International Journal of Pure and Applied Mathematics
Volume 79 No. 2 2012, 249-258
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

ON SEMICLOSED SUBSPACES OF HILBERT SPACES

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Abstract: Semiclosed subspaces possess many special features that distinguish them from arbitrary subspaces. Few properties of proper dense semiclosed subspaces of Hilbert spaces are discussed.

AMS Subject Classification: 46C07, 46C15
Key Words: semiclosed subspaces, operator ranges

1. Introduction

Let $\mathcal{H}$ denote a complex Hilbert space with the inner product $\langle .,. \rangle$ without the requirement of separability. A subspace M of $\mathcal{H}$ is called semiclosed [6] if there exists a Hilbert inner product $\langle .,. \rangle_*$ on M such that $(M, \langle .,. \rangle_*)$ is continuously embedded in $(\mathcal{H}, \langle .,. \rangle)$. That is, if there exists an inner product $\langle .,. \rangle_*$ on M such that $(M, \langle .,. \rangle_*)$ is Hilbert and there exists $k > 0$ with $\langle x, x \rangle \leq k \langle x, x \rangle_*$ for all $x \in M$. It is known that every subspace of $\mathcal{H}$ is closed if and only if $\mathcal{H}$ is of finite dimension. As every closed subspace is semiclosed, only infinite dimensional complex Hilbert spaces are discussed in the sequel. Semiclosed subspaces possess many special features that distinguish them from arbitrary subspaces. Dixmier [3] calls a semiclosed subspace a "Julia variety" ; "paraclosed subspace" by Foias in [5].

An operator on $\mathcal{H}$ will always be understood to be a linear transformation from $\mathcal{H}$ into itself. The set of bounded operators and the cone of positive bounded operators on $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})^+$ respectively. If M is a subspace of a Hilbert space $\mathcal{H}$, the closure of M in $\mathcal{H}$ and the orthogonal

Received: March 17, 2012

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complement of M in $\mathcal{H}$ are denoted by $\overline{M}$ and $M^\perp$ respectively, whereas the bar $\bar{z}$ indicates the complex conjugate of z.

2. Semiclosed Subspaces

An operator range [4] in a Hilbert space $\mathcal{H}$ is a subspace of $\mathcal{H}$ that is the range of some bounded operator on $\mathcal{H}$. Semiclosed subspaces are characterized by operator ranges. Indeed, if M is a semiclosed subspace of $\mathcal{H}$. Then there exists an inner product $\langle .,. \rangle_*$ on M such that $(M, \langle .,. \rangle_*)$ is Hilbert and there exists $k > 0$ with $\langle x, x \rangle \leq k \langle x, x \rangle_*$ for all $x \in M$. The inclusion map $J : (M, \langle .,. \rangle_*) \to \mathcal{H}$ is bounded. Now consider the polar decomposition of J^*, $J^* = U(JJ^*)^{1/2}$, then $U : \mathcal{H} \to (M, \langle .,. \rangle_*)$ is a partial isometry with final space $\overline{R(J^*)} = N(J)^\perp = M$. Considering U from $\mathcal{H}$ into $\mathcal{H}$, we have $R(U) = M$. Boundedness of U follows from $\langle Ux, Ux \rangle \leq k \langle Ux, Ux \rangle_* \leq k \langle x, x \rangle$ for all $x \in \mathcal{H}$. Hence M is an operator range.

Conversely, if M is an operator range. Then $M = R(T)$ for some $T \in \mathcal{B}(\mathcal{H})$. By closed graph theorem, T is closed. Now consider the operator $\tilde{T} = T|_{N(T)^\perp}$ which is an injective closed operator whose range equals M. The inverse of $\tilde{T}$ is a closed operator with domain M. Define $\langle x, y \rangle_* = \langle x, y \rangle + \langle \tilde{T}^{-1}x, \tilde{T}^{-1}y \rangle$ for $x, y \in M$. Then $(M, \langle .,. \rangle_*)$ is a Hilbert space and $\langle x, x \rangle \leq \langle x, x \rangle + \langle \tilde{T}^{-1}x, \tilde{T}^{-1}x \rangle = \langle x, x \rangle_*$ for all $x \in M$. Hence M is a semiclosed subspace.

