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On a Two-Step Kurchatov-Type Method in Banach Space

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Abstract. We present the semi-local convergence analysis of a two-step Kurchatov-type method to solve equations involving Banach space valued operators. The analysis is based on our ideas of recurrent functions and restricted convergence region. The study is completed using numerical examples.

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1. Introduction

Many mathematical equations are in the following form (or get reduced to):

$$F(x) = 0, \tag{1.1}$$

where $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ is a Fréchet-differentiable operator, $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach spaces, and D is a nonempty open convex subset of $\mathcal{B}_1$. Various iterative schemes are used to approximate the solution x^* of (1.1) [1–23]. Many of these methods are firmly based on various calculus and functional analysis concepts, and they can be effectively implemented by taking the advantage of the speed and power of modern computer technologies.

The study of convergence of iterative algorithms is usually centered into two categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find the estimates of the computed radii of the convergence balls.

In this study, we present the semi-local convergence of the two-step Kurchatov-type methods defined for each $x_0, y_0 \in D$ by the following:

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1} F(x_n) \\ y_{n+1} &= x_{n+1} - A_n^{-1} F(x_{n+1}), \end{aligned} \tag{1.2}$$

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where x_0 is an initial point, $A_n = [2y_n - x_n, x_n; F]$ and $[.,.; F] : D \times D \longrightarrow L(\mathcal{B}_1, \mathcal{B}_2)$ is finite difference of order one.

We find computable radii of convergence as well as error bounds on the distances based on Lipschitz-type conditions. The order of convergence is found using computable order of convergence (COC) or approximate computational order of convergence (ACOC) [3,23] (see Remark 2.3) that do not require usage of higher order derivatives.

The rest of the study is organized as follows: Sect. 2 contains the semi-local convergence of method (1.2), where, in the concluding Sect. 3, applications and numerical examples can be found.

2. Semi-local Convergence

We present the semi-local convergence of method (1.2) in this section. First, we need an auxiliary result on majorizing sequences for method (1.2).

Lemma 2.1. *Let $L_0 > 0, L > 0, s_0 \geq 0$ and $t_1 \geq 0$ be given parameters. Denote by α the only solution of equation:*

$$p(t) = 0, \text{ in } (0,1), \tag{2.1}$$

where $p(t) = 2L_0 t^3 + 3Lt - 3L$. Suppose that

$$0 < \frac{L(t_1 + 2s_0)}{1 - 2L_0 s_1} \leq \alpha \leq 1 - 2L_0 t_1, \tag{2.2}$$

where $s_1 = t_1 + L(t_1 + 2s_0)t_1$. Then, scalar sequence $\{t_n\}$ defined by

$$\begin{aligned} t_0 = 0, \ s_{n+1} &= t_{n+1} + \frac{L(t_{n+1} - t_n + 2(s_n - t_n))(t_{n+1} - t_n)}{1 - 2L_0 s_n} \\ t_{n+2} &= t_{n+1} + \frac{L(t_{n+1} - t_n + 2(s_n - t_n))(t_{n+1} - t_n)}{1 - 2L_0 s_{n+1}} \end{aligned} \tag{2.3}$$

is well-defined nondecreasing, bounded from above by

$$t^{**} = \frac{t_1}{1 - \alpha} \tag{2.4}$$

and converges to its unique least upper bound t^* which satisfies

$$t_n \leq t^* \leq t^{**}. \tag{2.5}$$

Moreover, the following estimates hold

$$0 < s_n - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n (t_1 - t_0) \tag{2.6}$$

and

$$0 < t_{n+1} - t_n \leq \alpha(t_n - t_{n-1}) \leq \alpha^n (t_1 - t_0). \tag{2.7}$$

