FRAMES FOR OPERATORS IN HILBERT AND BANACH SPACES

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

RAMU GEDDAVALASA

DEPARTMENT OF MATHEMATICAL & COMPUTATIONAL SCIENCES

NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA

SURATHKAL, MANGALORE - 575025

May, 2017

DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the research thesis entitled **"Frames for Operators in Hilbert and Banach Spaces"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

Place : NITK, Surathkal

Date : 31-05-2017

Ramu Geddavalasa
Reg. No. MA11F02
Department of MACS
NITK, Surathkal

CERTIFICATE

This is to **certify** that the research thesis entitled **"Frames for Operators in Hilbert and Banach Spaces"** submitted by **Ramu Geddavalasa**, (Register Number MA11F02) as the record of the research work carried out by him, is *accepted as the research thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. P. Sam Johnson

Research Guide

Chairman - DRPC

ACKNOWLEDGEMENT

I would like to take this opportunity to thank people who supported me in one or the other way to reach my goal. With a few words, I express my gratitude and heartfelt thanks to those who helped me in my research work.

One of the most important persons in the journey of my research work is, my research guide, *Dr. P. Sam Johnson*. Every moment he motivated me to analyze the things, which is helping me to understand any topic which I read. I remain grateful to my guide for all his support.

I extend thanks to RPAC members, *Dr. E. Sathyanarayana*, Department of Mathematical and Computational Sciences (MACS) and *Dr. Jeny Rajan*, Department of Computer Science and Engineering, for their valuable suggestions, which helped me to improve my presentations and my work.

I am thankful to *Prof. Santhosh George*, Head, Department of MACS, for providing a very nice working environment and all the facilities in the department.

I extend my thanks to all the teaching and non-teaching staff members of the Department of MACS. Everyone helped and guided me in one or the other way.

My stay at NITK was made pleasant and memorable by my research colleagues in the department ; we enjoyed a lot, shared the happy and sad moments with each other. Thanks to all my friends for making my stay memorable.

This acknowledgement will remain incomplete if I don't mention the names of two important persons, my mother, *Mrs. Vijaya Laxmi G.* and my father *Mr. Appala Naidu*, this work was impossible. I thank God for such a wonderful gift.

Place: NITK, Surathkal — Ramu Geddavalasa

Date: 31^{st} May, 2017

ABSTRACT

The notion of K-frames has been introduced by Laura Găvruţa to study the atomic systems with respect to a bounded linear operator K in a separable Hilbert space. K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K. Because of the higher generality of K-frames, many properties for ordinary frames may not hold for K-frames, such as the corresponding synthesis operator for K-frames is not surjective, the frame operator for K-frames is not isomorphic, the alternate dual reconstruction pair for K-frames is not interchangeable in general. Note that the frame operator S for a K-frame is semidefinite, so there is also $S^{1/2}$, but not positive. Operators that preserve K-frames and generating new K-frames from old ones by taking sums have been discussed. A close relation between K-frames and quotient operators is established using through operator-theoretic results on quotient operators and few characterizations are given.

A frame for a Banach space $\mathcal{X}$ was defined as a sequence of elements in $\mathcal{X}^*$, not of elements in the original space $\mathcal{X}$. However, semi-inner products for Banach spaces make possible the development of inner product type arguments in Banach spaces. The concept of a family of local atoms in a Banach space $\mathcal{X}$ with respect to a BK-space $\mathcal{X}_d$ was introduced by Dastourian and Janfada using a semi-inner product. This concept was generalized to an atomic system for an operator $K \in \mathcal{B}(\mathcal{X})$ called $\mathcal{X}_d^*$-*atomic system* and it has been led to the definition of a new frame with respect to the operator K, called $\mathcal{X}_d^*$-K-*frame.* Appropriate changes have been made in the definitions of $\mathcal{X}_d^*$-atomic systems and $\mathcal{X}_d^*$-K-frames to fit them for sequences in the dual space without using semi-inner products, called $\mathcal{X}_d$-*atomic systems* and $\mathcal{X}_d$-K-*frames* respectively. New $\mathcal{X}_d$-K-frames are generated from each $\mathcal{X}_d$-frame for a Banach space $\mathcal{X}$ and each operator $K \in \mathcal{B}(\mathcal{X}^*)$ and some characterizations are given. With some crucial assumptions, it is shown that frames for operators in Banach spaces share nice properties of frames for operators in Hilbert spaces.

Keywords : Frame ; K-frame ; $\mathcal{X}_d$-atomic system ; $\mathcal{X}_d$-K-frame.

Table of Contents

Chapter 1

PRELIMINARIES

1.1 General Introduction

Gabor (Gabor, 1946) introduced a method for reconstructing functions (signals) using a family of elementary functions. Later, Duffin and Schaeffer (Duffin and Schaeffer, 1952) presented a similar tool in the context of nonharmonic Fourier series and this is the starting point of frame theory. After some decades, Daubechies, Grossmann and Meyer (Daubechies et al., 1986) announced formally the definition of frame in the abstract Hilbert spaces. After their work, the theory of frames began to be studied widely and deeply.

Frames are generalizations of orthonormal bases. The linear independence property for a (Hamel) basis, which allows every vector to be uniquely represented as a linear combination is very restrictive for practical problems. Frames allow each element in the space to be written as a linear combination of the elements in the frame, but linear independence between the frame elements is not required. They provide basis-like, stable and usually non-unique representations of vectors in a Hilbert space.

Theoretical research of frames for Banach spaces is quite different from that of Hilbert spaces. Due to the lack of an inner product, frames for Banach spaces were simply defined as a sequence of linear functionals, rather than a sequence of elements in the space itself. Properties of Hilbert frames usually do not transfer automatically to Banach spaces.

Frames have been a focus of study for more than three decades in applications where redundancy plays a vital and useful role. The redundancy and flexibility offered by frames has spurred their applications in a variety of areas throughout mathematics and engineering, such as operator theory (Han and Larson, 2000), harmonic analysis (Gröchenig, 2001), pseudo-differential operators (Gröchenig and Heil, 1999), quantum computing (Eldar and Forney, 2002), signal and image processing (Donoho and Elad, 2003), wireless communication (Jr. and Paulraj, 2002), and so on.

Moreover, frames are now used to mitigate the effect of losses in pocket-based communication systems and hence to improve the robustness of data transmission (Casazza and Kovačević, 2003), and to design high-rate constellation with full diversity in multiple-antenna code design (Shokrollahi et al., 2001). For an introduction to the frame theory, we refer to (Christensen, 2003), (Daubechies, 1992), and (Mallat, 2009).

1.2 Schauder Bases

We consider linear spaces only over the complex number field $\mathbb{C}$. The set of all natural numbers is denoted by $\mathbb{N}$. Though a common index set I may be used for countable indexing, it is preferred to use $\mathbb{N}$ in place of I for convenience. The space of absolutely square summable sequences of complex numbers is denoted by ℓ_2.

Definition 1.2.1. *A subset E of a linear space $\mathcal{X}$ is said to be **linearly independent** if for all $f_1, f_2, \ldots, f_n \in E$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$, the equation*

$$\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = 0 \quad \textit{implies that} \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

*A subset E of $\mathcal{X}$ is called an **algebraic basis** or **(Hamel) basis** for $\mathcal{X}$ if E is a linearly independent set and a spanning set.*

When working with finite dimensional linear spaces, it is often convenient to take a Hamel basis. In an infinite dimensional linear space, the axiom of choice shows that we can still find a Hamel basis for any non-zero linear space : however, because it is now infinite, its use is often less. This is especially true in a Banach space, because such an algebraic basis takes no account of the extra structure induced by the norm. This leads to the notion of a Schauder basis.

Definition 1.2.2. *Let $\mathcal{X}$ be an infinite dimensional Banach space. A sequence of vectors $\{f_1, f_2, \ldots\}$ in $\mathcal{X}$ is called a* ***Schauder basis*** *for $\mathcal{X}$ if to each vector f in the space there corresponds a unique sequence of scalars $\{\alpha_1, \alpha_2, \ldots\}$ such that*

$$f = \sum_{i=1}^{\infty} \alpha_i f_i. \tag{1.2.1}$$

The convergence of the series is understood to be with respect to the norm topology of $\mathcal{X}$. In other words,

$$\left\| f - \sum_{i=1}^{n} \alpha_i f_i \right\| \to 0 \qquad as \quad n \to \infty.$$

The equation (1.2.1) is referred as the ***expansion of*** *f in the basis $\{f_i\}_{i=1}^{\infty}$.*

It is easy to see that a Banach space $\mathcal{X}$ that can be equipped with a Schauder basis is separable. Indeed, in this case, the countable set of finite linear combinations of basis elements with rational scalars is everywhere dense in $\mathcal{X}$.

Schauder bases were constructed for many Banach spaces. In 1927, Schauder formulated the well-known "problem of a basis" : Is it always possible to construct a Schauder basis in an arbitrary separable Banach space? This problem was solved in 1972 by P. Enflo who constructed an example of a separable Banach space without a Schauder basis. Some examples of Schauder bases are given below.

Example 1.2.3. *Let $\mathcal{X} = \ell_p$ for $1 \leq p < \infty$, or c_0. For $n \geq 1$, let $e_i \in \mathcal{X}$ be the sequence which is 0 except with a '1' in the ith position. Then $\{e_i\}$ is a Schauder basis for $\mathcal{X}$.*

Example 1.2.4. *$\{1, t, t^2, \ldots\}$ is a Schauder basis of $C[0,1]$ (with sup norm $\|.\|_\infty$) because span$\{1, t, t^2, \ldots\}$ is dense in $C[0,1]$. Similarly, $\{1, t, t^2, \ldots\}$ is a Schauder basis of $L_p[0,1]$, $1 \leq p < \infty$.*

Example 1.2.5. *(Limaye, 1996) Let* $1 \le p < \infty$*. For* $t \in [0,1]$*, let* $f_1(t) = 1$*,*

$$f_2(t) = \begin{cases} 1, & \text{if} \quad 0 \le t \le 1/2 \\ -1, & \text{if} \quad 1/2 < t \le 1 \end{cases}$$

and for $n = 1, 2, \ldots, \quad j = 1, 2, \ldots, 2^n$*,*

$$f_{2^n+j}(t) = \begin{cases} 2^{n/p}, & \text{if} \quad (2j-2)/2^{n+1} \le t \le (2j-1)/2^{n+1} \\ -2^{n/p}, & \text{if} \quad (2j-1)/2^{n+1} < t \le 2j/2^{n+1} \\ 0, & \text{otherwise.} \end{cases}$$

Then the ***Haar system*** $\{f_1, f_2, \ldots\}$ *is a Schauder basis for* $L_p[0,1]$*. Each* f_n *is a step function.*

Example 1.2.6. *(Limaye, 1996) For* $t \in \mathbb{R}$*, let* $g_0(t) = t,\ g_1(t) = 1 - t$*,*

$$g_2(t) = \begin{cases} 2t & \text{if} \quad 0 \le t \le 1/2 \\ 2 - 2t & \text{if} \quad 1/2 < t \le 1 \\ 0 & \text{if} \quad t < 0 \text{ or } t > 1 \end{cases}$$

and $g_{2^n+j}(t) = g_2(2^n t - j + 1)$ *for* $n = 1, 2, \ldots, \quad j = 1, 2, \ldots, 2^n$*. If* f_n *is the restriction of* g_n *on* $[0,1]$*, then* $\{f_0, f_1, \ldots\}$ *is a Schauder basis for* $C[0,1]$*. Each* f_n *is a non-negative piecewise linear continuous function, known as a* ***saw-tooth function****.*

Equation (1.2.1) merely means that the series

$$f = \sum_{i=1}^{\infty} \alpha_i f_i$$

converges with respect to the chosen order of the elements. If the series (1.2.1) converges unconditionally for each $f \in \mathcal{X}$, we say that $\{f_i\}_{i=1}^{\infty}$ is an **unconditional basis**.

If $\{e_i\}_{i=1}^{\infty}$ is a basis which is not unconditional, there exists a permutation σ for which $\{e_{\sigma(i)}\}_{i=1}^{\infty}$ is not a basis (Singer, 1970). It is known that every Banach space which has a basis also has a conditional basis (Pełczyński and Singer, 1965). In the sequel we make the convention that a basis shall be a Schauder basis, unless explicit reference is made.

Theorem 1.2.7. *(Christensen, 2003) Assume that $\{f_i\}_{i=1}^{\infty}$ is a basis for a Hilbert space $\mathcal{H}$. Then there exists a unique family $\{g_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ for which*

$$f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i, \text{ for all } f \in \mathcal{H}, \tag{1.2.2}$$

and $\{g_i\}_{i=1}^{\infty}$ is a basis for $\mathcal{H}$.

The basis $\{g_i\}_{i=1}^{\infty}$ satisfying (1.2.2) is called the **dual basis** or the **bi-orthogonal basis** associated to $\{f_i\}_{i=1}^{\infty}$.

1.3 Orthonormal Bases

Definition 1.3.1. *A family E of non-zero vectors in an inner product space $\mathcal{X}$ is called an **orthogonal set** if $\langle f, g \rangle = 0$, for any two distinct elements f and g of E. If, in addition, $\|f\| = 1$ for all $f \in E$, then E is called an **orthonormal set**.*

*A sequence of vectors which constitutes an orthonormal set is called **orthonormal sequence**.*

Example 1.3.2. *(Debnath and Mikusiński, 1999) The **Legendre polynomials** defined by*

$$\begin{aligned} P_0(x) &= 1, \\ P_n(x) &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad n = 1, 2, \ldots \end{aligned}$$

form an orthogonal set in $L^2[-1, 1]$.

Example 1.3.3. *(Debnath and Mikusiński, 1999) The **Hermite polynomials** of degree n are defined by*

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The functions

$$\psi_n(x) = e^{-x^2/2} H_n(x)$$

form an orthogonal set in $L^2(\mathbb{R})$.

When a sequence of non-zero vectors is orthogonal but not orthonormal, it is always possible to normalize the vectors and obtain an orthonormal sequence. It turns out that the same is possible if the original sequence of vectors in an inner product space is linearly independent, not necessarily orthogonal. The method of transforming such a sequence into an orthonormal sequence is called the **Gram-Schmidt orthonormalization process.**

Theorem 1.3.4. *(Debnath and Mikusiński, 1999) Let $f_1, f_2, \ldots, f_n$ be an orthonormal set of vectors in an inner product space $\mathcal{X}$. Then, for every $f \in \mathcal{X}$, we have*

$$\left\| f - \sum_{i=1}^{n} \langle f, f_i \rangle f_i \right\|^2 = \|f\|^2 - \sum_{i=1}^{n} |\langle f, f_i \rangle|^2 \tag{1.3.3}$$

and

$$\sum_{i=1}^{n} |\langle f, f_i \rangle|^2 \leq \|f\|^2.$$

The equality (1.3.3) can be generalized as follows :

$$\left\| f - \sum_{i=1}^{n} \alpha_i f_i \right\|^2 = \|f\|^2 - \sum_{i=1}^{n} |\langle f, f_i \rangle|^2 + \sum_{i=1}^{n} |\langle f, f_i \rangle - \alpha_i|^2 \tag{1.3.4}$$

for arbitrary complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$.

The expression (1.3.4) is minimized by taking $\alpha_i = \langle f, f_i \rangle$. This choice of $\alpha_i's$ minimizes

$$\left\| f - \sum_{i=1}^{n} \alpha_i f_i \right\|$$

and thus it provides the best approximation of f by a linear combination of vectors $f_1, f_2, \ldots, f_n$. This property of orthonormal sets is of fundamental importance for many approximation techniques.

Moreover, orthonormal sequences are weakly convergent to zero but not strongly convergent. If $\{f_i\}_{i=1}^{\infty}$ is an orthonormal sequence, then by letting $n \to \infty$ in (1.3.3), we obtain

$$\sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq \|f\|^2 \tag{1.3.5}$$

which implies that the series $\sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2$ converges for every $f \in \mathcal{X}$. In other words, the sequence $\{\langle f, f_i \rangle\}_{i=1}^{\infty}$ is an element of ℓ_2.

We can say that an orthonormal sequence in $\mathcal{X}$ induces a mapping from $\mathcal{X}$ into ℓ_2. The expansion

$$f \sim \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i \tag{1.3.6}$$

is called a **generalized Fourier series** of f. The scalars $\alpha_i = \langle f, f_i \rangle$ are called the **generalized Fourier coefficients** of f with respect to the orthogonal sequence $\{f_i\}_{i=1}^{\infty}$. As mentioned earlier, this set of coefficients gives the best approximation for any finite set of $f_i's$. In general, the series in (1.3.6) may not converge, however completeness of the space ensures the convergence, from the following theorem.