From the above discussions one can observe that the ranges of members of $\mathcal{B}(\mathcal{H})^+$ can alone characterize all semiclosed subspaces of $\mathcal{H}$. Operator ranges have been studied by many authors, most notoriously by J. Dixmier in [3] and by P. A. Fillmore and J. P. Williams in [4]. We start with a number of characterizations of semiclosed subspaces (operator ranges).

Theorem 1. *[4] Let M be a subspace of a Hilbert space $\mathcal{H}$. The following are equivalent:*

1. *M is a semiclosed subspace of $\mathcal{H}$.*
2. *M is the range of a bounded operator on $\mathcal{H}$.*
3. *M is the range of a closed operator on $\mathcal{H}$.*
4. *M is the domain of a closed operator on $\mathcal{H}$.*
5. *There is a sequence $\{\mathcal{H}_n : n \geq 0\}$ of closed mutually orthogonal subspaces*

of $\mathcal{H}$ such that

$$M = \left\{ \sum_{n=0}^{\infty} x_n : x_n \in \mathcal{H}_n \text{ and } \sum_{n=0}^{\infty} (2^n \|x_n\|)^2 < \infty \right\}.$$

As every semiclosed subspace in a Hilbert space $\mathcal{H}$ is the range of some bounded operator on $\mathcal{H}$, every semiclosed subspace is necessarily a Borel set, in fact, an F_σ-set and every proper semiclosed subspace is necessarily of first category by an application of the open mapping theorem ; these conditions are not sufficient, refer to [4].
The following examples show that all semiclosed subspaces are not closed and every subspace is not necessarily semiclosed.

Example 2. Consider the subspace

$$M = \{(x_1, x_2, \dots) : \sum_{n=1}^{\infty} (n|x_n|)^2 < \infty\}$$

of the space ℓ_2 of square-summable sequences. As M contains all sequences with finite support and $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ is not in M, it is a proper dense subspace of ℓ_2. So, M is not closed but it is semiclosed because M is the range of the bounded operator $T : \ell_2 \to \ell_2$ defined by $T(x_1, x_2, x_3, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, \dots)$.

Example 3. Consider in $\mathcal{H} = L_2[0,1]$, the subspace $L_p[0,1]$ $(2 < p < \infty)$ is not a semiclosed subspace of $\mathcal{H}$ because $L_p[0,1]$ cannot be the range of any bounded operator on L_2, by the following remark: If T is a continuous linear injection of a Banach space X in a Hilbert space $\mathcal{H}$, and if $T(X)$ is the range of a bounded operator on $\mathcal{H}$, then X is isomorphic to a Hilbert space. Since L_p is not isomorphic to a Hilbert space [9], it follows that L_p is not the range of any bounded operator on L_2.

The sum of closed subspaces of a Hilbert space need not be closed [1]. There are necessary and sufficient conditions for the sum of closed subspaces of a Hilbert space to be closed; specifically, the angle between the subspaces is not zero, or the projection of either space into the orthogonal complement of the other is closed.

In the case of the set $SC(\mathcal{H})$ of semiclosed subspaces of a Hilbert space $\mathcal{H}$, $SC(\mathcal{H})$ forms a complete lattice with respect to "intersection" and "sum", which is shown by the following propositions. Moreover, $SC(\mathcal{H})$ is the smallest lattice containing all closed subspaces of $\mathcal{H}$.