Proof. Notice that $p(0) = -3L < 0$ and $p(1) = 2L > 0$. It follows from the intermediate value theorem that $p(t)$ has at least one solution in $(0,1)$. Denote the only solution by α (since p' is increasing, p crosses the $x-$axis only once). If $t_1 = 0$, we have from (2.3) that $t_n = s_n = 0$ for each $n = 1, 2, \ldots$

and (2.6) and (2.7) are true for each $n = 1, 2, \ldots$. In what follows we suppose that $t_1 > 0$. It follows from (2.3) that (2.6) and (2.7) are true, if

$$0 < \frac{L(t_{n+1} - t_n + 2(s_n - t_n))}{1 - 2L_0 s_n} \leq \alpha, \tag{2.8}$$

$$0 < \frac{L(t_{n+1} - t_n + 2(s_n - t_n))}{1 - 2L_0 s_{n+1}} \leq \alpha \tag{2.9}$$

and

$$t_k \leq s_k. \tag{2.10}$$

Estimates (2.8)–(2.10) are true for $k = 0$ by $s \geq 0$ and the left-hand side inequality in (2.2). Suppose that they are true for all the values of $k = 0, 1, 2, \ldots, n$. We have by the induction hypotheses and (2.3) that

$$\begin{aligned} s_k &\leq t_k + \alpha^k (t_1 - t_0) \\ &\leq t_{k-1} + \alpha^{k-1}(t_1 - t_0) + \alpha^k (t_1 - t_0) \\ &\vdots \\ &\leq t_1 + \alpha(t_1 - t_0) + \cdots + \alpha^k (t_1 - t_0) \\ &= \frac{1 - \alpha^{k+1}}{1 - \alpha}(t_1 - t_0) < t^{**}, \end{aligned} \tag{2.11}$$

$$t_{n+1} \leq t_k + \alpha^k (t_1 - t_0) \leq \cdots \leq \frac{1 - \alpha^{k+1}}{1 - \alpha}(t_1 - t_0) < t^{**} \tag{2.12}$$

and

$$\frac{1}{1 - 2L_0 s_k} \leq \frac{1}{1 - 2L_0 s_{k+1}}. \tag{2.13}$$

Hence, instead of showing (2.8) and (2.9), we must show only (2.9). Evidently, (2.9) is true, if

$$0 < \frac{3L\alpha^k (t_1 - t_0)}{1 - 2L_0 \left[\frac{1-\alpha^{k+2}}{1-\alpha}(t_1 - t_0)\right]} \leq \alpha \tag{2.14}$$

or

$$3L\alpha^k (t_1 - t_0) + 2L_0 \alpha \left[\frac{1 - \alpha^{k+2}}{1 - \alpha}(t_1 - t_0)\right] - \alpha \leq 0. \tag{2.15}$$

Estimate (2.15) motivates us to define recurrent functions f_k on $[0, 1)$ by

$$f_k(t) = 3Lt^{k-1}(t_1 - t_0) + 2L_0 \frac{1 - t^{k+2}}{1 - t}(t_1 - t_0) - 1. \tag{2.16}$$

We need a relationship between two consecutive functions f_k :

$$f_{k+1}(t) = f_k(t) + p(t)t^{k-1}(t_1 - t_0). \tag{2.17}$$

In particular, we have that

$$f_{k+1}(\alpha) = f_k(\alpha). \tag{2.18}$$

Define function f_∞ on $(0, 1)$ by

$$f_\infty(t) = \lim_{k \longrightarrow \infty} f_k(t). \tag{2.19}$$

Then, by (2.19) and (2.16), we get that

$$f_\infty(t) = \frac{2L_0 t_1}{1-t} - 1. \tag{2.20}$$

In particular, we get by (2.2) that

$$f_\infty(\alpha) \le 0. \tag{2.21}$$

We also have that

$$f_\infty(\alpha) = \lim_{k \longrightarrow \infty} f_k(\alpha). \tag{2.22}$$

Then, (2.15) is true by (2.22) and (2.21). The induction is completed. It follows that sequence $\{t_n\}$ is increasing and bounded above by t^{**}, and as such, it converges to t^* satisfying (2.5). □

Denote by $U(w,\xi), \bar{U}(w,\xi)$, the open and closed balls in $\mathcal{B}_1$, respectively, with center $w \in \mathcal{B}_1$ and of radius $\xi > 0$.

Next, we present the local convergence analysis of method (1.2) using $\{t_n\}$ as a majorizing sequence.