Theorem 1.3.5. *(Debnath and Mikusiński, 1999) Let $\{f_i\}_{i=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$, and let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of complex numbers. Then the series $\sum_{i=1}^{\infty} \alpha_i f_i$ converges if and only if $\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty$ and in that case*

$$\left\| \sum_{i=1}^{\infty} \alpha_i f_i \right\|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2.$$

The above theorem and the inequality (1.3.5) imply that in a Hilbert space $\mathcal{H}$, the series $\displaystyle\sum_{i=1}^{\infty} \langle f, f_i \rangle f_i$ converges for every $f \in \mathcal{H}$. However, it can happen that it converges to an element different from f.

Example 1.3.6. *Let $\mathcal{H} = L^2[-\pi, \pi]$, and let*

$$f_i(t) = \frac{1}{\sqrt{\pi}} \sin it \quad \textit{for} \;\; i = 1, 2, \ldots .$$

The sequence $\{f_i\}_{i=1}^{\infty}$ is an orthonormal set in $\mathcal{H}$. On other hand, for $f(t) = \cos t$, we have

$$\sum_{i=1}^{\infty} \langle f, f_i \rangle \, f_i(t) = 0 \neq \cos t.$$

Definition 1.3.7. *An orthonormal sequence $\{f_i\}_{i=1}^{\infty}$ in an inner product space $\mathcal{X}$ is said to be **complete** if for every $f \in \mathcal{X}$ we have*

$$\lim_{n \to \infty} \left\| f - \sum_{i=1}^{n} \langle f, f_i \rangle f_i \right\| = 0,$$

written as $f = \displaystyle\sum_{i=1}^{\infty} \langle f, f_i \rangle f_i$. In other words, $\{f_i\}_{i=1}^{\infty}$ is complete if

$$span\{f_1, f_2, \ldots\} = \left\{ \sum_{i=1}^{n} \alpha_i f_i : n \in \mathbb{N}, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C} \right\} \textit{ is dense in } \mathcal{X}.$$

Definition 1.3.8. *An orthonormal set E in an inner product space $\mathcal{X}$ is called an **orthonormal basis** if for every $f \in \mathcal{X}$ has a unique representation*

$$f = \sum_{i=1}^{\infty} \alpha_i f_i,$$

where $\alpha_i \in \mathbb{C}$ and $f_i{}'s$ are distinct elements of E.

Example 1.3.9. *The sequences $\{e_n\}_{n=1}^{\infty}$ and $\left\{\frac{e^{int}}{\sqrt{2\pi}}\right\}$, $\quad n = 0, \pm 1, \pm 2, \ldots$ form orthonormal bases for ℓ_2 and $L_2[-\pi, \pi]$ respectively.*

The following theorem gives important characterizations of orthonormal bases in Hilbert spaces.

Theorem 1.3.10. *(Christensen, 2003) Let $\{f_i\}_{i=1}^{\infty}$ be an orthonormal sequence in a Hilbert space $\mathcal{H}$. Then the following are equivalent :*

1. *$\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$.*
2. *$f = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i$, for every $f \in \mathcal{H}$. [Fourier expansion]*
3. *$\langle f, g \rangle = \sum_{i=1}^{\infty} \langle f, f_i \rangle \langle f_i, g \rangle$, for every $f, g \in \mathcal{H}$.*
4. *$\sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 = \|f\|^2$, for every $f \in \mathcal{H}$. [Parseval formula]*
5. *$\{f_i\}_{i=1}^{\infty}$ is complete.*
6. *$\langle f, f_i \rangle = 0$, for all $i \in \mathbb{N}$ implies $f = 0$.*

Corollary 1.3.11. *If $\{f_i\}_{i=1}^{\infty}$ is an orthonormal basis, then each $f \in \mathcal{H}$ has an unconditionally convergent expansion*

$$f = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i.$$

A basis $\{g_i\}_{i=1}^{\infty}$ is called a **dual basis** of a basis $\{f_i\}_{i=1}^{\infty}$ if

$$f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i \quad \text{for every} \quad f \in \mathcal{H}.$$

Note that the dual basis of an orthonormal basis is itself.

Definition 1.3.12. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. A mapping $T : \mathcal{X} \to \mathcal{Y}$ is called a **linear operator** if for any $f, g \in \mathcal{X}$ and any scalars α, β*

$$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g).$$

*A linear operator $f : \mathcal{X} \to \mathbb{K}$ is called a **linear functional** on $\mathcal{X}$.*

*In general, the linear operator may not be defined on the whole space $\mathcal{X}$ and it may be defined on a proper subspace of $\mathcal{X}$. In that case we denote the **domain of definition** (simply **domain**) by $D(T)$.*

We denote the set of all linear operators from $\mathcal{X}$ into $\mathcal{Y}$ by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{L}(\mathcal{X}, \mathcal{X}) = \mathcal{L}(\mathcal{X})$. Every $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ gives rise to two important subspaces namely, the **null space** $N(T)$, defined by

$$N(T) = \{f \in D(T) : Tf = 0\}$$

and the **range space** $R(T)$ defined as

$$R(T) = \{Tf : f \in D(T)\}.$$

Definition 1.3.13. *An operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is called **bounded** if there exists $c > 0$ such that*

$$\|Tf\| \leq c\, \|f\| \text{ for all } f \in \mathcal{X}.$$

In this case, the quantity

$$\|T\| := \sup \left\{ \frac{\|Tf\|}{\|f\|} : f \in D(T), f \neq 0 \right\} < \infty,$$

*is called the **norm** of T.*

*We say that T is **bounded below** if $\|f\| \leq c\|Tf\|$ for all $f \in \mathcal{X}$, for some $c > 0$.*

If a linear operator is continuous at any point, then it is continuous at every point and moreover it is bounded. The set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{B}(\mathcal{X}, \mathcal{X}) = \mathcal{B}(\mathcal{X})$. The **dual space** $\mathcal{X}^*$ of $\mathcal{X}$ is the set of linear continuous functionals on $\mathcal{X}$.

Theorem 1.3.14. *(Limaye, 1996) Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ be fixed. Then there exists a unique operator $V \in \mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ that satisfies*

$$g(Tf) = (Vg)f \quad \textit{for all } f \in \mathcal{X}, g \in \mathcal{Y}^*.$$

*V is called the **adjoint** of T, denoted by T^*. Moreover, $\|T\| = \|T^*\|$. If $\mathcal{X}, \mathcal{Y}$ are reflexive, then $T^{**} = T$.*

Theorem 1.3.15. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. The following statements hold :*

1. *T is bounded below if and only if T is injective and $R(T)$ is closed.*
2. *T is invertible if and only if T and T^* are bounded below.*
3. *T is bounded below if and only if T^* is surjective.*
4. *T is surjective if and only if T^* is bounded below.*
5. *T has a bounded inverse on $R(T)$ if and only if $T^* : \mathcal{Y}^* \to \mathcal{X}^*$ is surjective.*

Every bounded linear operator between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, with domain $D(T)$ can be extended continuously to the closure of $D(T)$. Hence it can be extended to the whole space $\mathcal{H}$ by defining 0 on $D(T)^{\perp}$. Thus without loss of generality we assume that a bounded linear operator is an everywhere defined operator.

Theorem 1.3.16 (Riesz representation theorem)**.** *If ϕ is a bounded linear functional on a Hilbert space $\mathcal{H}$, then there exists exactly one g in $\mathcal{H}$ such that for every $f \in \mathcal{H}$ we have $\phi(f) = \langle f, g \rangle$. Moreover, $\|\phi\| = \|g\|$.*

Definition 1.3.17. *A Hilbert space $\mathcal{H}$ is said to be **isomorphic** to a Hilbert space $\mathcal{K}$ if there exists a one-to-one linear mapping T from $\mathcal{H}$ onto $\mathcal{K}$ such that*

$$\langle Tf, Tg \rangle = \langle f, g \rangle \quad \textit{for every} \quad f, g \in \mathcal{H}.$$

*Such a mapping is called a **Hilbert space isomorphism** of $\mathcal{H}$ onto $\mathcal{K}$.*

Definition 1.3.18. *Let $T \in \mathcal{B}(\mathcal{H})$. The unique element V of $\mathcal{B}(\mathcal{H})$ which satisfies*

$$\langle Tf, g\rangle = \langle f, Vg\rangle \quad \text{for all } f, g \in \mathcal{H}$$

*is called the **adjoint** of T it is denoted by T^*.*

Definition 1.3.19. *Let $T \in \mathcal{B}(\mathcal{H})$. Then T is called **isometry** if $T^*T = I$; **unitary** if $T^*T = I = TT^*$; **self-adjoint** if $T^* = T$; **invertible** if T^{-1} exists and belongs to $\mathcal{B}(\mathcal{H})$.*

*A linear operator T on $\mathcal{H}$ is said to be **positive** if $\langle Tf, f\rangle \geq 0$, for all $f \in \mathcal{H}$. The set of all positive operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})^+$.*

Definition 1.3.20. *Let $T \in \mathcal{B}(\mathcal{H})^+$. An operator $V \in \mathcal{B}(\mathcal{H})$ is said to be **square root** of T if $V^2 = T$.*

Every operator $T \in \mathcal{B}(\mathcal{H})^+$ has a unique square root, denoted by $T^{1/2}$, which commutes with every operator in $\mathcal{B}(\mathcal{H})$ that commutes with T.

The following theorem characterizes all orthonormal bases for $\mathcal{H}$ starting with an orthonormal basis.

Theorem 1.3.21. *Let $\{f_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then the orthonormal basis for $\mathcal{H}$ are precisely of the form $\{Tf_i\}_{i=1}^{\infty}$, where $T : \mathcal{H} \to \mathcal{H}$ is an unitary operator.*

Before concluding this section, we give some interesting characterizations of a Hilbert space having a countable orthonormal basis.

Theorem 1.3.22. *Let $\mathcal{H}$ be a non-zero Hilbert space. Then the following are equivalent:*

1. *$\mathcal{H}$ has a countable orthonormal basis.*
2. *$\mathcal{H}$ is isomorphic to $\mathbb{K}^n$ for some n, or ℓ_2.*
3. *$\mathcal{H}$ is separable.*

A countable orthonormal basis for $\mathcal{H}$ is, in particular, a Schauder basis for $\mathcal{H}$. Thus every separable Hilbert space has a Schauder basis.

1.4 Riesz Bases

All orthonormal bases are characterized in terms of unitary operators on a single orthonormal basis. A basis is introduced by weakening the condition on the operator as follows.

Definition 1.4.1. *Let $\mathcal{H}$ be a separable Hilbert space. A **Riesz basis** for $\mathcal{H}$ is a family of the form $\{Te_i\}_{i=1}^{\infty}$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ is bijective (not necessarily unitary).*

Example 1.4.2. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and consider the sequence $\{f_i\}_{i=1}^{\infty} = \left\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \dots\right\}$. Then $\{f_i\}_{i=1}^{\infty}$ is a Schauder basis but not a Riesz basis.*

Of course, every orthonormal basis is a Riesz basis. In fact, one can characterize Riesz bases in terms of bases satisfying extra conditions, as shown in the following result.

Theorem 1.4.3. *(Christensen, 2003) A sequence $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$ if and only if it is an unconditional basis for $\mathcal{H}$ and*

$$0 < \inf_i \|f_i\| \leq \sup_i \|f_i\| < \infty.$$

The dual basis associated to a Riesz basis is also a Riesz basis:

Theorem 1.4.4. *(Christensen, 2003) If a sequence $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$, there exists a unique sequence $\{g_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ such that for every $f \in \mathcal{H}$,*

$$f = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i, \tag{1.4.7}$$

$\{g_i\}_{i=1}^{\infty}$ is also a Riesz basis, and $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are bi-orthogonal (dual of each other). Moreover, the series in (1.4.7) converges unconditionally for all $f \in \mathcal{H}$.

Proposition 1.4.5. *(Christensen, 2003) If $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$, there exist two constants $0 < \lambda \leq \mu < \infty$ such that for every $f \in \mathcal{H}$,*

$$\lambda \|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq \mu \|f\|^2. \tag{1.4.8}$$

*Here λ and μ are called **upper** and **lower** bounds respectively. The largest possible value for λ is $\frac{1}{\|T^{-1}\|^2}$, and the smallest possible value for μ is $\|T\|^2$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $\{f_i\}_{i=1}^{\infty} = \{Te_i\}_{i=1}^{\infty}$.*

The next theorem gives equality conditions for $\{f_i\}_{i=1}^{\infty}$ being a Riesz basis.

Theorem 1.4.6. *Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

1. *$\{f_i\}_{i=1}^{\infty}$ is a Riesz basis for $\mathcal{H}$.*

2. *$\{f_i\}_{i=1}^{\infty}$ is complete in $\mathcal{H}$, and there exist two constants $0 < \lambda \leq \mu < \infty$ such that for every finite scalar sequence $\{c_i\}$ of scalars,*
$$\lambda \sum_{i=1}^{\infty} |c_i|^2 \leq \left\| \sum_{i=1}^{\infty} c_i f_i \right\|^2 \leq \mu \sum_{i=1}^{\infty} |c_i|^2.$$

Example 1.4.7. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Given a sequence $\{a_i\}_{i=1}^{\infty}$ of complex numbers with $\sup_i |a_i| < 1$, we consider a family of vectors $\{f_i\}_{i=1}^{\infty}$ defined by $f_i = e_i + a_i e_{i+1}$, $i \in \mathbb{N}$. Then $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis with bounds $(1-a)^2$ and $(1+a)^2$.*

1.5 Frames in Hilbert Spaces

Definition 1.5.1. *Let $\mathcal{H}$ be a (finite or infinite dimensional) Hilbert space. A sequence $\{f_i\}_{i=1}^{\infty}$ is called a **frame** (an **ordinary frame**) for $\mathcal{H}$ if there are two positive constants $0 < \lambda \leq \mu < \infty$ satisfying*
$$\lambda \|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq \mu \|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1.5.9}$$

The constants λ and μ are called **lower** and **upper frame bounds**, respectively. They are not unique. The optimal upper frame bound is the infimum over all upper frame bounds and the optimal lower frame bound is the supremum over all lower frame bounds. Note that the optimal bounds are actually frame bounds.

If $\lambda = \mu$, then this frame is called **λ-tight frame**, and if $\lambda = \mu = 1$, then it is called **Parseval frame**. From (1.5.9), it can be said that a sequence of vectors

$\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a frame if the norms $\|f\|_{\mathcal{H}}$ and $\|\{\langle f, f_i\rangle\}_{i=1}^{\infty}\|_{\ell_2}$ are equivalent.

Definition 1.5.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$. We call the frame $\{f_i\}_{i=1}^{\infty}$ is* ***inexact*** *(or,* ***redundant***, ***over complete****) if there is at least one element f_j that can be removed from the frame, so that the set $\{f_i\}_{\substack{i=1\\i\neq j}}^{\infty}$ is again a frame for $\mathcal{H}$.*

Definition 1.5.3. *A sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ satisfying the upper inequality in (1.5.9) is called* ***Bessel sequence****. If $\{f_i\}_{i=1}^{\infty}$ is a frame for the closed space $\overline{span}\{f_i\}_{i=1}^{\infty}$, we call it a* ***frame sequence****.*

Example 1.5.4. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then $\{e_1, e_1, e_2, e_3, \ldots\}$ is a frame with frame bounds $\lambda = 1$ and $\mu = 2$. But the sequence $\left\{e_1, \frac{e_2}{2}, \frac{e_3}{3}, \ldots\right\}$ is not a frame.*

Example 1.5.5. *Every Riesz basis is a frame by the inequality (1.4.8). The sequence $\left\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \ldots\right\}$ is a frame but not a Riesz basis.*

It is not difficult to prove that $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis if and only if $\{f_i\}_{i=1}^{\infty}$ is a frame and for $\{c_i\}_{i=1}^{\infty} \in \ell_2$, "$\sum_{i=1}^{\infty} c_i f_i = 0$ implies all c_i's are zero."

Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence and $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. In order to analyze an element (signal) $f \in \mathcal{H}$, the **analysis operator** $U : \mathcal{H} \to \ell_2$ given by

$$Uf = \{\langle f, f_i\rangle\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} \langle f, f_i\rangle e_i \tag{1.5.10}$$

is applied. The associated **synthesis operator** (or, **pre-frame operator**), denoted by L, which provides a mapping from the **representation space** ℓ_2, to $\mathcal{H}$, is defined to be adjoint of the analysis operator U, $L = U^* : \ell_2 \to \mathcal{H}$, given by

$$L(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i \quad (OR) \quad L(e_i) = f_i,\ i = 1, 2, \ldots. \tag{1.5.11}$$

We obtain the **frame operator** $S : \mathcal{H} \to \mathcal{H}$ given by $S = LU$. That is,

$$Sf = LUf = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i. \tag{1.5.12}$$

In particular, for all $f \in \mathcal{H}$,

$$\langle f, Sf\rangle = \left\langle f, \sum_{i=1}^{\infty}\langle f, f_i\rangle f_i \right\rangle = \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2.$$

Hence the frame operator S for the frame $\{f_i\}_{i=1}^{\infty}$ is a positive, self-adjoint and invertible operator. And it has a unique positive square root, denoted by $S^{1/2}$. Since for each $f \in \mathcal{H}$,

$$\lambda\|f\|^2 \leq \langle Sf, f\rangle \leq \mu\|f\|^2,$$

it follows that $\{f_i\}_{i=1}^{\infty}$ is a frame with frame bounds λ and μ if and only if $\lambda I \leq S \leq \mu I$. So $\{f_i\}_{i=1}^{\infty}$ is a Parseval frame if and only if $S = I$.

We have

$$f = S^{-1}Sf = \sum_{i=1}^{\infty}\langle f, f_i\rangle S^{-1}f_i,$$

and

$$f = SS^{-1}f = \sum_{i=1}^{\infty}\langle S^{-1}f, f_i\rangle f_i = \sum_{i=1}^{\infty}\langle f, S^{-1}f_i\rangle f_i.$$

Hence the frame operator S leads to the important **frame reconstruction formula**

$$f = \sum_{i=1}^{\infty}\langle f, f_i\rangle S^{-1}f_i = \sum_{i=1}^{\infty}\langle f, S^{-1}f_i\rangle f_i, \text{ for all } f \in \mathcal{H}. \tag{1.5.13}$$

Note that the above two series converge unconditionally for all $f \in \mathcal{H}$ [(Christensen, 2003), Theorem 5.1.6]. The collection $\{\widetilde{f}_i = S^{-1}f_i\}_{i=1}^{\infty}$ is also a frame for $\mathcal{H}$ and $\{\widetilde{f}_i\}_{i=1}^{\infty}$ is a dual frame of $\{f_i\}_{i=1}^{\infty}$. The sequence $\{S^{-1}f_i\}_{i=1}^{\infty}$ is called the **canonical dual frame** of $\{f_i\}_{i=1}^{\infty}$.

The canonical dual frame $\{S^{-1}f_i\}_{i=1}^{\infty}$ is the one having the least square property among all dual frames $\{\widetilde{f}_i\}_{i=1}^{\infty}$, that is, we have

$$\sum_{i=1}^{\infty} |\langle f, S^{-1}f_i\rangle|^2 \leq \sum_{i=1}^{\infty} |\langle f, \widetilde{f}_i\rangle|^2, \text{ for all } f \in \mathcal{H}.$$

If $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$, then $\{f_i\}_{i=1}^{\infty}$ has a unique dual frame if and only if it is exact (and in this case the unique dual is the canonical dual frame) (Han et al., 2008).

If $\{f_i\}_{i=1}^{\infty}$ is a frame sequence, by replacing $\mathcal{H}$ in (1.5.10 - 1.5.12) with $\overline{span}\{f_i\}_{i=1}^{\infty}$, the associated operators are still obtained. The corresponding synthesis operator and frame operator are respectively, surjective and invertible.

In this case, the following reconstruction formula is satisfied :

$$f = \sum_{i=1}^{\infty} \langle f, f_i \rangle S^{-1} f_i = \sum_{i=1}^{\infty} \langle f, S^{-1} f_i \rangle f_i, \text{ for all } f \in \overline{span}\{f_i\}_{i=1}^{\infty}.$$

Definition 1.5.6. *Two sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ in a Hilbert space are* ***equivalent*** *if there is an invertible linear operator T between their spans with $Tf_i = g_i$ for all $i \in \mathbb{N}$.*

Given any frame $\{f_i\}_{i=1}^{\infty}$, $\{S^{-1/2} f_i\}_{i=1}^{\infty}$ is a Parseval frame equivalent to $\{f_i\}_{i=1}^{\infty}$.

Frames can be described as images of an orthonormal basis by bounded linear operators in an infinite dimensional Hilbert space. They can be classified by the following result.

Theorem 1.5.7. *(Casazza and Kovačević, 2003) Let $\{e_i\}_{i=1}^{\infty}$ be an arbitrary infinite orthonormal basis for $\mathcal{H}$. The frames $\{f_i\}_{i=1}^{\infty}$ for $\mathcal{H}$ are the precisely the families $\{Te_i\}_{i=1}^{\infty}$, where $T \in \mathcal{B}(\mathcal{H})$ is surjective.*

This operator T is just the composition of the analysis operator of the orthonormal basis and the synthesis operator of the frame.

In the finite dimensional case, frames are equivalent to spanning sets. Here frames are the only feasible generalization of basis, if reconstruction is wanted. To be able to work with numerical methods, data and operators have to be discretized. Applications and algorithms always work with finite dimensional data. The typical properties of frames can be understood easily in the context of finite dimensional vector spaces.

For finite dimensional applications, Theorem 1.5.7 can be reformulated as:

Theorem 1.5.8. *The frames with m elements in $\mathbb{C}^n$ are exactly the images of an orthonormal basis in $\mathbb{C}^m$ by a surjective linear operator.*

Hence it is easy to recognize that the columns of "matrices with full rank" correspond exactly to frames. We have an infinite frame in any finite dimensional space.

Example 1.5.9. *Consider a basis $\{e_i\}_{i=1}^{n}$ in $\mathbb{C}^n$. Let $e_i^{(\ell)} = \dfrac{e_i}{\ell}$, for $\ell = 1, 2, \ldots$. Then $\left\{e_i^{(\ell)}\right\}_{i,\ell}$ is an infinite tight frame in $\mathbb{C}^n$.*

It is well known that for any orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of a Hilbert space $\mathcal{H}$, $\sum_{i=1}^{\infty} \|e_i\| < \infty$ if and only if the dimension of the space is finite. But it is not true with frame, as in the above example,

$$\sum_{i,\ell} \|e_i^{(\ell)}\| = \sum_{\ell=1}^{\infty} \sum_{i=1}^{n} \left\| \frac{e_i}{\ell} \right\| = \sum_{\ell=1}^{\infty} \frac{n}{\ell} = \infty.$$

But taking the sequence sum of the norms of frame elements of $\mathcal{H}$, there is an equivalent condition for $\mathcal{H}$ being finite dimensional:

Theorem 1.5.10. *(Balazs, 2008) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis and $\{f_i\}_{i=1}^{\infty}$ be a frame of a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

1. $\sum_{i=1}^{\infty} \|e_i\|^2 < \infty$.
2. $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$.
3. *the space $\mathcal{H}$ is finite dimensional.*

There are several generalizations of frame – all of these generalizations have been proved to be useful in many applications. Găvruţa (Găvruţa, 2012) recently presented a generalization of ordinary frames with a bounded linear operator K, called K-frames, when working on atomic systems for operators. Characterizations, methods of constructing K-frames and a class of operators in $\mathcal{B}(\mathcal{H})$ associated with a given atomic system are discussed in Chapter 2. Operators that preserve K-frames and generating new K-frames from old ones by taking sums have been discussed. Moreover, a close relation between K-frames and quotient operators is established using through operator-theoretic results on quotient operators in Chapter 3.

1.6 Frames in Banach Spaces

Extensions of the concept frame to Banach spaces were introduced during the last years and they became topic of investigation for many mathematicians. The properties of Hilbert frames usually do not transfer automatically to Banach spaces.

In Hilbert spaces, the norm equivalence hypothesis leads immediately to the reconstruction formula (1.5.13). This does not hold in Banach spaces in general.

Gröchenig (Gröchenig, 1991) generalized frames to Banach spaces and called them **atomic decompositions**. The main feature of frames that Gröchenig was trying to capture in a general Banach space was the unique association of a vector in a Hilbert space with the natural set of frame coefficients. Gröchenig also defined a more general notion for Banach spaces called a **Banach frame**. Atomic decompositions and Banach frames are defined with respect to certain sequence spaces.

A sequence space $\mathcal{X}_d$ is called a **BK-space** if it is a Banach space and the coordinate functionals are continuous on $\mathcal{X}_d$. That is, $x_n = \{\alpha_i^{(n)}\}_{i=1}^{\infty}$, $x = \{\alpha_i\}_{i=1}^{\infty}$ are elements in $\mathcal{X}_d$ and $\lim\limits_{n\to\infty} x_n = x$ imply that $\lim\limits_{n\to\infty} \alpha_i^{(n)} = \alpha_i$, for each $i = 1, 2, \ldots$.

If the canonical vectors form a Schauder basis for $\mathcal{X}_d$, then $\mathcal{X}_d$ is called a **CB-space** and its canonical basis is denoted by $\{e_i\}_{i=1}^{\infty}$. If $\mathcal{X}_d$ is reflexive and a CB-space, then $\mathcal{X}_d$ is called an **RCB-space**. Also the dual of $\mathcal{X}_d$ is denoted by $\mathcal{X}_d^*$. When $\mathcal{X}_d^*$ is a CB-space, then its canonical basis is denoted by $\{e_i^*\}_{i=1}^{\infty}$.

Definition 1.6.1. *(Gröchenig, 1991) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence of elements from $\mathcal{X}^*$ and $\{f_i\}_{i=1}^{\infty}$ be a sequence of elements of $\mathcal{X}$. If the following statements hold :*

1. *$\{g_i(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$, for each $f \in \mathcal{X}$;*

2. *the norms $\|f\|_{\mathcal{X}}$ and $\|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d}$ are equivalent, that is, there exist two constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in \mathcal{X}$*

$$\lambda\|f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu\|f\|_{\mathcal{X}}$$

3. *$f = \sum\limits_{i=1}^{\infty} g_i(f)\ f_i$, for each $f \in \mathcal{X}$;*

*then $(\{g_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty})$ is called an **atomic decomposition of $\mathcal{X}$ with respect to $\mathcal{X}_d$**. The constants λ and μ are called **lower** and **upper atomic bounds** for $(\{g_i\}_{i=1}^{\infty}, \{f_i\}_{i=1}^{\infty})$ respectively.*

Definition 1.6.2. *(Gröchenig, 1991) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence of elements from $\mathcal{X}^*$ and $S : \mathcal{X}_d \to \mathcal{X}$ be given. If the following statements hold :*

1. *$\{g_i(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$, for each $f \in \mathcal{X}$;*

2. *the norms $\|f\|_{\mathcal{X}}$ and $\|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d}$ are equivalent, that is, there exist two constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in \mathcal{X}$*

$$\lambda\|f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu\|f\|_{\mathcal{X}}$$

3. *S is a bounded linear operator from $\mathcal{X}_d$ to $\mathcal{X}$, and*

$$S(\{g_i(f)\}_{i=1}^{\infty}) = f, \text{for each } f \in \mathcal{X} \tag{1.6.14}$$

*then $(\{g_i\}_{i=1}^{\infty}, S)$ is a **Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$**. The mapping S is called the **reconstruction operator** (or the **pre-frame operator**). The constants λ, μ are called **lower** and **upper frame bounds** for $(\{g_i\}_{i=1}^{\infty}, S)$.*

The Banach frame $(\{g_i\}_{i=1}^{\infty}, S)$ is called **tight** if $\lambda = \mu$ and **normalized tight** if $\lambda = \mu = 1$. If the removal of one g_i renders the collection $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ no longer a Banach frame for $\mathcal{X}$, then $(\{g_i\}_{i=1}^{\infty}, S)$ is called an **exact Banach frame**.

Note that (1.6.14) can be considered as some kind of "generalized reconstruction formula" in the sense that it tells how to come back to $f \in \mathcal{X}$ based on the coefficients $\{g_i(f)\}_{i=1}^{\infty}$. The condition, however, does not imply reconstruction via an infinite series.

It turns out there is a natural relationship between these two definitions. Namely, a Banach frame is an atomic decomposition if and only if the unit vectors form a basis for the space $\mathcal{X}_d$. This result is stated formally in the next proposition.

Proposition 1.6.3. *(Casazza et al., 1999) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence of elements from $\mathcal{X}^*$ and $S : \mathcal{X}_d \to \mathcal{X}$ be given. Let $\{e_i\}_{i=1}^{\infty}$ be the unit vectors in $\mathcal{X}_d$, defined by $e_i(j) = \delta_{ij}$. Then the following are equivalent:*

1. *$(\{g_i\}_{i=1}^{\infty}, S)$ is a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ and $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis for $\mathcal{X}_d$.*

2. *$(\{g_i\}_{i=1}^{\infty}, \{S(e_i)\}_{i=1}^{\infty})$ is an atomic decomposition for $\mathcal{X}$ with respect to $\mathcal{X}_d$.*

Definition 1.6.4. *Let $\mathcal{X}$ be a Banach space. A sequence of vectors $\{g_i\}_{i=1}^{\infty}$ from $\mathcal{X}^*$ is said to be **total** on $\mathcal{X}$ if "$g_i(f) = 0$ for all $i \in \mathbb{N}$ implies $f = 0$."*

Every orthonormal basis of a separable Hilbert space is total.

Proposition 1.6.5. *(Singer, 1981) Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ be total on a Banach space $\mathcal{X}$. The linear space*

$$\mathcal{Z}_d := \Big\{\{g_i(f)\}_{i=1}^{\infty} : f \in \mathcal{X}\Big\}$$

with the norm $\|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{Z}_d} := \|f\|_{\mathcal{X}}$ is a BK-space, isometrically isomorphic to $\mathcal{X}$.

Proposition 1.6.6. *(Casazza et al., 2005) Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ be total on a Banach space $\mathcal{X}$. Then there exists a bounded linear operator $S : \mathcal{Z}_d \to \mathcal{X}$ such that $(\{g_i\}_{i=1}^{\infty}, S)$ is a Banach frame for $\mathcal{X}$ with respect to $\mathcal{Z}_d$.*

The following is an example of a Banach frame for a Hilbert space, which is not frame.

Example 1.6.7. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$. The sequence $\{e_i + e_{i+1}\}_{i=1}^{\infty}$ is not a frame for $\mathcal{H}$. But by Proposition 1.6.6, it is a Banach frame for $\mathcal{H}$ with respect to the BK-space*

$$\begin{aligned} \mathcal{Z}_d &= \Big\{\{\langle h, e_i + e_{i+1}\rangle\} : h \in \mathcal{H}\Big\} \\ &= \Big\{\{c_i + c_{i+1}\} : \{c_i\} \in \ell_2\Big\} \end{aligned}$$

with the norm $\|\{c_i + c_{i+1}\}\|_{\mathcal{Z}_d} = \|\{c_i\}\|_{\ell_2}$.

Peter G Casazza (Casazza et al., 1999) studied the relationship between frames and atomic decompositions and the various forms of the approximation properties. This study of Banach space frames answers the questions concerning which Banach spaces have frames or atomic decompositions.

1. Every separable Banach space has a normalized tight Banach frame. There are Banach spaces which do not have frames or atomic decompositions defined on them. Because they cannot be complemented in a space with a basis.

2. There are Banach spaces which have atomic decompositions, but no frames. The simplest examples for this are $L^1[0,1]$ or $C[0,1]$. Both these spaces have bases and hence atomic decompositions, but neither of them embeds as a complemented subspace of a Banach space with an unconditional basis.

3. There is a Banach space $\mathcal{X}$ with a frame so that $\mathcal{X}^*$ fails to have an atomic decomposition : The space ℓ_1 has an unconditional basis (hence a frame) while its dual is ℓ_∞, non-separable and hence ℓ_1^* has no atomic decompositions.

4. There is a Banach space $\mathcal{X}$ so that $\mathcal{X}^*$ has a frame, but $\mathcal{X}$ does not have a frame: Any pre-dual of ℓ_1 which is complemented in a space with an unconditional basis must be isomorphic to c_0. So if $\mathcal{X}$ is a pre-dual of ℓ_1 is not isomorphic to c_0, then $\mathcal{X}$ is a Banach space without any frames whose dual has an conditional basis (hence a frame).

A frame for a Banach space $\mathcal{X}$ was defined as a sequence of elements in $\mathcal{X}^*$, not of elements in the original space $\mathcal{X}$. However, semi-inner products for Banach spaces make possible the development of inner product type arguments in Banach spaces. A family of local atoms in a Banach space has been introduced and it has been generalized to an atomic system for operators in Banach spaces, which has been further led to introduce new frames for operators by Dastourian and Janfada, by making use of semi-inner products. Unlike the traditional way of considering sequences in the dual space, sequences in the original space are considered to study them. Appropriate changes have been made in the definitions of atomic systems and frames for operators to fit them for sequences in the dual space without using semi-inner products so that the new notion for Banach spaces can be thought of a generalization of Banach frames. With some crucial assumptions, we show in Chapter 4 that frames for operators in Banach spaces share nice properties of frames for operators in Hilbert spaces.