Proposition 4. *The intersection of two semiclosed subspaces of $\mathcal{H}$ is again a semiclosed subspace.*

Proof. Let M_1 and M_2 be semiclosed subspaces of $\mathcal{H}$. Then there are two Hilbert inner products $\langle .,. \rangle_1$ and $\langle .,. \rangle_2$ such that each $(M_i, \langle .,. \rangle_i)$ is continuously embedded in $\mathcal{H}$, $i = 1, 2$. Define $\langle .,. \rangle_*$ on $M_1 \cap M_2$ by $\langle x, y \rangle_* = \langle x, y \rangle_1 + \langle x, y \rangle_2$. Then $\langle .,. \rangle_*$ is a Hilbert inner product and is stronger than the usual inner product on $\mathcal{H}$. Hence $M_1 \cap M_2$ is a semiclosed subspace. □

Let T be a bounded operator on $\mathcal{H}$. The range of T can be given uniquely a Hilbert space structure, with norm $\|.\|_T$ as follows. Moreover, T becomes a coisometry from $\mathcal{H}$ to $(R(T), \|.\|_T)$. In fact, since T gives rise to a bijection from $N(T)^\perp = \overline{R(T^*)}$ is a Hilbert space, the inner product $\langle .,. \rangle_T$ on $R(T)$, defined by

$$\langle Ta, Tb \rangle_T = \langle Pa, Pb \rangle \text{ for } a, b \in \mathcal{H},$$

where P is the orthogonal projection to $N(T)^\perp$, makes $R(T)$ a Hilbert space and the uniqueness is obvious. Since $Ta = TPa$ and $\|Pa\| \leq \|a\|$, norm $\|u\|_T$ admits the description

$$\|u\|_T = \min\{\|a\| : Ta = u\} \text{ for } u \in R(T),$$

and the following inequality holds

$$\|u\| \leq \|T\| \ \|u\|_T \text{ for } u \in R(T).$$

Hence for each $u \in R(T)$ there is uniquely $a \in \overline{R(T^*)}$ such that

$$Ta = u \text{ and } \|a\| = \|u\|_T.$$

The space $R(T)$ equipped with the Hilbert space structure $\|.\|_T$ is denoted by $\mathcal{M}(T)$:

$$\mathcal{M}(T) \equiv (R(T), \|.\|_T).$$

$\mathcal{M}(T)$ is called de Branges space induced by T. These Hilbert spaces $\mathcal{M}(T)$ play a significant role in many areas, in particular in the de Branges complementation theory [2].

Conversely, suppose that a subspace M of a Hilbert space $\mathcal{H}$ is equipped with a Hilbert space structure $\|.\|_*$ such that $(M, \langle .,. \rangle_*)$ is continuously embedded in $\mathcal{H}$. Then there is uniquely a positive operator T on $\mathcal{H}$ such that $(M, \|.\|_*) = \mathcal{M}(T)$.

Theorem 5. *[2] For $T_1, T_2 \in \mathcal{B}(\mathcal{H})$, let $T = (T_1T_1^* + T_2T_2^*)^{1/2}$. Then $\|u_1+u_2\|_T^2 \leq \|u_1\|_{T_1}^2 + \|u_2\|_{T_2}^2$, for $u_1 \in R(T_1), u_2 \in R(T_2)$, and for any $u \in R(T)$, there are uniquely $u_1 \in R(T_1), u_2 \in R(T_2)$ such that $u = u_1 + u_2$ and*

$$\|u_1 + u_2\|_T^2 = \|u_1\|_{T_1}^2 + \|u_2\|_{T_2}^2.$$

Proposition 6. *The sum of two semiclosed subspaces of $\mathcal{H}$ is again a semiclosed subspace.*

Proof. Let M_1, M_2 be two semiclosed subspaces of $\mathcal{H}$. There are positive operators T_1, T_2 on $\mathcal{H}$ such that

$$(M_1, \|.\|_1) = \mathcal{M}(T_1) \text{ and } (M_2, \|.\|_2) = \mathcal{M}(T_2).$$