Theorem 2.2. *Let $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exists a divided difference $[.,.;F]$ of order one for operator F on $D \times D$. Moreover, suppose that these exist $x_0, y_0, 2y_0 - x_0 \in D, L_0 > 0, L > 0, s_0 \ge 0, t_1 \ge 0$, such that, for each $x, y \in D$:*

$$A_0^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1) \tag{2.23}$$

$$\|A_0^{-1}F(x_0)\| \le t_1, \tag{2.24}$$

$$\|x_0 - y_0\| \le s_0, \tag{2.25}$$

$$\|A_0^{-1}([x,y;F] - A_0)\| \le L_0(\|x - 2y_0 + x_0\| + \|y - x_0\|). \tag{2.26}$$

Let $D_0 = D \cap U(x_0, \frac{1}{2L_0})$.

$$\|A_0^{-1}([x,y;F] - [z,y;F])\| \le L\|x - z\| \text{ for each } x,y,z \in D_0, \tag{2.27}$$

$$\bar{U}(x_0, 3t^*) \subseteq D, \tag{2.28}$$

and hypotheses of Lemma 2.1 hold, where t^* is given in Lemma 2.1. Then, the sequence $\{x_n\}$ generated by method (1.2) is well defined, remains in $\bar{U}(x_0, 3t^*)$, and converges to a solution $x^* \in \bar{U}(x_0, 3t^*)$ of equation $F(x) = 0$. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$:

$$\|x_n - x^*\| \le t^* - t_n. \tag{2.29}$$

Furthermore, if, for $R \ge t^*$:

$$L_0(3t^* + R + 2s_0) < 1, \tag{2.30}$$

then the point x^* is the only solution of equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x_0, R)$.

Proof. We shall show using induction on k that

$$\|x_{k+1} - x_k\| \le t_{k+1} - t_k \tag{2.31}$$

and

$$\|x_{k+1} - y_{k+1}\| \le s_{k+1} - t_{k+1}. \tag{2.32}$$

Using (2.26), we have

$$\begin{aligned}\|A_0^{-1}(A_0-A_1)\| &\le L_0(\|2y_1-x_1-2y_0+x_0\|+\|x_1-x_0\|)\\ &\le L_0(\|y_1-x_1\|+\|y_1-y_0\|+\|y_0-x_0\|+\|x_1-x_0\|)\\ &\le L_0(s_1-t_1+s_1-s_0+s_0+t_1-t_0)\\ &\le 2L_0s_1. \qquad (2.33)\end{aligned}$$

It follows from the Banach lemma on invertible operators [1,13] that A_1^{-1} exists and

$$\|A_1^{-1}A_0\| \le \frac{1}{1-2L_0s_1}. \qquad (2.34)$$

Then, by (1.2), (2.4), and (2.34), we get that

$$\|x_2-x_1\| \le \|A_1^{-1}A_0\|\|A_0^{-1}F(x_1)\| \le \frac{L(t_1+2s_0)(t_1-t_0)}{1-2L_0s_1} = t_2-t_1, \qquad (2.35)$$

and

$$\|x_2-x_1\| \le \|x_2-x_1\|+\|x_1-x_0\| \le t_2-t_1+t_1-t_0 = t_2 \le t^*,$$

so $x_2 \in \bar{U}(x_0,3t^*)$ and (2.31) holds for $k=1$.

We also have from (1.2), (2.4), (2.26), and the estimate

$$F(x_2)=F(x_2)-F(x_1)-A_1(x_2-x_1)=([x_2,x_1;F]-[2y_1-x_1,x_1;F])(x_2-x_1) \qquad (2.36)$$

that

$$\|A_0^{-1}F(x_2)\| \le L(\|x_2-x_1\|+2\|y_1-x_1\|)\|x_2-x_1\| \le L(t_2-t_1+2(s_1-t_1)(t_2-t_1), \qquad (2.37)$$

so

$$\begin{aligned}\|y_2-x_2\| &\le \|A_1^{-1}A_0\|\|A_0^{-1}F(x_2)\| \le \frac{L(t_2-t_1+2(s_1-t_1))(t_2-t_1)}{1-2L_0s_1} = s_2-t_2\\ \|y_2-x_0\| &\le \|y_2-x_2\|+\|x_2-x_0\| \le s_2-t_2+t_2-t_0 = s_2 \le t^* \qquad (2.38)\end{aligned}$$

and

$$\|2y_2-x_1-x_0\| \le \|2(y_2-x_0)-(x_1-x_0)\| \le 2\|y_2-x_0\|+\|x_1-x_0\| \le 3t^*,$$

which show (2.32) and $2y_2-x_1, y_2 \in \bar{U}(x_0,3t^*)$. By simply replacing y_0, x_1, y_1, x_2, y_2 by $y_k, x_{k+1}, y_{k+1}, x_{k+2}, y_{k+2}$, we complete the induction for (2.31). Hence, $\{x_k\}$ converges to some $x^* \in \bar{U}(x_0,3t^*)$. In view of the estimate (2.32) and (2.33), we have

$$\|A_0^{-1}F(x_{k+1})\| \le L(t_{k+1}-t_k+2(s_k-t_k))(t_{k+1}-t_k), \qquad (2.39)$$

and by letting $k \longrightarrow \infty$ in (2.39), we obtain $F(x^*)=0$. Estimate (2.29) follows from the standard majorizing techniques [1–3,13].