Throughout the thesis $\mathcal{H}$ and $\mathcal{K}$ will denote separable Hilbert spaces, $\mathcal{X}$ and $\mathcal{Y}$ will denote separable Banach spaces, $\mathcal{X}^*$ the dual space of $\mathcal{X}$, $\mathcal{X}_d$ a Banach sequence space and $\mathcal{X}_d^*$ the dual of $\mathcal{X}_d$. The analysis, synthesis and frame operators are denoted by U, L and S respectively. All spaces are nontrivial ; operators are non-zero.

Chapter 2

K-FRAMES IN HILBERT SPACES

There are several generalizations of frame – all of these generalizations have been proved to be useful in many applications. Some generalizations of frame significance have been presented such as fusion frames (frame of subspaces) (Casazza and Kutyniok, 2004), generalized frames (Sun, 2006; Xiao et al., 2015), continuous frames (Fornasier and Rauhut, 2005) and continuous fusion frames (Faroughi and Ahmadi, 2010). In the sequel, we discuss results on one such generalization of frames, called *K-frames*.

The notion of K-frames have been recently introduced by Laura Găvruţa (Găvruţa, 2012) to study the atomic systems with respect to a bounded operator K in Hilbert spaces. It is known that K-frames are more general than ordinary frames – in the sense that the lower frame bound only holds for the elements in the range of K.

Because of the higher generality of K-frames, many properties for ordinary frames may not hold for K-frames. Several methods to construct K-frames and the stability of perturbations for the K-frames have been discussed in (Xiao et al., 2013). In this chapter, we construct a frame sequence for the closed subspace $R(K)$ (the range of K) from an atomic system for a closed range operator K. In the end, we find a class of bounded operators in which a given Bessel sequence is an atomic system for every member in the class.

2.1 Atomic Systems

Definition 2.1.1. *(Găvruţa, 2012) Let $K \in \mathcal{B}(\mathcal{H})$. A sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ is called an* ***atomic system*** *for K, if the following conditions are satisfied :*

1. *$\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence ;*

2. *there exists $c > 0$ such that for every $f \in \mathcal{H}$ there exists $a_f = \{a_i\}_{i=1}^{\infty} \in \ell_2$ such that*
$$\|a_f\|_{\ell_2} \leq c\|f\| \text{ and } Kf = \sum_{i=1}^{\infty} a_i f_i.$$

Every operator $K \in \mathcal{B}(\mathcal{H})$ has an atomic system (Găvruţa, 2012). One may ask whether every Bessel sequence $\{f_i\}_{i=1}^{\infty}$ has an operator K which makes $\{f_i\}_{i=1}^{\infty}$ an atomic system for K. The answer is in the affirmative by the following proposition.

Proposition 2.1.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$. Then $\{f_i\}_{i=1}^{\infty}$ is an atomic system for the frame operator S.*

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$, we have a bounded operator
$$L : \ell_2 \to \mathcal{H} \text{ defined by } L(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i,$$
with bound μ and its adjoint
$$U : \mathcal{H} \to \ell_2 \text{ defined by } Uf = \{\langle f, f_i\rangle\}_{i=1}^{\infty}.$$
Let $S = LU$. Then
$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i \tag{2.1.1}$$
is a bounded operator on $\mathcal{H}$. Let $a_f = \{a_i\}_{i=1}^{\infty} = \{\langle f, f_i\rangle\}_{i=1}^{\infty} \in \ell_2$.
Now
$$\|a_f\|_{\ell_2} = \|\{\langle f, f_i\rangle\}_{i=1}^{\infty}\|_{\ell_2} = \left(\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2\right)^{1/2} \leq \sqrt{\mu}\|f\| \tag{2.1.2}$$
Since $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, from the relations (2.1.1) and (2.1.2), $\{f_i\}_{i=1}^{\infty}$ is an atomic system for S. □

Theorem 2.1.3. *(Găvruţa, 2012) Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{H}$ and $K \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent:*

1. $\{f_i\}_{i=1}^{\infty}$ *is an atomic system for* K *;*

2. *there exist two constants* $0 < \lambda \le \mu < \infty$ *such that*

$$\lambda \|K^* f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \le \mu \|f\|^2, \text{ for all } f \in \mathcal{H}$$

3. *there exists a Bessel sequence* $\{g_i\}_{i=1}^{\infty}$ *such that* $Kf = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i$.

Each atomic system is associated with a bounded operator on $\mathcal{H}$. We now find a class of bounded linear operators in which a given Bessel sequence is an atomic system for every member in the class.

Theorem 2.1.4. *Let* $K_1, K_2 \in \mathcal{B}(\mathcal{H})$*. If* $\{f_i\}_{i=1}^{\infty}$ *is an atomic system for* K_1 *and* K_2*, and* α, β *are scalars, then* $\{f_i\}_{i=1}^{\infty}$ *is an atomic system for* $\alpha K_1 + \beta K_2$ *and* $K_1 K_2$.

Proof. It is given that $\{f_i\}_{i=1}^{\infty}$ is an atomic system for K_1 and K_2, then there are positive constants $0 < \lambda_n \le \mu_n < \infty$ $(n = 1, 2)$ such that

$$\lambda_n \|K_n^* f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \le \mu_n \|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{2.1.3}$$

By simple calculations, now we have

$$\begin{aligned} \|(\alpha K_1 + \beta K_2)^* f\|^2 &\le |\alpha|^2 \|{K_1}^* f\|^2 + |\beta|^2 \|{K_2}^* f\|^2 \\ &\le |\alpha|^2 \left[\frac{1}{\lambda_1} \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \right] + |\beta|^2 \left[\frac{1}{\lambda_2} \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \right] \\ &= \left[\frac{|\alpha|^2}{\lambda_1} + \frac{|\beta|^2}{\lambda_2} \right] \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2. \end{aligned}$$

It follows that

$$\frac{\lambda_1 \lambda_2}{\lambda_2 |\alpha|^2 + \lambda_1 |\beta|^2} \|(\alpha K_1 + \beta K_2)^* f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2.$$

Hence $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition. And from inequalities (2.1.3), we get

$$\sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \le \left(\frac{\mu_1 + \mu_2}{2} \right) \|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Therefore $\{f_i\}_{i=1}^{\infty}$ is an atomic system for $\alpha K_1 + \beta K_2$.

Now for each $f \in \mathcal{H}$, we have

$$\|(K_1K_2)^*f\|^2 = \|K_2{}^*K_1{}^*f\|^2 \leq \|K_2{}^*\|^2\|K_1{}^*f\|^2.$$

Since $\{f_i\}_{i=1}^{\infty}$ is an atomic system for K_1,

$$\frac{\|(K_1K_2)^*f\|^2}{\|K_2{}^*\|^2} \leq \|K_1{}^*f\|^2 \leq \frac{1}{\lambda_1}\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq \frac{\mu_1}{\lambda_1}\|f\|^2.$$

This implies that

$$\frac{\lambda_1}{\|K_2{}^*\|^2}\|(K_1K_2)^*f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq \mu_1\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Therefore $\{f_i\}_{i=1}^{\infty}$ is an atomic system for K_1K_2. $\square$

Corollary 2.1.5. *If $\{f_i\}_{i=1}^{\infty}$ is an atomic system for $\mathcal{A}$, where $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, then $\{f_i\}_{i=1}^{\infty}$ is an atomic system for any operator in the subalgebra generated by $\mathcal{A}$.*

Corollary 2.1.6. *If $\{f_i\}_{i=1}^{\infty}$ is an atomic system for a normal operator K, then $\{f_i\}_{i=1}^{\infty}$ is an atomic system for any operator in the subalgebra generated by K and K^*.*

Definition 2.1.7. *(Limaye, 1996) Let $\mathcal{H}$ be a Hilbert space, and suppose that $T \in \mathcal{B}(\mathcal{H})$ has a closed range. Then there exists an operator $T^\dagger \in \mathcal{B}(\mathcal{H})$ for which*

$$N(T^\dagger) = R(T)^\perp, \quad R(T^\dagger) = N(T)^\perp, \quad TT^\dagger f = f, \quad f \in R(T).$$

*We call the operator $T^\dagger$ the **pseudo-inverse** of T. This operator is uniquely determined by these properties. In fact, if T is invertible, then we have $T^{-1} = T^\dagger$.*

Let $T \in \mathcal{B}(\mathcal{H})$ have a closed range. Then the following holds:

1. The orthogonal projection of $\mathcal{H}$ onto $R(T)$ is given by $TT^\dagger$.

2. The orthogonal projection of $\mathcal{H}$ onto $R(T^\dagger)$ is given by $T^\dagger T$.

3. T^* has a closed range, and $(T^*)^\dagger = (T^\dagger)^*$.

4. On $R(T)$, the operator $T^\dagger$ is given explicitly by $T^\dagger = T^*(TT^*)^{-1}$.

Traditionally, frames have been studied for the whole space or for the closed subspace. A frame sequence for the closed subspace $R(K)$ has been constructed from an atomic system for a closed range operator K.

Theorem 2.1.8. *Let $\{f_i\}_{i=1}^{\infty}$ be an atomic system for an operator K having a closed range. Then there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ such that $\{(K^{\dagger}|_{R(K)})^* g_i\}_{i=1}^{\infty}$ is a frame sequence for $R(K)$.*

Proof. As $\{f_i\}_{i=1}^{\infty}$ is an atomic system, by Theorem 2.1.3, there exists a Bessel sequence $\{g_i\}_{i=1}^{\infty}$ such that

$$Kf = \sum_{i=1}^{\infty} \langle f, g_i \rangle f_i. \tag{2.1.4}$$

Since $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence, there exists $\mu > 0$ such that

$$\sum_{i=1}^{\infty} |\langle f, g_i \rangle|^2 \leq \mu \|f\|^2, \text{ for every } f \in \mathcal{H}.$$

Hence

$$\sum_{i=1}^{\infty} |\langle f, K^{\dagger^*} g_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle K^{\dagger} f, g_i \rangle|^2 \leq \gamma \|f\|^2, \text{ where } \gamma = \mu \|K^{\dagger}\|^2. \tag{2.1.5}$$

Using the definition of pseudo-inverse and equation (2.1.4), for any $f \in R(K)$,

$$f = KK^{\dagger} f = \sum_{i=1}^{\infty} \langle K^{\dagger} f, g_i \rangle f_i = \sum_{i=1}^{\infty} \langle f, K^{\dagger^*} g_i \rangle f_i.$$

Now

$$\begin{aligned}
\|f\|^4 = |\langle f, f \rangle|^2 &= \left(\langle f, \sum_{i=1}^{\infty} \langle f, K^{\dagger^*} g_i \rangle f_i \rangle \right)^2 \\
&= \left(\sum_{i=1}^{\infty} \overline{\langle f, K^{\dagger^*} g_i \rangle} \langle f, f_i \rangle \right)^2 \\
&\leq \sum_{i=1}^{\infty} |\langle f, K^{\dagger^*} g_i \rangle|^2 \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \\
&\leq \sum_{i=1}^{\infty} |\langle f, K^{\dagger^*} g_i \rangle|^2 \mu \|f\|^2.
\end{aligned}$$

Therefore

$$\frac{1}{\mu} \|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, K^{\dagger^*} g_i \rangle|^2, \text{ for all } f \in R(K).$$

That is

$$\mu_1\|f\|^2 \le \sum_{i=1}^{\infty} |\langle f, K^{\dagger^*} g_i\rangle|^2, \quad \text{where } \mu_1 = \frac{1}{\mu} \tag{2.1.6}$$

Thus from equations (2.1.5) and (2.1.6), we get that $\{(K^\dagger|_{R(K)})^* g_i\}_{i=1}^{\infty}$ is a frame sequence for $R(K)$. □

2.2 Characterizations and Generating New K-frames

Definition 2.2.1. *(Găvruţa, 2012) Let $K \in \mathcal{B}(\mathcal{H})$. A sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ is called a K-**frame** for $\mathcal{H}$ if there exist two constants $0 < \lambda \le \mu < \infty$ such that*

$$\lambda\|K^* f\|^2 \le \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \le \mu\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

*We call λ, μ the **lower** and **upper bounds** for the K-frame $\{f_i\}_{i=1}^{\infty}$ respectively. If the above inequalities hold only for $f \in \overline{span}\{f_i\}_{i=1}^{\infty}$, then $\{f_i\}_{i=1}^{\infty}$ is said to be a K-**frame sequence**.*

If K is equal to I, the identity operator on $\mathcal{H}$, then K-frames and K-frame sequences are just ordinary frames and frame sequences, respectively.

Definition 2.2.2. *(Ding et al., 2013) Let $K \in \mathcal{B}(\mathcal{H})$. A sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ is said to be a **tight** K-**frame** with bound λ if*

$$\lambda\|K^* f\|^2 = \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2, \text{ for all } f \in \mathcal{H}. \tag{2.2.7}$$

*When $\lambda = 1$, it is called a **Parseval** K-**frame**.*

There are many essential differences between K-frames and ordinary frames due to the involved operator K. For instance, we know that that an important equivalent characterization of ordinary frames is that the corresponding synthesis operators are bounded and surjective. But for K-frames, it is required that the corresponding synthesis operators are bounded and the range of K is included in the ranges of the synthesis operators.

Theorem 2.2.3. *(Găvruţa, 2012) Let $\mathcal{H}$ be a Hilbert space. Then $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$ if and only if there exists a bounded operator $L : \ell_2 \to \mathcal{H}$ such that $f_i = Le_i$ and $R(K) \subseteq R(L)$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for ℓ_2.*

Every frame generates a K-frame by forming image of the frame elements under K: $\{f_i\}_{i=1}^{\infty}$ is an ordinary frame for $\mathcal{H}$, then $\{Kf_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$ (Xiao et al., 2013). But we prove that every frame is a K-frame, for any $K \in \mathcal{B}(\mathcal{H})$.

Theorem 2.2.4. *Let $K \in \mathcal{B}(\mathcal{H})$ with $\|K\| \geq 1$. Then every ordinary frame is a K-frame for $\mathcal{H}$.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$. Then there exist two constants $0 < \lambda \leq \mu < \infty$ such that

$$\lambda\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq \mu\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{2.2.8}$$

For $K \in \mathcal{B}(\mathcal{H})$, we have $\|K^*f\| \leq \|K\|\|f\|$, for all $f \in \mathcal{H}$. This implies that $\frac{1}{\|K\|}\|K^*f\| \leq \|f\|$, for all $f \in \mathcal{H}$. From the inequality (2.2.8), we have

$$\frac{\lambda}{\|K\|^2}\|K^*f\|^2 \leq \lambda\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq \mu\|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Therefore $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. $\square$

Given $K \in \mathcal{B}(\mathcal{H})$, every ordinary frame is a K-frame but converse need not be true. The following example illustrates that there exists a sequence $\{f_i\}_{i=1}^{\infty}$ which is a K-frame but not a frame for $\mathcal{H}$.

Example 2.2.5. *Suppose that $\mathcal{H} = \mathbb{C}^3$, $\{e_i\}_{i=1}^{3} = \{e_1, e_2, e_3\}$, where e_1, e_2, e_3 is an orthonormal basis for $\mathcal{H}$. Define $K \in \mathcal{B}(\mathcal{H})$ as follows $K : \mathcal{H} \to \mathcal{H}$,*

$$Ke_1 = e_1, Ke_2 = e_1, Ke_3 = e_2.$$

Let $f_i = Ke_i$, for $i = 1, 2, 3$ Obviously, $\{e_i\}_{i=1}^{3}$ is an ordinary frame for $\mathcal{H}$. Then $\{f_i\}_{i=1}^{3}$ is a K-frame for $\mathcal{H}$ but not a frame for $\mathcal{H}$ because $\overline{span}\{f_i\} \neq \mathcal{H}$.

Given a closed range operator $K \in \mathcal{B}(\mathcal{H})$, every frame sequence is a K-sequence (Zhong and Yong, 2016). But the converse is not true in general.

Example 2.2.6. *(Zhong and Yong, 2016) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and define $K : \mathcal{H} \to \mathcal{H}$ as follows :*

$$Kf = \sum_{i=1}^{\infty} \langle f, e_{2i} \rangle e_{2i}.$$

Clearly, K is well-defined, bounded linear operator with

$$K^* f = \sum_{i=1}^{\infty} \langle f, e_{2i} \rangle e_{2i}.$$

Define $f_i = e_i$ when i is even and $f_i = \frac{e_i}{i}$ when i is odd. Then $\{f_i\}_{i=1}^{\infty}$ is a K-frame sequence but it is not a frame sequence.