Moreover, there are positive numbers k_1 and k_2 such that

$$\|u_1\| \leq k_1\|u_1\|_1 \text{ and } \|u_2\| \leq k_2\|u_2\|_2 \text{ for all } u_1 \in M_1, u_2 \in M_2.$$

Let $T = (T_1T_1^* + T_2T_2^*)^{1/2}$. Then by theorem 5, for any $u \in R(T)$, there are uniquely $u_1 \in R(T_1), u_2 \in R(T_2)$ such that $u = u_1 + u_2$ and

$$\|u\|_T^2 = \|u_1 + u_2\|_T^2 = \|u_1\|_1^2 + \|u_2\|_2^2.$$

Hence $\|.\|_T$ is a Hilbert inner product on $M_1 + M_2$ such that $(M_1 + M_2, \langle ., .\rangle_T)$ is continuously embedded in $(\mathcal{H}, \langle ., .\rangle)$ because for any $u \in M_1 + M_2$,

$$\|u\|^2 \leq \|u_1\|^2 + \|u_2\|^2 \leq k^2(\|u_1\|_1^2 + \|u_2\|_2^2) = k^2\|u\|_T^2,$$

where $k = \max\{k_1, k_2\}$. Thus $M_1 + M_2$ is a semiclosed subspace of $\mathcal{H}$. □

To appreciate an application of Riesz representation theorem for Hilbert spaces, we reproduce proof of the result in [8] which reveals that semiclosed subspaces can be characterized alone by the ranges of members of $\mathcal{B}(\mathcal{H})^+$. For a fixed inner product $\langle ., .\rangle_*$ on M, some properties for the operator A corresponding to inner product $\langle ., .\rangle_*$ are discussed below. Unless and otherwise specified, $\mathcal{H}$ and M have Hilbert inner products $\langle ., .\rangle$ and $\langle ., .\rangle_*$ respectively. We denote $\mathbb{C}$ for the set of complex numbers and the conjugate of $\langle x, y\rangle$ by $\overline{\langle x, y\rangle}$.

Theorem 7. *Let M be a semiclosed subspace of a Hilbert space $\mathcal{H}$. For each Hilbert inner product $\langle ., .\rangle_*$ on M such that $(M, \langle ., .\rangle_*)$ is continuously embedded in $\mathcal{H}$, there is a unique $A \in \mathcal{B}(\mathcal{H})^+$ such that $\langle x, y\rangle = \langle x, Ay\rangle_*$ for all $x \in M, y \in \mathcal{H}$.*

Proof. Given that $(M, \langle ., .\rangle_*)$ is a Hilbert space and there exists $k > 0$ such that $\langle x, x\rangle \leq k\langle x, x\rangle_*$ for all $x \in M$. Let $y \in \mathcal{H}$. Define $f_y : \mathcal{H} \to \mathbb{C}$ by $f_y(x) = \langle x, y\rangle$. The restriction of f_y to M is bounded on $(M, \langle ., .\rangle_*)$ because for $x \in M$,

$$|f_y(x)| \leq \|x\| \, \|y\| \leq k\|x\|_* \, \|y\|.$$

By Riesz representation theorem for Hilbert spaces, there exists a unique $z \in M$ so that $f_y(x) = \langle x, z\rangle_*$ for all $x \in M$.

Define $A : (\mathcal{H}, \langle ., .\rangle) \to (M, \langle ., .\rangle_*)$ by $Ay = z$. Then $\langle x, y\rangle = \langle x, Ay\rangle_*$ for all $x \in M, y \in \mathcal{H}$. Clearly $A(\mathcal{H}) \subset M$ and the uniqueness of A follows from the Riesz representation theorem.