To complete the proof, we show the uniqueness of the solution in $\bar{U}(x_0,R)$. Let $y^* \in \bar{U}(x_0,R)$ be such that $F(y^*)=0$. By (2.26), we have in turn that

$$\begin{aligned}\|A_0^{-1}([x^*, y^*; F] - A_0)\| &\le L_0(\|x^* - 2y_0 + x_0\| + \|y^* - y_0\|)\\ &\le L_0(\|x^* - x_0\| + \|y^* - x_0\| + 2\|x_0 - y_0\|)\\ &\le L_0(3t^* + R + 2s_0)\\ &= L_0(3t^* + R + 2s_0) < 1. \end{aligned} \tag{2.40}$$

It follows that $[x^*, y^*; F]^{-1}$ exists. Then, from the identity

$$[x^*, y^*; F](x^* - y^*) = F(x^*) - F(y^*) = 0,$$

we obtain that $x^* = y^*$. □

Remark 2.3.
- The limit point t^* can be replaced by t^{**} given in closed form by (2.4) in Lemma 2.1.
- Condition (2.27) can be replaced by the stronger but popular hypothesis for the study of secant-type methods:

$$\|A_n^{-1}([x, y; F] - [z, w; F])\| \le M(\|x - z\| + \|y - w\|). \tag{2.41}$$

  From (2.36) and (2.41), we have

$$L_0 \le M$$

  hold in general and $\frac{M}{L_0}$ can be arbitrarily large [1–3].
- It follows from the proof of Lemma 2.1 that some other convergence criteria can be introduced. Define quadratic polynomial p_1 on $(0, 1)$ by $p_1(t) = 2L_0t^2 + 3Lt - 3L$ and parameter α_0 by $\alpha_0 = \frac{6L}{3L+\sqrt{9L^2+24L_0L}}$. In view of (2.17), we have that

$$f_{k+1}(\alpha) \le f_k(\alpha) \le \cdots \le f_1(\alpha),$$

  since $p(t) \le p_1(t)$ for each $t \in [0, 1]$. Therefore, the proof of Lemma 2.1 goes through, if $f_(\alpha) \le 0$. Then, the convergence criteria are:

$$0 < \frac{L(t_1 + 2s_0)}{1 - 2L_0s_1} \le \alpha_0 \tag{2.42}$$

  and

$$3L + 2L_0(1 + \alpha_0 + \alpha_0^2))t_1 \le 1$$

  instead of (2.2).
- It is worth noticing that a similar value of $\bar{\alpha} = \frac{2L}{L+\sqrt{L^2+8L_0L}}$ also appears in our studies of Newton's method under Lipschitz and center Lipschitz conditions [1–3].

So far, we presented results based on divided difference of order one. Next, we present the corresponding results based on divided difference of order two.

Lemma 2.4. *Let $L_0 > 0, L_1 > 0, s_0 \ge 0, t_1 \ge 0$, and $K \ge 0$ be given parameters. Denote by β the smallest solution in $(0, 1)$ of the following equation:*

$$\bar{p}(t) = 2L_0t^3 + Kt_1t^2 + \left(\frac{L_1}{2} - Kt_1\right)t - \frac{L_1}{2} = 0.$$