For a fixed $K \in \mathcal{B}(\mathcal{H})$, it is obvious that every K-frame for $\mathcal{H}$ is a K-frame sequence, but not conversely. The following proposition gives a condition under which a K-frame sequence is a K-frame.

Example 2.2.7. *(Zhong and Yong, 2016) Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and let N be a fixed natural number. Define $K \in \mathcal{B}(\mathcal{H})$ as follows : $Ke_i = ie_i$ for $1 \leq i \leq N$ and $Ke_i = e_i$ for $i > N$. Then $\{e_i\}_{i=N+1}^{\infty}$ is not a K-frame for $\mathcal{H}$ but it is a K-frame for $\overline{span}\{e_i\}_{i=N+1}^{\infty}$.*

Proposition 2.2.8. *(Zhong and Yong, 2016) Let $K \in \mathcal{B}(\mathcal{H})$ be a closed range. Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame sequence in $\mathcal{H}$ with bounds λ and μ. If $R(K) \subseteq \overline{span}\{f_i\}_{i=1}^{\infty}$, then $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$.*

Characterizations for Bessel sequences to become K-frames are given after the statement of famous Douglas' factorization theorem which finds a relation between range inclusion, majorization and factorization of bounded operators on Hilbert spaces.

Definition 2.2.9. *(Barnes, 2005) Assume that $T, V \in \mathcal{B}(\mathcal{H})$. Then V **majorizes** T if there exists $\lambda > 0$ such that $\|Tf\| \leq \lambda \|Vf\|$ for all $f \in \mathcal{H}$.*

Theorem 2.2.10 (Douglas' factorization theorem). *(Douglas, 1966) Let $\mathcal{H}$ be a Hilbert space and $T, V \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

1. *$R(T) \subseteq R(V)$.*

2. *$TT^* \leq \alpha^2 VV^*$ for some $\alpha > 0$ [V^* majorizes T^*].*

3. *$T = VW$ for some $W \in \mathcal{B}(\mathcal{H})$.*

Theorem 2.2.11. *(Xiao et al., 2013) Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$ and $K \in \mathcal{B}(\mathcal{H})$. Then $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$ if and only if there exists $\lambda > 0$ such that $S \geq \lambda KK^*$, where S is the frame operator for $\{f_i\}_{i=1}^{\infty}$.*

Theorem 2.2.12. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$. Then $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$ if and only if $K = S^{1/2}W$, for some $W \in \mathcal{B}(\mathcal{H})$.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a K-frame, by Theorem 2.2.11, there exist two constants $0 < \lambda \leq \mu < \infty$ such that

$$\lambda KK^* \leq S^{1/2}S^{1/2^*}.$$

Then by definition of inner product, for each $f \in \mathcal{H}, \|K^*f\|^2 \leq \lambda^{-1}\|S^{1/2^*}f\|^2$. Therefore $S^{1/2^*}$ majorizes K^*. Then by Douglas' factorization theorem,
$K = S^{1/2}W$, for some $W \in \mathcal{B}(\mathcal{H})$.

On the other hand, let $K = S^{1/2}W$, for some $W \in \mathcal{B}(\mathcal{H})$. Then by Douglas' factorization theorem, $S^{1/2}$ majorizes K^*. Then there is a positive number λ such that

$$\|K^*f\| \leq \lambda\|S^{1/2}f\|, \text{ for all } f \in \mathcal{H}$$

which implies that $S \geq \lambda^2 KK^*$. Hence by Theorem 2.2.11, $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. $\square$

The following two examples illustrate that a Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is a K-frame but it is not the same for other operator R.

Example 2.2.13. *Let $\mathcal{H} = \mathbb{C}^3$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathcal{H}$. Define $K : \mathcal{H} \to \mathcal{H}$ by $Ke_1 = e_1$, $Ke_2 = e_1$, $Ke_3 = e_2$. Then $\{f_i\}_{i=1}^{3} = \{e_1, e_1, e_2\}$*

is a K-frame for $\mathcal{H}$. By using the definition

$$Sf = \sum_{i=1}^{3} \langle f, f_i \rangle f_i.$$

We get

$$Se_1 = \sum_{i=1}^{3} \langle e_1, f_i \rangle f_i = \langle e_1, e_1 \rangle e_1 + \langle e_1, e_1 \rangle e_1 + \langle e_1, e_2 \rangle e_2 = 2e_1,$$

$$Se_2 = \sum_{i=1}^{3} \langle e_2, f_i \rangle f_i = \langle e_2, e_1 \rangle e_1 + \langle e_2, e_1 \rangle e_1 + \langle e_2, e_2 \rangle e_2 = e_2,$$

$$Se_3 = \sum_{i=1}^{3} \langle e_3, f_i \rangle f_i = \langle e_3, e_1 \rangle e_1 + \langle e_3, e_1 \rangle e_1 + \langle e_3, e_2 \rangle e_2 = 0.$$

The frame operator is $S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and its square root is* $S^{1/2} = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Let $R = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ *and $f = e_3 \in \mathcal{H}$. Then* $\sum_{i=1}^{3} |\langle f, f_i \rangle|^2 = 0$ *and* $\|L^* f\|^2 = 4$.
Hence $\{f_i\}_{i=1}^{3}$ is not a R-frame for $\mathcal{H}$.

In the example, the matrix R is not of the form $S^{1/2}W$, for any matrix W of order 3, because R has a column which is not a linear combination of columns of $S^{1/2}$.

Example 2.2.14. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis in ℓ_2. Define operators T and K on ℓ_2 by $Te_i = e_{i-1}$ for $i > 1$ and $Te_1 = 0$ and $Ke_i = e_{i+1}$ respectively. It is clear that $\{Ke_i\}_{i=1}^{\infty}$ is a K-frame for ℓ_2. Suppose $\{Ke_i\}_{i=1}^{\infty}$ is a T-frame. Then by Theorem 2.2.11, there exists $\lambda > 0$ such that $KK^* \geq \lambda TT^*$. Hence by Douglas' factorization theorem, $R(T) \subseteq R(K)$. But this is contradiction to $R(T) \nsubseteq R(K)$, since $e_1 \in R(T)$ but $e_1 \notin R(K)$.*

Proposition 2.2.15. *Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame for $\mathcal{H}$. Let $T \in \mathcal{B}(\mathcal{H})$ with $R(T) \subseteq R(K)$. Then $\{f_i\}_{i=1}^{\infty}$ is a T-frame for $\mathcal{H}$.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. Then there are positive constants $0 < \lambda \leq \mu < \infty$ such that

$$\lambda\|K^*f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq \mu\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{2.2.9}$$

Since $R(T) \subseteq R(K)$, by Douglas' factorization theorem, there exists $\alpha > 0$ such that $TT^* \leq \alpha^2 KK^*$.

From the inequality (2.2.9), we have

$$\frac{\lambda}{\alpha^2}\|T^*f\|^2 \leq \lambda\|K^*f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2, \text{ for all } f \in \mathcal{H}.$$

Hence $\{f_i\}_{i=1}^{\infty}$ is a T-frame for $\mathcal{H}$. □

2.3 Sums of K-frames

Necessary and sufficient conditions on Bessel sequences $\{f_i\}_{i=1}^{\infty}$, $\{g_i\}_{i=1}^{\infty}$ and operators T_1, T_2 on $\mathcal{H}$ so that $\{T_1 f_i + T_2 g_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$, have been discussed in (Obeidat et al., 2009).

It is a consequence of Douglas' factorization theorem that the operators A and $(AA^*)^{1/2}$ have the same range. This fact leads the following result.

Theorem 2.3.1. *(Fillmore and Williams, 1971) Let $T, V, W \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

1. *$R(T) \subseteq R(V) + R(W)$.*

2. *$TT^* \leq \alpha^2(VV^* + WW^*)$ for some $\alpha > 0$.*

3. *$T = VC + WD$ for some $C, D \in \mathcal{B}(\mathcal{H})$.*

Theorem 2.3.2. *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be K-frames for $\mathcal{H}$ and the corresponding synthesis operators be L_1 and L_2 respectively. If $L_1 L_2^*$ and $L_2 L_1^*$ are positive operators, then $\{f_i + g_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$.*

Proof. Suppose that $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are K-frames for $\mathcal{H}$, then by Theorem 2.2.3, there exist bounded operators L_1 and L_2 such that $L_1 e_i = f_i$, $L_2 e_i = g_i$ and $R(K) \subseteq R(L_1), R(K) \subseteq R(L_2)$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for ℓ_2.

So $R(K) \subseteq R(L_1) + R(L_2)$, by Theorem 2.3.1,

$$KK^* \leq \alpha^2(L_1L_1^* + L_2L_2^*), \text{ for some } \alpha > 0.$$

Now, for each $f \in \mathcal{H}$,

$$\begin{aligned}
\sum_{i=1}^{\infty} |\langle f, f_i + g_i\rangle|^2 &= \sum_{i=1}^{\infty} |\langle f, L_1e_i + L_2e_i\rangle|^2 \\
&= \|(L_1 + L_2)^* f\|^2 \\
&= \langle L_1L_1^* f, f\rangle + \langle L_1L_2^* f, f\rangle + \langle L_2L_1^* f, f\rangle + \langle L_2L_2^* f, f\rangle \\
&\geq \big\langle (L_1L_1^* + L_2L_2^*)f, f\big\rangle \qquad [\because L_1L_2^* \text{ and } L_2L_1^* \text{ are positive.}] \\
&\geq \frac{1}{\alpha^2}\langle KK^* f, f\rangle = \frac{1}{\alpha^2}\|K^* f\|^2. \qquad (2.3.10)
\end{aligned}$$

Suppose μ_1 and μ_2 are Bessel bounds of $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ respectively. Then, applying Minkowski's inequality, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} |\langle f, f_i + g_i\rangle|^2\right)^{\frac{1}{2}} &\leq \left(\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{\infty} |\langle f, g_i\rangle|^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{\mu_1}\|f\| + \sqrt{\mu_2}\|f\| \\
&= (\sqrt{\mu_1} + \sqrt{\mu_2})\|f\|. \qquad (2.3.11)
\end{aligned}$$

From (2.3.10) and (2.3.11), $\{f_i + g_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. □

Corollary 2.3.3. *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be K-frames for $\mathcal{H}$ with frame operators S_1 and S_2 respectively. Then $K = S_1^{1/2}C + S_2^{1/2}D$, for some $C, D \in \mathcal{B}(\mathcal{H})$.*

Proof. Since $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are K-frames for $\mathcal{H}$, by Theorem 2.2.11, there are positive constants λ_1 and λ_2 such that

$$S_1 \geq \lambda_1 KK^* \quad \text{and} \quad S_2 \geq \lambda_2 KK^*.$$

Hence by Douglas' factorization theorem, we get $R(K) \subseteq R(S_1^{1/2})$ and $R(K) \subseteq R(S_2^{1/2})$. Hence $R(K) \subseteq R(S_1^{1/2}) + R(S_2^{1/2})$. By Theorem 2.3.1, there exist $C, D \in \mathcal{B}(\mathcal{H})$ such that $K = S_1^{1/2}C + S_2^{1/2}D$. □

Theorem 2.3.4. *Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame for $\mathcal{H}$ with the frame operator S and let T be a positive operator. Then $\{f_i + Tf_i\}_{i=1}^{\infty}$ is a K-frame. Moreover for any natural number n, $\{f_i + T^n f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. Then by Theorem 2.2.11, there exists $\lambda > 0$ such that $S \geq \lambda KK^*$. The frame operator for $\{f_i + Tf_i\}_{i=1}^{\infty}$ is $(I+T)S(I+T)^*$ because for each $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{i=1}^{\infty}\Big\langle f, (f_i + Tf_i)\Big\rangle(f_i + Tf_i) &= (I+T)\sum_{i=1}^{\infty}\langle f, (I+T)f_i\rangle f_i \\ &= (I+T)S(I+T)^* f.\end{aligned}$$

As we have

$$(I+T)S(I+T)^* = S + ST^* + TS + TST^* \geq S \geq \lambda KK^*,$$

again by Theorem 2.2.11, we can conclude that $\{f_i + Tf_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. For any natural number n, the frame operator for $\{f_i + T^n f_i\}_{i=1}^{\infty}$ is $(I+T^n)S(I+T^n)^* \geq S$. Thus $\{f_i + T^n f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. $\square$

Corollary 2.3.5. *Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame for $\mathcal{H}$ with the frame operator S and let $\{I_1, I_2\}$ be a partition of $\mathbb{N}$. For $j = 1, 2$, let S_j be the K-frame operator for the Bessel sequence $\{f_i\}_{i\in I_j}$. Then*

$$\{f_i + S_1^m f_i\}_{i\in I_1} \cup \{f_i + S_2^n f_i\}_{i\in I_2}$$

is a K-frame for $\mathcal{H}$ for any natural numbers m and n.

Proof. For any natural number m, we can define S^m by

$$S^m f = \sum_{i=1}^{\infty}\langle f, S^{\frac{m-1}{2}} f_i\rangle S^{\frac{m-1}{2}} f_i.$$

For each $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{i=1}^{\infty}\langle f, f_i + S_1^m f_i\rangle(f_i + S_1^m f_i) &= (I + S_1^m)\sum_{i=1}^{\infty}\langle f, f_i + S_1^m f_i\rangle f_i \\ &= (I + S_1^m)\sum_{i=1}^{\infty}\langle f, (I + S_1^m) f_i\rangle f_i \\ &= (I + S_1^m)S_1(I + S_1^m)^* f \\ &= (I + S_1^m)\left(S_1 + S_1^{(1+m)}\right) f \\ &= \left(S_1 + 2S_1^{(1+m)} + S_1^{(1+2m)}\right) f.\end{aligned}$$

Thus the frame operators for $\{f_i + S_1^m f_i\}_{i\in I_1}$ and $\{f_i + S_2^n f_i\}_{i\in I_2}$ are $S_1 + 2S_1^{(1+m)} + S_1^{(1+2m)}$ and $S_2 + 2S_2^{(1+n)} + S_2^{(1+2n)}$ respectively. Let S_0 be the frame operator for $\{f_i + S_1^m f_i\}_{i\in I_1} \cup \{f_i + S_2^n f_i\}_{i\in I_2}$.

Since $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$, there exists $\lambda > 0$ such that $S \geq \lambda K K^*$ and $S_0 \geq S_1 + S_2 = S \geq \lambda K K^*$. Hence $\{f_i + S_1^m f_i\}_{i\in I_1} \cup \{f_i + S_2^n f_i\}_{i\in I_2}$ is a K-frame for $\mathcal{H}$. $\square$

Theorem 2.3.6. *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be Parseval K-frames for $\mathcal{H}$, with synthesis operators L_1 and L_2 respectively. If $L_1 L_2^* = 0$ then $\{f_i + g_i\}_{i=1}^{\infty}$ is a 2-tight K-frame for $\mathcal{H}$.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are two Parseval K-frames for $\mathcal{H}$. Then there are synthesis operators $L_1, L_2 \in \mathcal{B}(\mathcal{H})$ such that $L_1 e_i = f_i$ and $L_2 e_i = g_i$ with $R(K) = R(L_1), R(K) = R(L_2)$ respectively. For each $f \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} |\langle f, f_i + g_i\rangle|^2 &= \sum_{i=1}^{\infty} |\langle f, L_1 e_i + L_2 e_i\rangle|^2 \\
&= \|(L_1 + L_2)^* f\|^2 \\
&= \|L_1{}^* f\|^2 + \langle L_2 L_1^* f, f\rangle + \langle L_1 L_2^* f\rangle + \|L_2{}^* f\|^2 \\
&= \|L_1{}^* f\|^2 + \|L_2{}^* f\|^2 \\
&= 2\|K^* f\|^2.
\end{aligned}$$

$\square$

Theorem 2.3.7. *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be K-frames for $\mathcal{H}$, and let L_1 and L_2 be synthesis operators of $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ respectively, such that $L_1 L_2^* = 0$ and let $T_j \in \mathcal{B}(\mathcal{H})$ with $R(L_j) \subseteq R(T_j L_j)$, for $j = 1, 2$. Then $\{T_1 f_i + T_2 g_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$.*

Proof. Suppose that $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are K-frames for $\mathcal{H}$. Then by Theorem 2.2.3, there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ in ℓ_2 such that $L_1 e_i = f_i$, $L_2 e_i = g_i$

and $R(K) \subseteq R(L_1)$, $R(K) \subseteq R(L_2)$. For each $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle f, T_1 f_i + T_2 g_i\rangle|^2 &= \sum_{i=1}^{\infty} |\langle f, T_1 L_1 e_i + T_2 L_2 e_i\rangle|^2 \\ &= \|(T_1 L_1 + T_2 L_2)^* f\|^2 \\ &= \|(T_1 L_1)^* f\|^2 + \langle T_2 L_2 L_1^* T_1^* f, f\rangle + \langle T_1 L_1 L_2^* T_2{}^* f\rangle + \|(T_2 L_2)^* f\|^2 \\ &= \|(T_1 L_1)^* f\|^2 + \|(T_2 L_2)^* f\|^2. \quad [\because L_1 L_2^* = 0].\end{aligned}$$

We have that $R(K) \subseteq R(L_j) \subseteq R(T_j L_j)$ for $j = 1, 2$. So by Douglas' factorization theorem, for each $j = 1, 2$, there exists $\alpha_j > 0$ such that

$$KK^* \leq \alpha_j (T_j L_j)(T_j L_j)^*.$$

Then from the above inequality, for each $f \in \mathcal{H}$

$$\sum_{i=1}^{\infty} |\langle f, T_1 f_i + T_2 g_i\rangle|^2 = \|(T_1 L_1)^* f\|^2 + \|(T_2 L_2)^* f\|^2 \geq \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \|K^* f\|^2.$$

Hence $\{T_1 f_i + T_2 g_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. □

2.4 Conclusion

Frame for operators in Hilbert spaces introduced by Găvruţa is one of the generalizations of frames available in literature. Because of its relationship with operators, it has been chosen and analyzed in the thesis. Frame sequences are constructed and a class of operators associated with a given Bessel sequence, making it a frame for each operator in the class is also explored.