For each $x \in \mathcal{H}$,

$$\begin{aligned}\|Ax\| \leq k\|Ax\|_* = k \sup\{|\langle z, Ax\rangle_*| : \|z\|_* = 1\} \\ = k \sup\{|\langle z, x\rangle| : \|z\|_* = 1\} \leq k\|x\|,\end{aligned}$$

hence A is bounded. From the relation $\langle Ax, y\rangle = \langle Ax, Ay\rangle_* = \overline{\langle Ay, Ax\rangle_*} = \overline{\langle Ay, x\rangle} = \langle x, Ay\rangle$, we get $\langle x, Ay\rangle = \langle Ax, y\rangle$ for all $x, y \in \mathcal{H}$. The positiveness of the operator A comes from $\langle Ax, x\rangle = \langle Ax, Ax\rangle_* = \|Ax\|_*^2 \geq 0$ for all $x \in \mathcal{H}$. □

Proposition 8. *Let M be a semiclosed subspace of $\mathcal{H}$ and A_M be the restriction of A to M with the Hilbert norm $\|.\|_*$. Then $A_M : M \to M$ is a bounded positive self-adjoint operator on M.*

Proof. The boundedness of A_M comes from the following:

$$\begin{aligned}\|A_M x\|_* &= \sup\{|\langle z, A_M x\rangle_*| : \|z\|_* = 1\} \\ &= \sup\{|\langle z, x\rangle| : \|z\|_* = 1\} \leq \|x\| \leq k\|x\|_*.\end{aligned}$$

For each $x, y \in M$,

$$\langle A_M x, y\rangle_* = \overline{\langle y, A_M x\rangle_*} = \overline{\langle y, x\rangle} = \langle x, y\rangle = \langle x, A_M y\rangle_*$$

hence we get $\langle x, A_M y\rangle_* = \langle A_M x, y\rangle_*$ for all $x, y \in M$. A_M is positive because for $x \in M$, $\langle A_M x, x\rangle_* = \|x\|^2 \geq 0$. □

In addition to the notational conventions mentioned in the introduction, we denote the positive square roots of A and A_M by $A^{1/2}$ and $A_M^{1/2}$ respectively. To differentiate the convergence of a sequence in M with respect to the new norm $\|.\|_*$, we denote $\lim_* x_n = x$ for the sequence (x_n) in M converging to x with respect to the norm $\|.\|_*$. The closure of a subset N of M corresponding to the norm $\|.\|_*$ is denoted by $\overline{N}^*$ whereas $\overline{N}$ denotes the closure of N with the usual inner product on $\mathcal{H}$.

Theorem 9. *[7] For each self-adjoint operator $A \in \mathcal{B}(\mathcal{H})^+$, there exists a unique self-adjoint $B \in \mathcal{B}(\mathcal{H})^+$ such that $B^2 = A$ and B is strong limit of the sequence given by the recursive relation $A_0 = 0, A_n = [(I - A) - A_{n-1}^2], n \geq 1$.*

Proposition 10. *For $x \in M$, $\|x\| = \|A_M^{1/2}x\|_*$ and $A^{1/2}$ agrees with $A_M^{1/2}$ on M.*

Proof. Let $x \in M$. Then $\|x\|^2 = \langle x, A_M x\rangle_* = \langle A_M^{1/2}x, A_M^{1/2}x\rangle_* = \|A_M^{1/2}x\|_*^2$. We next claim that $A^{1/2}$ and $A_M^{1/2}$ are same at every point of M. If $x \in M$, then $A_M^{1/2}x = \lim_* A_n x$. As $\|.\|_*$ is stronger than $\|.\|$, we get $\lim A_n x = A_M^{1/2}x$, hence $A^{1/2}x = A_M^{1/2}x$, where $\{A_n\}_{n\geq 1}$ are as given in the theorem 9. □

For each such inner product $\langle ., .\rangle_*$ on the semiclosed subspace M, there corresponds a topology on M. Interestingly all such inner products are equivalent which is shown by the following theorem. Hence the topology on M is unique. This topology coincides with the induced topology if the subspace M is closed in $\mathcal{H}$.