Suppose that

$$0 < \frac{\frac{L_1}{2}t_1 + Ks_0^2}{1-2L_0s_1} \leq \beta \text{ and } \bar{f}_1(\beta) \leq 0, \tag{2.43}$$

where

$$s_1 = t_1 + \left(\frac{L_1}{2}t_1 + Ks_0^2\right)t_1, \tag{2.44}$$

and $\bar{f}_1$ *is defined below. Then, scalar sequence* $\{t_n\}$ *defined by*

$$t_0 = 0,\ s_{n+1} = t_{n+1} + \frac{(\frac{L_1}{2}(t_{n+1}-t_n) + K(s_n-t_n)^2)(t_{n+1}-t_n)}{1-2L_0s_n}$$

$$t_{n+2} = t_{n+1} + \frac{(\frac{L_1}{2}(t_{n+1}-t_n) + K(s_n-t_n)^2)(t_{n+1}-t_n)}{1-2L_0s_{n+1}} \tag{2.45}$$

is well defined, bounded from above by

$$t^{**} = \frac{t_1}{1-\beta} \tag{2.46}$$

and converges to its unique least upper bound t^* *which satisfies*

$$t_1 \leq t^* \leq t^{**}. \tag{2.47}$$

Moreover, the following estimates hold:

$$0 < s_n - t_n \leq \beta(t_n - t_{n-1}) \leq \beta^n(t_1 - t_0) \tag{2.48}$$

$$0 < t_{n+} - t_n \leq \beta(t_n - t_{n-1}) \leq \beta^n(t_1 - t_0). \tag{2.49}$$

Proof. The proof follows along the lines of Lemma 2.1. We must have this time β (2.43)–(2.49):

$$0 < \frac{\frac{L_1}{2}(t_{n+1}-t_n) + K(s_n-t_n)^2}{1-2L_0s_n} \leq \beta,$$

$$0 < \frac{\frac{L_1}{2}(t_{n+1}-t_n) + K(s_n-t_n)^2}{1-2L_0s_{n+1}} \leq \beta,$$

$$0 < \frac{\frac{L_1}{2}\beta^k t_1 + K(\beta^k t_1)^2}{1-2L_0\frac{1-\beta^{k+2}}{1-\beta}t_1} \leq \beta,$$

$$\frac{L_1}{2}\beta^k t_1 + K\beta^{k+1}(t_1-t_0)^2 + 2L_0\beta\frac{1-\beta^{k+2}}{1-\beta}t_1 - \beta \leq 0,$$

$$\bar{f}_k(t) = \frac{L_1}{2}\beta^{k-1}t_1 + K\beta^k t_1^2 + 2L_0\frac{1-\beta^{k+2}}{1-\beta}t_1 - 1,$$

$$\bar{f}_{k+1}(t) = f_k(t) + p(t)\beta^{k-1}t_1.$$

Another set of convergence criteria (as in Remark 2.3) is given by

$$0 < \frac{\frac{L_1}{2}t_1 + Ks_0^2}{1-2L_0s_1} < \beta_0 \leq 1 - 2L_0t_1, \tag{2.50}$$

where β_0 is given by

$$\beta_0 = \frac{L_1}{\frac{L_1}{2} - Kt_1 + \sqrt{(\frac{L_1}{2} - Kt_1)^2 + 2L(Kt_1 + 2L_0)}} \tag{2.51}$$

and for

$$\bar{p}_1(t) = (Kt_1 + 2L_0)t^2 + (\frac{L_1}{2} - Kt_1)t - \frac{L_1}{2},\ \bar{p}_1(\beta_0) = 0$$

replacing the corresponding items in Lemma 2.1. $\square$

Theorem 2.5. *Let $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exists a divided difference $[.,.;F], [.,.,.;F]$ of order one and two, respectively, for operator F on $D \times D$ and $D \times D \times D$, respectively. Moreover, suppose that these exist $x_0, y_0 \in D, L_0 > 0, L_1 > 0, s_0 \geq 0, t_1 \geq 0$ and $K \geq 0$, such that for each $x, y, 2y_0 - x_0 \in D$:*

$$\begin{aligned} A_0^{-1} &\in L(\mathcal{B}_2, \mathcal{B}_1) \\ \|A_0^{-1}F(x_0)\| &\leq t_1, \\ \|x_0 - y_0\| &\leq s_0, \\ \|A_0^{-1}([x, y; F] - A_0)\| &\leq L_0(\|x - 2y_0 + x_0\| + \|y - x_0\|), \quad (2.52) \\ \|A_0^{-1}(F'(x) - F'(y))\| &\leq L_1\|x - y\| \text{ for each } x, y \in D_0 \\ \|A_0^{-1}([u, x, y; F] - [z, x, y; F])\| &\leq K\|u - z\|, \text{ for each } x, y, z, u \in D_0 \\ \bar{U}(x_0, 3t^*) &\subseteq D, \quad (2.53) \end{aligned}$$

and hypotheses of Lemma 2.4 hold, where $t^*, \{t_n\}$ are given in Lemma 2.4. Then, the conclusions of Theorem 2.2 hold for method (1.2).

