		
	$p = 0.1$ and $q = (10\pi/h)^2$ $O(h^4)$	$p = 0.1$ and $q = (100\pi/h)^2$ $O(h^4)$	$p = 0.1$ and $q = (1000\pi/h)^2$ $O(h^4)$
2^{-1}	0.41243749(−07)	0.41233861(−09)	0.41228132(−11)
2^{-2}	0.21014416(−08)	0.21678548(−10)	0.21638247(−12)
2^{-3}	0.11713330(−09)	0.12501111(−11)	0.12545520(−13)

Note: A number listed in Table 1–4 as $a(-b)$ means $a \cdot 10^{-b}$.

In solving the problems, the initial approximations $y_{n+1}^{(0)}$ and $y_{n+1}^{\prime(0)}$ are obtained from the exact solution. These values are used in finding the approximate value to y_{n+1} from the specified methods (i.e. (3.2), (3.5), (3.8)) and they are used to find the next iterates to y_{n+1}. The successive Picard's iteration process is carried out with an error tolerance $e = 1 \times 10^{-10}$ to find the numerical solution at each step.

We can also apply the methods to the special second-order initial value problem (1.3) by assuming the coefficient of y' is zero (i.e., $p = 0$).

Problem 4. Consider the nonlinear equation (see [2])

$$y'' = 50y^3, \quad y(1) = \tfrac{1}{6},\ y'(1) = -\tfrac{5}{36}. \tag{6.4}$$

Its exact solution is $y(t) = 1/(1 + 5t)$.

The problem is solved using the method (3.2) of $O(h^4)$ by taking $p = 0$ and $q = (100\pi/h)^2$. The absolute errors of the numerical solutions at $t = 10$ with $h = 0.5$ are presented in Table 5.

We have also compared the results obtained by the method (3.2) to the Numerov method (1.4) and the P-stable method of $O(h^4)$:

$$y_{n+1} - 2y_n + y_{n+1} = \tfrac{1}{12}h^2(y''_{n+1} + 10y''_n + y''_{n-1}) - \tfrac{1}{144}h^4(y^{\mathrm{iv}}_{n+1} - 2y^{\mathrm{iv}}_n + y^{\mathrm{iv}}_{n-1}) \tag{6.5}$$

Table 5
Absolute and relative errors for the problem (6.4) at $t = 10$

Methods	Absolute error $h = 0.5$	Relative error $h = 0.5$
Method (3.2)	5.17(−14)	2.62(−12)
E-stable (1.6)	3.21(−04)	1.63(−02)
Hairer (6.5)	3.47(−02)	1.77(+00)
Numerov (1.4)	7.80(−02)	3.98(+00)

proposed by Hairer [6] with minimum truncation error are also presented in Table 5 for comparison.

The truncation error entirely depends on q, We can decrease the truncation error by increasing the value of n. This is a new approach in the truncation error.

7. Conclusions

The numerical results presented for linear and nonlinear problems show that the methods are absolutely stable when the parameters p and q are selected in such a way that $0 < p < 2\sqrt{q}$. From Table 4, it is observed that, as q becomes very large the truncation error becomes smaller and we get more accurate results. From Table 5, one can see that, the fourth-order Obrechkoff method gives better results than the other fourth-order Numerov, P-stable Hairer and E-stable methods.

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