Theorem 11. *Let M be a semiclosed subspace of $\mathcal{H}$. Then all inner products $\langle ., .\rangle_*$ such that $(M, \langle ., .\rangle_*)$ is continuously embedded, generate the same topology on M.*

Proof. Suppose M has two such Hilbert space inner products $\langle ., .\rangle_1$ and $\langle ., .\rangle_2$ respectively. Then by theorem 7, there are $A_1, A_2 \in \mathcal{B}(\mathcal{H})^+$ such that

$$\langle x, y\rangle = \langle x, A_1 y\rangle_1 \text{ and } \langle x, y\rangle = \langle x, A_2 y\rangle_2 \text{ for all } x \in M, y \in \mathcal{H}.$$

Let $B_1 = \{y \in M : \|y\|_1 = 1\}$. Then for $y \in B_1$, then linear functional $F_y(x) = \langle x, y\rangle = \langle x, A_1 y\rangle_1 = \langle x, A_2 y\rangle_2$, for all $x \in M, y \in \mathcal{H}$. As $|F_y(x)| \leq \|x\|_1\|y\|_1$ and $|F_y(x)| \leq \|x\|_2\|y\|_2$, F_y is a bounded linear functional on both $(M_1, \|.\|_1)$ and $(M_2, \|.\|_2)$.

For its corresponding operator norms $\|F_y\|_1$ and $\|F_y\|_2$, we obtain $\|F_y\|_1 = \|y\|_1$ and $\|F_y\|_2 = \|y\|_2$. For all $x \in M$, we have $\sup_{y\in B_1} |F_y(x)| \leq \|x\|_1 < \infty$. By the uniform boundedness principle, there exists $c > 0$ such that $\sup_{y\in B_1} \|F_y\|_2 \leq c$. This proves that $\|y\|_2 \leq c\|y\|_1$ for all $y \in M$. By interchanging the role of $\|.\|_1$ and $\|.\|_2$, we obtain that these norms are equivalent. Thus each semiclosed subspace of $\mathcal{H}$ has a unique topology. □

3. Dense Semiclosed Subspaces

Every subspace of a Hilbert space which is dense and proper, is never closed in the strong topology. But semiclosed subspaces possess many special features that distinguish them from closed subspaces. Few properties of operators associated with proper dense semiclosed subspaces are discussed in the section.

Theorem 12. *Let M be a dense semiclosed subspace of a Hilbert space $\mathcal{H}$ and A be the operator corresponding to the inner product $\langle .,. \rangle_*$. Then A has dense range in $\mathcal{H}$ and A is injective.*

Proof. Suppose $y_0 \in M$ such that $\langle Ax, y_0 \rangle_* = 0$ for each $x \in \mathcal{H}$. Then $0 = \langle Ay_0, y_0 \rangle_* = \langle y_0, y_0 \rangle$, so $y_0 = 0$, hence $R(A)$ is dense in M with respect to $\langle .,. \rangle_*$. As $M = \overline{R(A)}^* \subset \overline{R(A)}$ and M is dense in $\mathcal{H}$, we get $\mathcal{H} = \overline{M} \subset \overline{R(A)} \subset \mathcal{H}$, $R(A)$ is dense in $\mathcal{H}$.

Suppose that for some $x \in \mathcal{H}$ with $Ax = 0$. Then for each $y \in \mathcal{H}$, $\langle x, Ay \rangle = \langle Ax, y \rangle = 0$, so x is in the orthogonal complement of $R(A)$. The denseness of $R(A)$ in $\mathcal{H}$ gives that $x = 0$. Hence A is injective. □

Theorem 13. *Let M be a dense semiclosed subspace of a Hilbert space $\mathcal{H}$ and A be the operator corresponding to the inner product $\langle .,. \rangle_*$. Then $R(A^{1/2}) = M$.*

Proof. Suppose $x \in \mathcal{H}$. Since M is dense in $\mathcal{H}$, there exists a sequence (x_n) in M such that $\lim x_n = x$. By Proposition 10, for each n, $\|x_n\| = \|A_M^{1/2} x_n\|_*$. The boundedness of A gives that $A^{1/2}$ is bounded from $\mathcal{H}$ to $\mathcal{H}$. Then

$$\|x\| = \lim \|x_n\| = \lim \|A^{1/2} x_n\|_* = \|A^{1/2} x\|_*.$$

Thus if $x \in \mathcal{H}$ and $\|x\| = \|A^{1/2} x\|_*$.