Proof. Simply use the proof of Theorem 2.2 and the approximation

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - [2y_k - x_k, x_k; F](x_{k+1} - x_k) \\ &= F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) \\ &\quad +(F'(x_k) - [2y_k - x_k, x_k; F])(x_{k+1} - x_k) \\ &= \left\{ \int_0^1 [F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)]\mathrm{d}\theta \right. \\ &\quad +[x_k, x_k; F] - [x_k, x_{k-1}; F] + [x_k, x_{k-1}; F] \\ &\quad \left. -[2y_k - x_k, x_k; F] \right\} (x_{k+1} - x_k) \\ &= \left\{ \int_0^1 [F'(x_k + \theta(x_{k+1} - x_k), y_k; F])(y_k - x_k) \right\} (x_{k+1} - x_k) \end{aligned} \quad (2.54)$$

instead of (2.36). Indeed this way, we obtain [using (2.52) and (2.53) instead of (2.27)] that

$$\begin{aligned} \|A_0^{-1}F(x_{k+1})\| &\leq \| \int_0^1 A_0^{-1}[F'(x_k + \theta(x_{k+1} - x_k)) - F'(x_k)](x_{k+1} - x_k)\mathrm{d}\theta\| \\ &\quad +\|A_0^{-1}([y_k, x_k, y_k; F] \quad (2.55) \\ &\quad -[2y_k - x_k, x_k, y_k; F])(y_k - x_k)(x_{k+1} - x_k)\| \\ &\leq \frac{L_1}{2}\|x_{k+1} - x_k\|^2 + K\|y_k - x_k\|^2\|x_{k+1} - x_k\| \\ &\leq \frac{L_1}{2}(t_{n+1} - t_k)^2 + K(s_k - t_k)^2(t_{k+1} - t_k), \quad (2.56) \end{aligned}$$

instead of (2.37). $\square$

3. Numerical Examples

We shall use the divided difference given by $[x, y; F] = \frac{1}{2}(F'(x) + F'(y))$ in both examples.

Example 3.1. Let $D = \bar{U}(x_0, 1-\gamma), x_0 = 1, y_0 = x_0 + 10^{-3}, \gamma \in [0, 1)$. Define function F on D by

$$F(x) = x^3 - \gamma.$$

We have that

$$[x, y; F] - [z, y; F] = \frac{3}{2}(x+z)(x-z)$$

and

$$[2y_0 - x_0, x_0; F] = \frac{3}{2}\left(2(y_0 - x_0)^2 + x_0^2\right),$$

so by (2.27) and (2.28), we get that, for $\gamma = 0.95$:

$$L_0 = \frac{2(2-\gamma)}{(2y_0 - x_0)^2 + x_0^2} = 1.0479$$

and

$$L = \frac{3(1-\gamma)+2}{(2y_0 - x_0)^2 + x_0^2} = 1.0729,$$

so that (2.1) and (2.2) are satisfied, so the results apply.

Example 3.2. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^3$, $\Omega_0 = \Omega = (-1, 1)^3$ and define $F = (F_1, F_2, F_3)^T$ on Ω by

$$F(x) = F(x_1, x_2, x_3) = (e^{x_1} - 1, {x_2}^2 + x_2, x_3)^T. \tag{3.1}$$

For the points $u = (u_1, u_2, u_3)^T, v = (v_1, v_2, v_3)^T \in \Omega$, we get

$$[u, v; F] = \begin{pmatrix} \frac{e^{u_1} - e^{v_1}}{u_1 - v_1} & 0 & 0 \\ 0 & u_2 + v_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\overline{y}_0 = (0.1, 0.1, 0.1)^T$, $\overline{x}_0 = (0.11, 0.11, 0.11)^T$ be two initial points for the Kurchatov method (1.2). Here, we use $\overline{x}_n$ instead of x_n to distinct iterative points with its component for some integer $n \geq -1$. Then, we have

$$2\overline{x}_0 - \overline{y}_0 = (0.12, 0.12, 0.12), \quad t_0 = s_0 = 0.01,$$

$$A_0 \approx \begin{pmatrix} 1.116296675 & 0 & 0 \\ 0 & 1.22 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_0^{-1} \approx \begin{pmatrix} 0.895819205 & 0 & 0 \\ 0 & 0.819672131 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$t_1 = t_0 + \|A_0^{-1} F(\overline{x}_0)\| = 0.12, \quad \overline{x}_1 \approx (0.005835871, 0.009918033, 0).$$