Chapter 3

FRAME OPERATORS OF K-FRAMES

Many properties for ordinary frames may not hold for K-frames, such as the corresponding synthesis operator for K-frames is not surjective, the frame operator for K-frames is not isomorphic, the alternate dual reconstruction pair for K-frames is not interchangeable in general.

A central object in frame theory, both from the theoretical and applications points of view, is the frame operator. It is known that the frame operator of a frame is a bounded positive invertible operator. However, the design of a frame corresponding to a given bounded positive invertible operator on a Hilbert space is of considerable practical importance.

A question here is : which bounded operators on $\mathcal{H}$ can arise as frame operators of frames in $\mathcal{H}$? The answer (Easwaran Nambudiri and Parthasarathy, 2012) in the context of abstract frames turns out to be simple: Every bounded positive invertible operator on a separable Hilbert space $\mathcal{H}$ is the frame operator of a suitable frame in $\mathcal{H}$. Indeed, for a given bounded positive invertible operator A on $\mathcal{H}$, there is a bounded positive invertible operator B on $\mathcal{H}$ such that $A = B^2$, that is, B is the positive square root of A. Choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for $\mathcal{H}$. Then $\{Be_i\}_{i=1}^{\infty}$ is a Riesz basis. The frame operators of the orthonormal basis $\{e_i\}_{i=1}^{\infty}$ and the Riesz basis (frame) are I (the identity operator) and BIB respectively. Hence A is the frame operator of the frame $\{Be_i\}_{i=1}^{\infty}$.

The results in this chapter are organized as follows. Operators that preserve K-frames and generating new K-frames from old ones by taking sums have been discussed. Moreover, a close relation between K-frames and quotient operators is established using through operator-theoretic results on quotient operators and few characterizations are given.

3.1 Operators for Frames

Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{H}$. Consider the synthesis operator $L : \ell_2 \to \mathcal{H}$ defined by

$$L(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i.$$

As the domain $D(L)$ contains c_{00} (all sequences with finitely many non-zero terms), $D(L)$ is dense in ℓ_2. Also, the range $R(L)$ lies in between $span\{f_i\}_{i=1}^{\infty}$ and $\overline{span}\{f_i\}_{i=1}^{\infty}$. The linear operator L may be unbounded. When L has a closed range, then $R(L) = \overline{span}\{f_i\}_{i=1}^{\infty}$. As L has a dense domain, the adjoint L^* of L exists. If only the upper inequality in (1.5.9) is satisfied, then $D(L) = \ell_2$ and L is bounded with the norm $\|L\| \leq \lambda$.

The analysis operator $U : \mathcal{H} \to \ell_2$, mapped by $f \mapsto \{\langle f, f_i\rangle\}_{i=1}^{\infty}$, is well-defined and linear. The domain

$$D(U) = \left\{f \in \mathcal{H} : \{\langle f, f_i\rangle\}_{i=1}^{\infty} \in \ell_2\right\}$$

is a subspace (not necessarily closed) of $\mathcal{H}$. If $\{f_i\}_{i=1}^{\infty}$ satisfies only the lower inequality in (1.5.9), then U is bounded on $D(U)$ with the norm $\|U\| \leq \frac{1}{\mu}$. Note that in this case, $\{f_i\}_{i=1}^{\infty}$ is total in $\mathcal{H}$.

It is well known that frames are equivalent to spanning sets in a finite dimensional cases. So we can consider an infinite dimensional separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$. The sequence $\{\frac{e_i}{i}\}_{i=1}^{\infty}$ satisfies only the upper inequality, whereas the sequence $\{ie_i\}_{i=1}^{\infty}$ satisfies only the lower inequality. Moreover, if $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$, then $U^* = L$ and $L^* = U$.

Theorem 3.1.1. *(Antoine and Balazs, 2012) Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$, with the synthesis operator L. The following statements hold:*

1. *$\{f_i\}_{i=1}^{\infty}$ is a frame sequence if and only if L has a closed range.*

2. *$\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$ if and only if L is surjective.*

3. *$\{f_i\}_{i=1}^{\infty}$ is a Riesz sequence if and only if L is injective and has a closed range.*

4. *$\{f_i\}_{i=1}^{\infty}$ is a Riesz basis of $\mathcal{H}$ if and only if L is invertible.*

A classification of finite dimensional spaces with frames is as follows:

Theorem 3.1.2. *(Christensen, 2003) Let $\{f_i\}_{i=1}^{\infty}$ be a frame in $\mathcal{H}$. The frame operator S is compact if and only if $\mathcal{H}$ is finite dimensional.*

3.2 Operators Preserving K-frames

The frame operator S for a K-frame is semidefinite, so there is also $S^{1/2}$, but not positive. In general, it is not invertible, shown below by an example. But in the case of K having a closed range, it is proved to be invertible on $R(K)$.

Example 3.2.1. *Let $\mathcal{H} = \mathbb{C}^3$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for $\mathcal{H}$. Define $K : \mathcal{H} \to \mathcal{H}$ by $Ke_1 = e_1$, $Ke_2 = e_1$, $Ke_3 = e_2$. Then $\{f_i\}_{i=1}^{3} = \{e_1, e_1, e_2\}$ is a K-frame for $\mathcal{H}$ with the frame operator $S = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which is not invertible.*

Theorem 3.2.2. *Let $K \in \mathcal{B}(\mathcal{H})$ have a closed range. The frame operator of a K-frame is invertible on the subspace $R(K)$ of $\mathcal{H}$.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. Then there is some $\lambda > 0$ such that

$$\sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \geq \lambda \|K^* f\|^2, \text{ for all } f \in \mathcal{H}. \tag{3.2.1}$$

Since $R(K)$ is closed, then $KK^{\dagger} f = f$, for all $f \in R(K)$. That is,

$$KK^{\dagger}|_{R(K)} = I_{R(K)},$$

we have $I_{R(K)}^* = (K^{\dagger}|_{R(K)})^* K^*$.

For any $f \in R(K)$, we obtain

$$\|f\| = \|(K^\dagger|_{R(K)})^* K^* f\| \leq \|K^\dagger\|.\|K^* f\|,$$

hence $\|K^* f\|^2 \geq \|K^\dagger\|^{-2}\|f\|^2$. Combined with (3.2.1), we have

$$\sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \geq \lambda \|K^* f\|^2 \geq \lambda \|K^\dagger\|^{-2}\|f\|^2, \text{ for all } f \in R(K).$$

So, from the definition of K-frame, we have

$$\lambda \|K^\dagger\|^{-2}\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq \mu \|f\|^2, \text{ for all } f \in R(K).$$

Hence

$$\lambda \|K^\dagger\|^{-2}\|f\| \leq \|Sf\| \leq \mu \|f\|, \text{ for all } f \in R(K).$$

Thus $S : R(K) \to R(S)$ is a bounded linear operator and invertible on $R(K)$. □

Theorem 3.2.3. *Let $K \in \mathcal{B}(\mathcal{H})$ be with a dense range. Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame and $T \in \mathcal{B}(\mathcal{H})$ have a closed range. If $\{Tf_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$, then T is surjective.*

Proof. Suppose $\{Tf_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$ with frame bounds λ and μ. Then for any $f \in \mathcal{H}$,

$$\lambda \|K^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, Tf_i\rangle|^2 \leq \mu \|f\|^2. \tag{3.2.2}$$

As K has a dense range, $\mathcal{H} = \overline{R(K)}$, so K^* is injective. Then from (3.2.2), T^* is injective since $N(T^*) \subseteq N(K^*)$. Moreover, $R(T) = N(T^*)^\perp = \mathcal{H}$. Thus T is surjective. □

Theorem 3.2.4. *Let $K \in \mathcal{B}(\mathcal{H})$ and let $\{f_i\}_{i=1}^{\infty}$ be a K-frame for $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ has a closed range with $TK = KT$, then $\{Tf_i\}_{i=1}^{\infty}$ is a K-frame for $R(T)$.*

Proof. Since T has a closed range, it has the pseudo-inverse $T^\dagger$ such that $TT^\dagger = I$. Now $I = I^* = T^{\dagger *} T^*$. Then for each $f \in R(T)$, $K^* f = T^{\dagger *} T^* K^* f$, so we have

$$\|K^* f\| = \|T^{\dagger *} T^* K^* f\| \leq \|T^{\dagger *}\| \; \|T^* K^* f\|.$$

Therefore $\|T^{\dagger *}\|^{-1}\|K^* f\| \leq \|T^* K^* f\|$.

Now for each $f \in R(T)$,

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle f, Tf_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, f_i \rangle|^2 &\geq \lambda \|K^* T^* f\|^2 \\ &= \lambda \|T^* K^* f\|^2 \\ &\geq \lambda \|T^{\dagger *}\|^{-2} \|K^* f\|^2.\end{aligned}$$

Since $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence with bound μ, for each $f \in R(T)$ we have

$$\sum_{i=1}^{\infty} |\langle f, Tf_i \rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, f_i \rangle|^2 \leq \mu \|T^* f\|^2 \leq \mu \|T\|^2 \|f\|^2.$$

Therefore $\{Tf_i\}_{i=1}^{\infty}$ is a K-frame for $R(T)$. $\square$

Remark 3.2.5. *From the above Theorems 3.2.3 and 3.2.4, we conclude the following : Let $K \in \mathcal{B}(\mathcal{H})$ be with a dense range. Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame for $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ have a closed range with $TK = KT$. Then $\{Tf_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$ if and only if T is surjective.*

Theorem 3.2.6. *Let $K \in \mathcal{B}(\mathcal{H})$ be with a dense range. Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame and let $T \in \mathcal{B}(\mathcal{H})$ have a closed range. If $\{Tf_i\}_{i=1}^{\infty}$ and $\{T^* f_i\}_{i=1}^{\infty}$ are K-frames for $\mathcal{H}$, then T is invertible.*

Proof. Suppose $\{Tf_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$ with frame bounds λ_1 and μ_1. Then for any $f \in \mathcal{H}$,

$$\lambda_1 \|K^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, Tf_i \rangle|^2 \leq \mu_1 \|f\|^2. \tag{3.2.3}$$

As K has a dense range, K^* is injective. Then from (3.2.3), T^* is injective since $N(T^*) \subseteq N(K^*)$. Moreover $R(T) = N(T^*)^{\perp} = \mathcal{H}$. Then T is surjective.

Suppose $\{T^* f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$ with frame bounds λ_2 and μ_2. Then for any $f \in \mathcal{H}$,

$$\lambda_2 \|K^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, T^* f_i \rangle|^2 \leq \mu_2 \|f\|^2. \tag{3.2.4}$$

As K has a dense range, K^* is injective. Then from (3.2.4), T is injective since $N(T) \subseteq N(K^*)$. Therefore T is bijective. By Bounded Inverse Theorem, T is invertible. $\square$

Theorem 3.2.7. *Let $K \in \mathcal{B}(\mathcal{H})$ and let $\{f_i\}_{i=1}^{\infty}$ be a K-frame for $\mathcal{H}$ and let $T \in \mathcal{B}(\mathcal{H})$ be co-isometry with $TK = KT$. Then $\{Tf_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. Now for each $f \in \mathcal{H}$

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle f, Tf_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, f_i\rangle|^2 &\geq \lambda \|K^* T^* f\|^2 \\ &= \lambda \|T^* K^* f\|^2 \\ &= \lambda \|K^* f\|^2 \quad [\because T \text{ is co-isometry }]\end{aligned}$$

It is clear that $\{Tf_i\}_{i=1}^{\infty}$ is a Bessel sequence. Since $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence, for each $f \in \mathcal{H}$ we have

$$\sum_{i=1}^{\infty} |\langle f, Tf_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, f_i\rangle|^2 \leq \mu \|T\|^2 \|f\|^2.$$

Therefore $\{Tf_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. □

3.3 Frame Operators for K-frames

Let $A, B \in \mathcal{B}(\mathcal{H})$. The **quotient** $[A/B]$ is a map from $R(B)$ to $R(A)$ defined by $Bf \mapsto Af$. We note that $T = [A/B]$ is a linear operator on $\mathcal{H}$ if and only if $N(B) \subseteq N(A)$. In this case $D(T) = R(B), R(T) \subseteq R(A)$ and $TB = A$. The quotient $[A/B]$ is called a **semiclosed operator** and its collection is closed under sum and product. Moreover, it is the smallest collection which contains all closed operators, their sum and product (Kaufman, 1979). We present few results on K-frames using techniques on quotients of bounded operators.

Theorem 3.3.1. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$ with the frame operator S and $K \in \mathcal{B}(\mathcal{H})$. Then $\{f_i\}_{i=1}^{\infty}$ is a K-frame if and only if the quotient operator $\left[K^*/S^{1/2}\right]$ is bounded.*

Proof. Suppose that $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. Then there exists a constant $\lambda > 0$ such that

$$\lambda \|K^* f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2, \text{ for all } f \in \mathcal{H}.$$

That is, $\lambda\|K^*f\|^2 \leq \|S^{1/2}f\|^2$, for all $f \in \mathcal{H}$. Define

$$Q : R(S^{1/2}) \to R(K^*) \text{ by } Q(S^{1/2}f) = K^*f.$$

Then Q is well-defined because $N(S^{1/2}) \subseteq N(K^*)$. As for all $f \in \mathcal{H}$,

$$\|QS^{1/2}f\| = \|K^*f\| \leq \frac{1}{\sqrt{\lambda}}\|S^{1/2}f\|,$$

Q is bounded. From the notion of quotient of bounded operators, Q can be expressed as $\left[K^*/S^{1/2}\right]$.

Conversely, suppose that the quotient operator $\left[K^*/S^{1/2}\right]$ is bounded. Then there exists $\mu > 0$ such that

$$\|K^*f\|^2 \leq \mu\|S^{1/2}f\|^2, \text{ for all } f \in \mathcal{H}. \tag{3.3.5}$$

Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then for each $f \in \mathcal{H}$,

$$f = \sum_{i=1}^{\infty} \langle f, e_i \rangle e_i.$$

Then we have

$$\|S^{1/2}f\|^2 = \sum_{i=1}^{\infty} |\langle f, S^{1/2}e_i \rangle|^2, \text{ for all } f \in \mathcal{H}.$$

From the equation (3.3.5),

$$\|K^*f\|^2 \leq \mu\|S^{1/2}f\|^2 = \mu \sum_{i=1}^{\infty} |\langle f, S^{1/2}e_i \rangle|^2.$$

Define $f_i = S^{1/2}e_i$ for $i = 1, 2, \ldots$. Then $\frac{1}{\mu}\|K^*f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2$. Therefore $\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$. □

Corollary 3.3.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$ with the frame operator S. Then $\{f_i\}_{i=1}^{\infty}$ is a frame if and only if the frame operator S is invertible.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$. As $S \leq \mu I$, for some μ, it is enough to show that $\lambda I \leq S$, for some $\lambda > 0$. Since $\{f_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{H}$, then there exist constants $0 < \lambda \leq \mu < \infty 0$ such that

$$\lambda\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq \mu\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{3.3.6}$$

Then $\lambda I \leq S \leq \mu I$. The other way is trivial. □

Theorem 3.3.3. *Let $\{f_i\}_{i=1}^{\infty}$ be a K-frame with the frame operator S and $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent :*

1. *$\{Tf_i\}_{i=1}^{\infty}$ is a TK-frame ;*

2. *$\left[(TK)^*/S^{1/2}T^*\right]$ is bounded ;*

3. *$\left[(TK)^*/(TST^*)^{1/2}\right]$ is bounded.*

Proof. (1)⇒(2); Suppose that $\{Tf_i\}_{i=1}^{\infty}$ is a TK-frame. Then there exists $\lambda > 0$ such that

$$\lambda\|(TK)^*f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, Tf_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle T^*f, f_i\rangle|^2 = \|S^{1/2}(T^*f)\|^2.$$

Therefore $\left[(TK)^*/S^{1/2}T^*\right]$ is bounded.