Suppose $(A^{1/2} x_n)$ converges in M to y. Since $\|A^{1/2} x_n\| = \|x_n\|$, we have that (x_n) is Cauchy in $\mathcal{H}$. Let $\lim_* x_n = x$. Then $A^{1/2} x = y$ since

$$\|A^{1/2} x - y\| \leq \|A^{1/2} x - A^{1/2} x_n\| + \|A^{1/2} x_n - y\|$$

and given $\varepsilon > 0$, there exists an n so that the right side of the inequality is less than ε. Hence $A^{1/2} x = y$ and A^{-1} exists, by Theorem 12.

Suppose $A^{1/2} x = 0$. Then $Ax = A^{1/2} A^{1/2} x = 0$ and so $x = 0$, hence $A^{1/2}$ is injective. To show the range of $A^{1/2}$ is dense in M. Suppose there exists $y \in M$ so that $\langle A^{1/2} x, y \rangle_* = 0$ for all $x \in \mathcal{H}$. Then if we let $x = A^{1/2} y$,

$$0 = \langle A^{1/2} A^{1/2} y, y \rangle_* = \langle A^{1/2} y, A^{1/2} y \rangle_* = \|A^{1/2} y\|_*.$$

This implies that $A^{1/2} y = 0$, so $y = 0$. Thus $R(A^{1/2})$ is a dense and closed subspace of M, hence it is equal to M. □

Theorem 14. *Let M be a semiclosed subspace of $\mathcal{H}$ and A be the operator corresponding to the inner product $\langle .,. \rangle_*$. Then A from $\mathcal{H}$ to $\mathcal{H}$ is compact if and only if A from $\mathcal{H}$ to M is compact.*

Proof. Suppose $A : \mathcal{H} \to M$ is compact. Then for a bounded sequence (x_n) in $\mathcal{H}$, (Ax_n) has a convergent subsequence in M. Since the norm $\|.\|_*$ is stronger than the usual norm $\|.\|$ on $\mathcal{H}$, (Ax_n) has a convergent subsequence in $\mathcal{H}$. On the other hand, suppose that $A : \mathcal{H} \to \mathcal{H}$ is compact. Then we get $A^{1/2} : \mathcal{H} \to \mathcal{H}$ is compact. Thus if (x_n) is a bounded sequence in $\mathcal{H}$, then $(A^{1/2}x_n)$ has a convergent subsequence in $\mathcal{H}$; we call the subsequence by the same name as the original sequence. By theorems 12 and 13, for each $x \in \mathcal{H}$, $\|A_M^{1/2}x\|_* = \|x\|$, we get that $\|Ax_n\|_* = \|A^{1/2}A^{1/2}x_n\|_* = \|A^{1/2}x_n\|$. Since $(A^{1/2}x_n)$ is convergent in $\mathcal{H}$, it is a Cauchy sequence. Using the above relation, we get (Ax_n) is Cauchy in M. Hence it converges in M. $\square$

Lemma 15. *[8] If A is a positive self-adjoint bounded operator on a Hilbert space $\mathcal{H}$ and $z \in \mathcal{H}$, then $z \in A^{1/2}(\mathcal{H})$ iff there exists $b > 0$ such that $|\langle x, z\rangle|^2 \le b\langle x, Ax\rangle$ for each $x \in \mathcal{H}$. Moreover, $\|A^{-1/2}z\|^2$ is the least of b.*