Note that, for any $x=(x_1,x_2,x_3)^T, y=(y_1,y_2,y_3)^T, z=(z_1,z_2,z_3)^T$, $v=(v_1,v_2,v_3)^T \in \Omega$, we have

$$[x,y;F]-[z,v;F]=\begin{pmatrix} \frac{e^{x_1}-e^{y_1}}{x_1-y_1}-\frac{e^{z_1}-e^{v_1}}{z_1-v_1} & 0 & 0 \\ 0 & x_2+y_2-z_2-v_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

In view of

$$\begin{aligned} &|\tfrac{e^{x_1}-e^{y_1}}{x_1-y_1}-\tfrac{e^{z_1}-e^{v_1}}{z_1-v_1}| = |\int_0^1 (e^{y_1+t(x_1-y_1)}-e^{v_1+t(z_1-v_1)})\mathrm{d}t| \\ &\quad = |\int_0^1\int_0^1 e^{v_1+t(z_1-v_1)+\theta(y_1+t(x_1-y_1)-v_1-t(z_1-v_1))} \\ &\qquad \big(y_1+t(x_1-y_1)-v_1-t(z_1-v_1)\big)\mathrm{d}\theta\mathrm{d}t| \\ &\quad \le \int_0^1\int_0^1 e|t(x_1-z_1)+(1-t)(y_1-v_1)|\mathrm{d}\theta\mathrm{d}t \\ &\quad \le \tfrac{e}{2}(|x_1-z_1|+|y_1-v_1|), \end{aligned}$$

we have

$$\begin{aligned} &\|A_0^{-1}([x,y;F]-[z,v;F])\| \\ &\quad \le \max(\tfrac{e\times 0.895819205}{2}(|x_1-z_1|+|y_1-v_1|), 0.819672131 \\ &\qquad \times(|x_2-z_2|+|y_2-v_2|)) \\ &\quad \le \max(\tfrac{e\times 0.895819205}{2}(\|x-z\|+\|y-v\|), 0.819672131(\|x-z\|+\|y-v\|)) \\ &\quad = \tfrac{e\times 0.895819205}{2}(\|x-z\|+\|y-v\|). \end{aligned} \tag{3.3}$$

In particular, set $z=2\overline{x}_0-\overline{y}_0$ and $v=\overline{y}_0$ in (3.3), we have

$$\|A_0^{-1}([x,y;F]-A_0)\| \le \tfrac{e\times 0.895819205}{2}(\|x-(2\overline{x}_0-\overline{y}_0)\|+\|y-\overline{y}_0\|). \tag{3.4}$$

That is, we can choose constants $L_0=L_1=K\approx \frac{e\times 0.895819205}{2}\approx 1.217544533$ in Theorem 2.2.

Using method (1.2), we get that $t_2\approx 0.148267584$, $t_3\approx 0.161640408$, $t_4\approx 0.163517484$, $t_5\approx 0.163626179$ and $t_6\approx 0.163627029$. That is to say, we have $t^\star\approx 0.163627029$. Then, we have $r_0=\max(2(t_1-t_0), t^\star-t_0)=0.22$.

Next. we verify that all conditions of Lemma 2.1 hold. In fact, by the definition of polynomial p, we get that $\alpha\approx 0.769178231264085243$. We also have

$$\begin{aligned} 0<\frac{L(t_1+2s_0)}{1-2L_0s_1} &= 0.04921445723287308504\approx \\ &\le \alpha\le 1-2L_0t_1 = 0.97564910934000004, \end{aligned}$$

i.e., (2.2) satisfies and

$$0<\frac{L(t_1+2s_0)}{1-2L_0s_1}=0.04921445723287308504\approx\alpha_0=0.7298292456330510 7135$$

i.e., (2.42) satisfies.

By now, we see that all conditions of Theorem 2.2 are satisfied, so Theorem 2.2 applies.

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