(2)⇒(3); Suppose $\left[(TK)^*/S^{1/2}T^*\right]$ is bounded. Then there exists $\mu > 0$ such that

$$\|(TK)^*f\|^2 \leq \mu\|S^{1/2}T^*f\|^2, \text{ for all } f \in \mathcal{H}.$$

Now

$$\begin{aligned}\|(TST^*)^{1/2}f\|^2 &= \left\langle (TST^*)^{1/2}f, (TST^*)^{1/2}f\right\rangle = \langle (TST^*)f, f\rangle \\ &= \langle ST^*f, T^*f\rangle = \left\|S^{1/2}T^*f\right\|^2 \geq \frac{1}{\mu}\|(TK)^*f\|^2\end{aligned}$$

Therefore $[(TK)^*/(TST^*)^{1/2}]$ is bounded.

(3)⇒(1); Suppose $\left[(TK)^*/(TST^*)^{1/2}\right]$ is bounded. Then there exists $\mu > 0$ such that

$$\|(TK)^*f\|^2 \leq \mu\|(TST^*)^{1/2}f\|^2 \text{ for all } f \in \mathcal{H}.$$

Consider

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle f, Tf_i\rangle|^2 &= \sum_{i=1}^{\infty} |\langle T^*f, f_i\rangle|^2 = \|S^{1/2}T^*f\|^2 \\ &= \langle S^{1/2}T^*f, S^{1/2}T^*f\rangle \\ &= \langle TST^*f, f\rangle\end{aligned}$$

because TST^* is positive and self-adjoint, its square root exists, and it is denoted by $(TST^*)^{1/2}$. Hence for each $f \in \mathcal{H}$,

$$\sum_{i=1}^{\infty} |\langle f, Tf_i \rangle|^2 = \|(TST^*)^{1/2} f\|^2 \geq \frac{1}{\mu} \|(TK)^* f\|^2.$$

Thus $\{Tf_i\}_{i=1}^{\infty}$ is a TK-frame for $\mathcal{H}$. □

Corollary 3.3.4. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$ and let $K \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

1. *$\{Kf_i\}_{i=1}^{\infty}$ is a K-frame frame $\mathcal{H}$;*

2. *$\left[K^*/S^{1/2}\right]$ is bounded.*

Corollary 3.3.5. *Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ and let $K \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

1. *$\{Ke_i\}_{i=1}^{\infty}$ is a K-frame frame $\mathcal{H}$;*

2. *$[K^*/I]$ is bounded.*

Theorem 3.3.6. *Let $\{f_i\}_{i=1}^{\infty}$ be a Bessel sequence in $\mathcal{H}$ with the frame operator S and let $K \in \mathcal{B}(\mathcal{H})$ with $N(K) = N(S^{1/2})$. Then $\{f_i\}_{i=1}^{\infty}$ is a K-tight frame if and only if the quotient operator $\left[K^*/S^{1/2}\right]$ is invertible.*

Proof. Suppose $\{f_i\}_{i=1}^{\infty}$ is a K-tight frame for $\mathcal{H}$, then $R(K) = R(S^{1/2})$. As $N(K) = N(S^{1/2})$, by (Fillmore and Williams, 1971, Corollary 1) there exists an invertible operator Q on $\mathcal{H}$ such that $K = S^{1/2}Q$. From the notion of quotient operator, $\left[K^*/S^{1/2}\right]$ is invertible. □

3.4 Conclusion

Given a Bessel sequence in a Hilbert space, one can have synthesis, analysis, and frame operators. The study of these associated operators is quite useful to understand some properties of the Bessel sequence. For instance, if the synthesis operator is invertible, then the sequence is a Riesz basis and vice versa. Nevertheless, the frame operator for K-frame gives a quotient of bounded operators. Using operator-theoretic results on quotient operators, new K-frames are generated from old ones and few characterizations are derived.

Chapter 4

K-FRAMES IN BANACH SPACES

The theoretical research of frames for Banach spaces is quite different from that of Hilbert spaces. Due to the lack of an inner product, frames for Banach spaces were simply defined as a sequence of linear functionals in $\mathcal{X}^*$, the dual of $\mathcal{X}$, rather than a sequence of basis-like elements in $\mathcal{X}$ itself. The concept of p-frame was introduced by (Aldroubi et al., 2001) and some abstract theories for it were studied by Chirstensen and Stoeva (Christensen and Stoeva, 2003; Stoeva, 2006).

A sequence $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ is a p**-frame** $(1 \leq p \leq \infty)$ if the norm $\|.\|_{\mathcal{X}}$ is equivalent to the ℓ_p-norm of the sequence $\{g_i(.)\}_{i=1}^{\infty}$, that is, if there are constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in \mathcal{X}$,

$$\lambda\|f\|_{\mathcal{X}} \leq \left(\sum_{i=1}^{\infty} |g_i(f)|^p\right)^{1/p} \leq \mu\|f\|_{\mathcal{X}}.$$

For the case $p = \infty$, $\left(\sum_{i=1}^{\infty} |g_i(f)|^p\right)^{1/p}$ is replaced by $\sup_i |g_i(f)|$. If there exists a 2-frame for a Banach space, then the Banach space is isomorphic to a Hilbert space.

Casazza, Christensen and Stoeva (Casazza et al., 2005) defined an $\mathcal{X}_d$-frame which is a natural generalization of Hilbert frames to Banach frames.

Definition 4.0.1. *(Casazza et al., 2005) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. A sequence $\{g_i\}_{i=1}^{\infty}$ of elements in $\mathcal{X}^*$, which satisfies*

1. $\{g_i(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$, *for all* $f \in \mathcal{X}$,

2. *there are constants* $0 < \lambda \leq \mu < \infty$ *such that for each* $f \in \mathcal{X}$

$$\lambda \|f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu \|f\|_{\mathcal{X}} \tag{4.0.1}$$

*is called an $\mathcal{X}_d$**-frame** for $\mathcal{X}$. The constants λ and μ are called **lower** and **upper bounds** respectively for $\{g_i\}_{i=1}^{\infty}$. When $\{g_i\}_{i=1}^{\infty}$ satisfies (1) and the upper inequality in (4.0.1) for all $f \in \mathcal{X}$, $\{g_i\}_{i=1}^{\infty}$ is called an $\mathcal{X}_d$**-Bessel sequence** for $\mathcal{X}$.*

Note that the definition of $\mathcal{X}_d$-frame is a part of the definition of a Banach frame introduced by Gröchenig (Gröchenig, 1991). An ℓ_p-frame for a Banach space is exactly a p-frame (here the sequence space $\mathcal{X}_d$ is ℓ_p). Since a Banach space $\mathcal{X}$ can be identified with a subspace of the bidual space $\mathcal{X}^{**}$ of $\mathcal{X}$, for a given sequence in $\mathcal{X}$, the $\mathcal{X}_d$-Bessel sequence (respectively, frame) for $\mathcal{X}^*$ can be analogously defined. In a similar way, $\mathcal{X}_d^*$-Bessel sequence for $\mathcal{X}^*$ can be defined for a sequence $\{f_i\}_{i=1}^{\infty}$ of elements of $\mathcal{X}$: if there exists a constant $\mu > 0$ such that

$$\|\{g(f_i)\}_{i=1}^{\infty}\|_{\mathcal{X}_d^*} \leq \mu \|g\|_{\mathcal{X}^*}, \text{ for all } g \in \mathcal{X}^*.$$

If $\mathcal{X}$ is a Hilbert space and $\mathcal{X}_d = \ell_2$, (4.0.1) means that $\{g_i\}_{i=1}^{\infty}$ is a frame, and in this case it is well-known that there exists a sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{X}$ such that

$$f = \sum_{i=1}^{\infty} \langle f, f_i \rangle \; g_i = \sum_{i=1}^{\infty} \langle f, g_i \rangle \; f_i.$$

Similar reconstruction formulas are not always available in the Banach space setting.

The following proposition answers the question of existence of an $\mathcal{X}_d$-frame for $\mathcal{X}$ with respect to a given BK-space $\mathcal{X}_d$.

Proposition 4.0.2. *(Casazza et al., 2005) Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Then there exists an $\mathcal{X}_d$-frame for $\mathcal{X}$ if and only if $\mathcal{X}$ is isomorphic to a subspace of $\mathcal{X}_d$.*

Lemma 4.0.3. *(Casazza et al., 2005) Let $\mathcal{X}_d$ be a BK-space for which the canonical unit vectors $\{e_i\}_{i=1}^{\infty}$ form a Schauder basis. Then the space $\mathcal{Y}_d = \big\{\{F(e_i)\}_{i=1}^{\infty} : F \in \mathcal{X}_d^*\big\}$ with norm $\|\{F(e_i)\}_{i=1}^{\infty}\|_{\mathcal{Y}_d} = \|F\|_{\mathcal{X}_d^*}$ is a BK-space isometrically isomorphic to $\mathcal{X}_d^*$. Also, every continuous linear functional F on $\mathcal{X}_d$ has the form*

$$F(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i d_i,$$

where $\{d_i\} = F(e_i)$, is uniquely determined by $d_i = F(e_i)$, and $\|F\|_{\mathcal{X}_d^*} = \|\{d_i\}\|_{\mathcal{Y}_d}$.

Lemma 4.0.4. *(Casazza et al., 2005) Let $\mathcal{X}_d$ be a BK-space and $\mathcal{X}_d^*$ be a CB-space. If $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ is an $\mathcal{X}_d$-Bessel sequence for $\mathcal{X}$ with bound μ, then the operator $L : \{d_i\}_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} d_i g_i$ is well-defined (hence bounded) from $\mathcal{X}_d^*$ into $\mathcal{X}^*$ and $\|L\| \leq \mu$. If $\mathcal{X}_d$ is reflexive, the converse is also true.*

4.1 Operators for $\mathcal{X}_d$-frames

Let $\mathcal{X}_d$ be a BK-space and $\{g_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{X}^*$. If $\{g_i\}_{i=1}^{\infty}$ satisfies only the upper inequality in (4.0.1), the **analysis operator** U from $\mathcal{X}$ to $\mathcal{X}_d$ mapped by $f \mapsto \{g_i(f)\}_{i=1}^{\infty}$, is well-defined and linear, having domain

$$D(U) = \big\{f \in \mathcal{X} : \{g_i(f)\}_{i=1}^{\infty} \in \mathcal{X}_d\big\}.$$

The domain $D(U)$ is a subspace (not necessarily closed) of $\mathcal{X}$. If $\{g_i\}_{i=1}^{\infty}$ is a $\mathcal{X}_d$-Bessel sequence for $\mathcal{X}$, then $D(U) = \mathcal{X}$ and U is bounded with the norm $\|U\| \leq \mu$.

If only the lower inequality in (4.0.1) is satisfied by $\{g_i\}_{i=1}^{\infty}$, then U is bounded below on $D(U)$. Thus if $\{g_i\}_{i=1}^{\infty}$ satisfies the $\mathcal{X}_d$-frame inequalities (4.0.1), we get that U is bounded and bounded below on $D(U)$. Hence $R(U)$ is closed in $\mathcal{X}_d$ and the inverse $U^{-1} : R(U) \to D(U)$ is also bounded with the norm $\|U^{-1}\| \leq \frac{1}{\lambda}$. We can conclude that given an $\mathcal{X}_d$-frame $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ for $\mathcal{X}$, the analysis operator $U : \mathcal{X} \to \mathcal{X}_d$ defined by

$$Uf = \{g_i(f)\}_{i=1}^{\infty}$$

is an isomorphism of $\mathcal{X}$ onto $R(U)$.

Given a sequence $\{g_i\}_{i=1}^{\infty}$ in $\mathcal{X}^*$, we now consider a function $L : \mathcal{X}_d^* \to \mathcal{X}^*$, called the **synthesis operator**, mapped as $\{d_i\}_{i=1}^{\infty} \mapsto \sum_{i=1}^{\infty} d_i g_i$ is well-defined and linear on the domain

$$D(L) = \left\{ \{d_i\}_{i=1}^{\infty} \in \mathcal{X}_d^* : \sum_{i=1}^{\infty} d_i g_i \text{ converges in } \mathcal{X}^* \right\}.$$

If $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ is an $\mathcal{X}_d$-Bessel sequence in $\mathcal{X}$ with bound μ and if $\mathcal{X}_d^*$ is a CB-space, then L is bounded from $\mathcal{X}_d^*$ to $\mathcal{X}^*$ and $\|L\| \leq \mu$, by Lemma 4.0.4. If $\mathcal{X}_d$ is a CB-space, then $U^* = L$. If $\mathcal{X}_d$ is reflexive and $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-frame for $\mathcal{X}$, then $U = L^*$ because $\mathcal{X}$ is isomorphic to a closed subspace of $\mathcal{X}_d$ and every closed subspace of a reflexive space is reflexive. Hence $\mathcal{X}$ is also reflexive.

In the Hilbert frame case, the frame operator $S = LU$ exists and this operator S is helpful to produce the reconstruction formula. But in the $\mathcal{X}_d$-frame case the corresponding operators $U : \mathcal{X} \to \mathcal{X}_d$ and $L : {\mathcal{X}_d}^* \to \mathcal{X}^*$ cannot be composed.

A map from $\mathcal{X}_d$ into ${\mathcal{X}_d}^*$ is needed and for this the duality mapping on $\mathcal{X}_d$ is used : The mapping $\phi_{\mathcal{X}}$ from $\mathcal{X}$ into the the set of subsets of $\mathcal{X}^*$, determined by

$$\phi_{\mathcal{X}}(f) = \left\{ g \in \mathcal{X}^* : g(f) = \|f\|^2 = \|g\|^2 \right\}$$

is called the **duality mapping on $\mathcal{X}$.** By the Hahn-Banach Theorem, $\phi_{\mathcal{X}}(f)$ is a non-empty set for all $f \in \mathcal{X}$ and $\phi_{\mathcal{X}}(0) = 0$.
In general, the duality mapping is set-valued, but for certain spaces it is single-valued and such spaces are called smooth.

Definition 4.1.1. *A Banach space $\mathcal{X}$ is called **strictly convex** whenever*

$$\|f_1 + f_2\|_{\mathcal{X}} = \|f_1\|_{\mathcal{X}} + \|f_2\|_{\mathcal{X}}$$

where $f_1, f_2 \neq 0$ then $f_1 = \alpha f_2$ for some $\alpha > 0$.

Proposition 4.1.2. *(Stoeva, 2008) Let $\mathcal{X}$ be a Banach space. Then the following statements hold :*

1. *If $\mathcal{X}^*$ is strictly convex, then for every $f \in \mathcal{X}$, $\phi_{\mathcal{X}}(f)$ is single-valued.*

2. *If $\mathcal{X}$ and $\mathcal{X}^*$ are strictly convex and $\mathcal{X}$ is reflexive, then $\phi_{\mathcal{X}}$ is bijective.*

3. *If $\mathcal{H}$ is a Hilbert space and $\mathcal{H}^*$ is identified with $\mathcal{H}$ by the Riesz representation theorem, then $\phi_{\mathcal{H}}(f) = f$, for every $f \in \mathcal{H}$.*

Definition 4.1.3. *(Stoeva, 2008) Let $\mathcal{X}_d^*$ be strictly convex and $\{g_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{X}^*$. If $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-frame for a Banach space $\mathcal{X}$, then the bounded map $S : \mathcal{X} \to \mathcal{X}^*$, $S := L\phi_{\mathcal{X}_d}U$ is called the $\mathcal{X}_d$-**frame map** for $\{g_i\}_{i=1}^{\infty}$.*

When $\mathcal{X} = \mathcal{H}$ is a Hilbert space and $\{g_i\}_{i=1}^{\infty}$ is a frame for $\mathcal{X}$, the ℓ_2-frame map

$$S = L\phi_{\mathcal{X}_d}U$$

gives the Hilbert frame operator $S = LU$.