Theorem 16. *Let M be a proper dense semiclosed subspace of $\mathcal{H}$. Then there exists a proper subspace N of $\mathcal{H}$ such that M is a semiclosed subspace of N.*

Proof. The semiclosedness of M gives a Hilbert inner product $\langle ., .\rangle_*$ on M such that for some $k > 0$, $\langle x, x\rangle \le k\langle x, x\rangle_*$ for each $x \in M$. Then by theorems 7, 12 and 13, there exists $A \in \mathcal{B}(\mathcal{H})^+$ such that $\langle x, y\rangle = \langle x, Ay\rangle_*$ for all $x \in M, y \in \mathcal{H}$ and $M = R(A^{1/2})$. As M is proper in $\mathcal{H}$, choose u_0 in $\mathcal{H}$ not in M such that $\|u_0\| = 1$. Define $B : \mathcal{H} \to \mathcal{H}$ by $Bx = Ax + \langle x, u_0\rangle u_0$, $x \in \mathcal{H}$. Clearly $B \in \mathcal{B}(\mathcal{H})^+$ is injective and hence $B^{-1/2}$ exists.

Let $N = R(B^{1/2})$ and $\langle ., .\rangle_1$ be the inner product on N defined by $\langle B^{1/2}x, B^{1/2}y\rangle_1 = \langle x, y\rangle$ for each $x, y \in \mathcal{H}$. Clearly $\langle ., .\rangle_1$ is a Hilbert inner product on N. Let $z \in M$. For each $x \in \mathcal{H}$,

$$|\langle x, z\rangle|^2 = |\langle Ax, z\rangle_*|^2 \le \|Ax\|_*^2\|z\|_*^2 \le \|z\|_*^2\langle x, Ax\rangle \le \|z\|_*^2\langle x, Bx\rangle$$

so by Lemma 15, $\|z\|_1 \le \|z\|_*$. Let $z \in N$. Then for each $x \in \mathcal{H}$,

$$\begin{aligned} |\langle x, z\rangle|^2 &= |\langle B^{1/2}x, B^{1/2}z\rangle_1|^2 = |\langle Bx, z\rangle_1|^2 \\ &\le \|z\|_1^2\ \|Bx\|_1^2 \le \|z\|_1^2\ \langle x, Bx\rangle \\ &\le \|z\|_1^2\ (\langle x, Ax\rangle + \langle x, x\rangle) = \|z\|_1^2\ (k+1)\langle x, x\rangle \end{aligned}$$

so $\|z\| \le (k+1)^{1/2}\|z\|_1$. We proved that $(M, \langle ., .\rangle_*)$ is a semiclosed subspace of $(N, \langle ., .\rangle_1)$ which is a semiclosed subspace of $\mathcal{H}$.

We next claim that N is a proper subspace of $\mathcal{H}$. Suppose $B^{1/2}(\mathcal{H}) = \mathcal{H}$. Then $\mathcal{B}(\mathcal{H}) = \mathcal{H}$. Therefore there exists $x_0 \in \mathcal{H}$ such that $Bx_0 = u_0$. Then

$$A^{1/2}(A^{1/2})x_0 = Ax_0 = [1 - \langle x_0, u_0\rangle]u_0.$$

Since $u_0 \notin M = R(A^{1/2})$, $A^{1/2}x_0 = 0$ so that $x_0 = 0$ implies that $u_0 = 0$, which is a contradiction to $u_0 \neq 0$, hence $B^{1/2}(\mathcal{H}) \neq \mathcal{H}$. Note that $A \leq B$ because $\langle x, Ax\rangle \leq \langle x, Bx\rangle$ for all $x \in \mathcal{H}$. □

Acknowledgments

The present work of first author was partially supported by SERC, DST, Government of India, through the Fast Track Scheme for Young Scientists (D.O. No. SR/FTP/MS-050/2011) and the second author thanks the National Institute of Technology Karnataka (NITK), Surathkal for giving financial support.

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