Theorem 4.1.4. *(Stoeva, 2008) Let $\mathcal{X}_d$ be a RCB-space such that $\mathcal{X}_d^*$ is strictly convex. Let $\{g_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d$-frame for $\mathcal{X}$ with lower and upper bounds λ and μ respectively. The $\mathcal{X}_d$-frame map S has the following properties:*

1. *$S = U^*\phi_{\mathcal{X}_d}U$.*

2. *$\lambda^2\|f\|_{\mathcal{X}}^2 \leq Sf(f) \leq \mu^2\|f\|_{\mathcal{X}}^2$, for all $f \in \mathcal{X}$.*

Definition 4.1.5. *(Limaye, 1996) Let M be a closed subspace of $\mathcal{X}$. We say that M is called **(topologically) complemented** if there exists a closed subspace N of $\mathcal{X}$ such that $M \cap N = \{0\}$ and $M + N = \mathcal{X}$. In this case we write $\mathcal{X} = M \oplus N$. Subspace N is called a **complement** of M.*

Examples 4.1.6. 1. *Every finite dimensional subspace (or, a subspace with finite codimension) is complemented. Every ℓ_p ($p > 1,\ p \neq 2$) has a closed subspace which is not complemented. Also c_0 is not complemented in ℓ_∞.*

2. *Every infinite dimensional complemented subspace of ℓ_∞ is isomorphic to ℓ_∞. This also holds if ℓ_∞ is replaced by ℓ_p, $1 \leq p < \infty$, c_0, or c.*

3. *Every infinite dimensional Banach space which is not isomorphic to a Hilbert space contains a closed subspace which has no complement. In a Hilbert space every closed subspace has a complement (its orthogonal complement).*

4. *A closed subspace M of $\mathcal{X}$ has a complement if and only if there exists a continuous projection operator onto M.*

Proposition 4.1.7. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and let $B \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then there exists a bounded linear operator $A : \mathcal{Y} \to \mathcal{X}$ satisfying $AB = I$ if and only if B is bounded below and $R(B)$ is a complemented subspace of $\mathcal{Y}$.*

Proof. Let $A : \mathcal{Y} \to \mathcal{X}$ be a bounded linear operator satisfying $AB = I$. Suppose $Bf_1 = Bf_2$ for some $f_1, f_2 \in \mathcal{X}$ with $f_1 \neq f_2$. Then $ABf_1 = ABf_2$, so, $f_1 = f_2$. This contradiction proves that B is injective.

Since $AB = I$, $(BA)^2 = BA$ and $R(B) = R(BAB) \subseteq R(BA) \subseteq R(B)$. Hence BA is a projection onto $R(BA) = R(B)$, and $R(B)$ is a complemented subspace of $\mathcal{Y}$. Also, $R(B) = N(I - BA)$, which is closed. Hence B is bounded below.

Conversely, let $P \in \mathcal{B}(\mathcal{Y})$ be a projection onto $R(B)$. Since B is bounded below, B is injective, hence $B^{-1} : R(B) \to \mathcal{X}$ is bounded. Define $A_1 = B^{-1}P$. Then A_1 is bounded and $A_1 B = I$. □

Proposition 4.1.8. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and let $B \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then there exists a bounded linear operator $C : \mathcal{Y} \to \mathcal{X}$ satisfying $BC = I$ if and only if B is surjective and $N(B)$ is a complemented subspace of $\mathcal{X}$.*

Proof. Let $C : \mathcal{Y} \to \mathcal{X}$ be a bounded linear operator satisfying $BC = I$. Since $B(R(C)) = \mathcal{Y}$, B is surjective. As $(CB)^2 = CB$, CB is a projection onto $R(CB)$. Hence $\mathcal{X} = R(CB) \oplus N(CB)$. It is easy to see that $N(B) \subseteq N(CB)$. Let $f \in N(BC)$. Then $CBf = 0$, so $BCBf = 0$, hence $f \in N(B)$. Therefore $N(B) = N(CB)$. Thus $N(B)$ is a complemented subspace of $\mathcal{X}$.

Conversely, let B be a surjective and let $\mathcal{X} = N(B) \oplus M$ for some closed subspace M of $\mathcal{X}$. Consider the operator $B|_M : M \to \mathcal{Y}$, the restriction of B to M. $(B|_M)^{-1}$ exists and it is bijective, by Bounded Inverse Theorem, $(B|_M)^{-1}$ is bounded. Define $B_1 = J(B|_M)^{-1}$, where J is the natural embedding. Then B_1 is bounded and $BC = I$. □

Definition 4.1.9. *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and let $B \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. A is called* ***left-inverse*** *of B if $A \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $AB = I$; C is called* ***right-inverse*** *of B if $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $BC = I$.*

We have seen that every Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$ is an $\mathcal{X}_d$-frame for $\mathcal{X}$. One may ask a question whether every $\mathcal{X}_d$-frame $\{g_i\}_{i=1}^{\infty}$ has a bounded linear operator $S : \mathcal{X}_d \to \mathcal{X}$ such that $(\{g_i\}_{i=1}^{\infty}, S)$ is a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. The answer is in affirmative. This guarantees a natural existence of reconstruction operator for any $\mathcal{X}_d$-frame for $\mathcal{X}$.

Proposition 4.1.10. *Let $\mathcal{X}_d$ be a BK-space and $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ be an $\mathcal{X}_d$-frame for $\mathcal{X}$. If $R(U)$ is complemented in $\mathcal{X}_d$, then there exists a bounded linear operator $V : \mathcal{X}_d \to \mathcal{X}$ such that $(\{g_i\}_{i=1}^{\infty}, V)$ is a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$.*

Proof. Since $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-frame for $\mathcal{X}$, there are constants $0 < \lambda \le \mu < \infty$ such that

$$\lambda \|f\|_{\mathcal{X}} \le \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \le \mu \|f\|_{\mathcal{X}}, \text{ for all } f \in \mathcal{X}.$$

Since $R(U)$ is complemented in $\mathcal{X}_d$, there exists a closed subspace M of $\mathcal{X}_d$ such that

$$\mathcal{X}_d = R(U) \oplus M. \tag{4.1.2}$$

Since U is an isomorphism from $\mathcal{X}$ onto $R(U)$, U^{-1} is a bounded linear from $R(U)$ into $\mathcal{X}$. From (4.1.2), U^{-1} can be extended to $\mathcal{X}_d$ by defining $U^{-1}f = 0$ for $f \in M$. Again from (4.1.2), we can find a projection of $\mathcal{X}_d$ onto $R(U)$, say P.

Define $V : \mathcal{X}_d \to \mathcal{X}$ by $V = U^{-1}P$. Each coordinate functional on $\mathcal{X}_d$ is continuous and it is denoted by e_i, for $i = 1, 2, \ldots$. Let $g_i = U^* e_i$. Then for each $f \in X$ we have

$$g_i(f) = (U^* e_i)(f) = e_i(Uf), \text{ for all } i = 1, 2, \ldots .$$

Hence the image of f under U is the sequence $\{g_i(f)\}_{i=1}^{\infty}$ in $\mathcal{X}_d$. Since U is an isomorphism, for each $f \in \mathcal{X}$, $S(\{g_i(f)\}_{i=1}^{\infty}) = f$. Thus $(\{g_i\}_{i=1}^{\infty}, V)$ is a Banach frame for $\mathcal{X}$ with respect to $\mathcal{X}_d$. □

Theorem 4.1.11. *Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ be an $\mathcal{X}_d$-frame for $\mathcal{X}$ and let $V : \mathcal{X}_d \to \mathcal{X}$ be a bounded linear operator satisfying $V(\{g_i(f)\}_{i=1}^{\infty}) = f$, for each $f \in \mathcal{X}$, then $R(U)$ is complemented.*

Proof. The hypothesis shows that V is a left-inverse of U. By Proposition 4.1.7, $R(U)$ is complemented. $\square$

Analysis operator U coming from a Banach frame of a Banach space $\mathcal{X}$ decomposes $\mathcal{X}_d$ a direct sum of two closed subspaces of $\mathcal{X}_d$.

Example 4.1.12. *Let $\mathcal{X} = c_0$, $\mathcal{X}_d = \ell_\infty$, and $\{g_i\}_{i=1}^\infty$ be the canonical unit vectors as a basis of ℓ_1. Then $\{g_i\}_{i=1}^\infty$ is an $\mathcal{X}_d$-frame for c_0. Note that*

1. *$\mathcal{X}_d = \ell_\infty$ does not have the canonical unit vectors as a basis.*
2. *$R(U)$ is not complemented in $\mathcal{X}_d = \ell_\infty$.*

The section ends with a result connecting majorization, factorization and range inclusion for operators on Banach spaces.

Theorem 4.1.13. *(Barnes, 2005) Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Banach spaces and let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y}), B \in \mathcal{B}(\mathcal{Z}, \mathcal{Y})$. Then the following statements hold :*

1. *If $A = BT$ for some $T \in \mathcal{B}(\mathcal{X}, \mathcal{Z})$, then B^* majorizes A^*. Converse is true when $N(B)$ is complemented in $\mathcal{Z}$, and $\mathcal{Z}$ is reflexive.*
2. *If $R(A) \subseteq R(B)$, then B^* majorizes A^*. Converse is true when $\mathcal{Z}$ is reflexive.*

4.2 $\mathcal{X}_d$-Atomic Systems and $\mathcal{X}_d$-K-frames

A frame for a Banach space $\mathcal{X}$ was defined as a sequence of elements in $\mathcal{X}^*$, not of elements in the original space $\mathcal{X}$. However, semi-inner products for Banach spaces make possible the development of inner product type arguments in Banach spaces. Frame (sequence of elements in $\mathcal{X}$) for Banach spaces via semi-inner product were defined by (Zhang and Zhang, 2011).

The concept of a family of local atoms in a Banach space $\mathcal{X}$ with respect to a BK-space $\mathcal{X}_d$ was introduced by Dastourian and Janfada (Dastourian and Janfada, 2016) using a semi-inner product. This concept was generalized to an atomic system for an operator $K \in \mathcal{B}(\mathcal{X})$ called $\mathcal{X}_d^*$*-atomic system* and it has been led to the definition of a new frame with respect to the operator K, called $\mathcal{X}_d^*$-K*-frame.* Unlike the traditional way of considering sequences in the dual space $\mathcal{X}^*$, sequences in the original space $\mathcal{X}$ are considered in (Dastourian and Janfada, 2016) to study a family of $\mathcal{X}_d^*$-local atoms and $\mathcal{X}_d^*$-atomic systems by making use of semi-inner products. Here $\mathcal{X}$ is assumed to be a reflexive separable Banach space.

Appropriate changes have been made in the definitions of $\mathcal{X}_d^*$-atomic systems and $\mathcal{X}_d^*$-K-frames to fit them for sequences in the dual space without using semi-inner products, called $\mathcal{X}_d$*-atomic systems* and $\mathcal{X}_d$-K*-frames* respectively. Thus the notion of $\mathcal{X}_d$-K-frames for Banach spaces can be thought of a generalization of $\mathcal{X}_d$-frames.

Definition 4.2.1. *Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $K \in \mathcal{B}(\mathcal{X}^*)$ and $\{g_i\}_{i=1}^\infty \subseteq \mathcal{X}^*$. We say that $\{g_i\}_{i=1}^\infty$ is an* $\mathcal{X}_d$***-atomic system for*** $\mathcal{X}$ ***with respect to*** K *if the following statements hold :*

1. $\sum\limits_{i=1}^{\infty} d_i g_i$ *converges in $\mathcal{X}^*$ for all $d = \{d_i\}_{i=1}^\infty$ in $\mathcal{X}_d^*$ and there exists $\mu > 0$ such that* $\left\| \sum\limits_{i=1}^{\infty} d_i g_i \right\|_{\mathcal{X}^*} \leq \mu \|d\|_{\mathcal{X}_d^*}$ *;*
2. *there exists $c > 0$ such that for every $g \in \mathcal{X}^*$ there exists $a_g = \{a_i\}_{i=1}^\infty \in \mathcal{X}_d$ such that $\|a_g\|_{\mathcal{X}_d} \leq c\|g\|_{\mathcal{X}^*}$ and*

$$Kg = \sum_{i=1}^{\infty} a_i g_i.$$

When $\mathcal{X}_d$ is reflexive, the condition (1) in Definition 4.2.1 actually says that $\{g_i\}_{i=1}^\infty$ is an $\mathcal{X}_d$-Bessel sequence for $\mathcal{X}$ with bound μ, by Lemma 4.0.4. We find a necessary condition for a sequence $\{g_i\}_{i=1}^\infty \subseteq \mathcal{X}^*$ to be an $\mathcal{X}_d$-atomic system for $\mathcal{X}$ with respect to a given operator K if the associated sequence space satisfies the following crucial property: For each $\{g_i\}_{i=1}^\infty, \{h_i\}_{i=1}^\infty \in \mathcal{X}_d$,

$$\left| \sum_{i=1}^{\infty} g_i h_i \right| \leq \|\{g_i\}_{i=1}^\infty\|_{\mathcal{X}_d} \ \|\{h_i\}_{i=1}^\infty\|_{\mathcal{X}_d}. \tag{4.2.3}$$

For instance, let $\{g_i\}_{i=1}^{\infty}, \{h_i\}_{i=1}^{\infty} \in \ell_p$ and $p \in (1,2]$. Then the conjugate of p, q lies in $[2,\infty)$. Hence by Hölder's inequality, the sequence space ℓ_p for $1 < p \leq 2$ satisfies (4.2.3).

Theorem 4.2.2. *Let $\mathcal{X}_d$ be a BK-space. Let $\{g_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{X}^*$ and $K \in \mathcal{B}(\mathcal{X}^*)$. If $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-atomic system for $\mathcal{X}$ with respect to K and the sequence space $\mathcal{X}_d$ satisfies the inequality (4.2.3), then there exists a constant $\lambda > 0$ such that*

$$\|K^*f\|_{\mathcal{X}} \leq \lambda \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \quad \textit{for each } f \in \mathcal{X}.$$

Proof. Suppose $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-atomic system for $\mathcal{X}$ with respect to K. Then there is some $c > 0$ such that for every $g \in \mathcal{X}^*$ there exists $a_g = \{a_i\}_{i=1}^{\infty} \in \mathcal{X}_d$ such that

$$\|a_g\|_{\mathcal{X}_d} \leq c\ \|g\|_{\mathcal{X}^*}$$

and $Kg = \sum_{i=1}^{\infty} a_i g_i$. Hence for each $f \in \mathcal{X}$,

$$\begin{aligned}
\|K^*f\|_{\mathcal{X}} &= \sup_{g\in\mathcal{X}^*,\ \|g\|=1} |g(K^*f)| \\
&= \sup_{g\in\mathcal{X}^*,\ \|g\|=1} |(Kg)(f)| \\
&= \sup_{g\in\mathcal{X}^*,\ \|g\|=1} \Big|\sum_{i=1}^{\infty} a_i g_i(f)\Big| \\
&\leq \sup_{g\in\mathcal{X}^*,\ \|g\|=1} \|\{a_i\}_{i=1}^{\infty}\|_{\mathcal{X}_d}\ \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \\
&= \sup_{g\in\mathcal{X}^*,\ \|g\|=1} \|a_g\|_{\mathcal{X}_d}\ \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \\
&\leq c \sup_{g\in\mathcal{X}^*,\ \|g\|=1} \|g\|_{\mathcal{X}^*}\ \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \quad [\because\ \|a_g\|_{\mathcal{X}_d} \leq c\ \|g\|_{\mathcal{X}^*}]
\end{aligned}$$

Thus for some $c > 0$, $\|K^*f\|_{\mathcal{X}} \leq c\ \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d}$, for each $f \in \mathcal{X}$. □

Definition 4.2.3. *Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $K \in \mathcal{B}(\mathcal{X}^*)$ and $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$. We say that $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-K**-frame for** $\mathcal{X}$ if the following statements hold:*

1. $\{g_i(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$, *for each* $f \in \mathcal{X}$;

2. *there exist two constants* $0 < \lambda \leq \mu < \infty$ *such that*

$$\lambda\|K^*f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu\|f\|_{\mathcal{X}}, \textit{ for each } f \in \mathcal{X}.$$

The elements λ and μ are called the ***lower*** *and* ***upper $\mathcal{X}_d$-K-frame bounds.***

We say that an $\mathcal{X}_d$-frame for $\mathcal{X}$ is an $\mathcal{X}_d$-I-frame for $\mathcal{X}$, where I is the identity operator on $\mathcal{X}^*$. The set of all $\mathcal{X}_d$-frames for $\mathcal{X}$ can be considered as a subset of $\mathcal{X}_d$-K-frames for $\mathcal{X}$. Thus $\mathcal{X}_d$-K-frame is a generalization of $\mathcal{X}_d$-frame for a Banach space $\mathcal{X}$. We present an example for an $\mathcal{X}_d$-K-frame which is not an $\mathcal{X}_d$-frame for $\mathcal{X}$.

Example 4.2.4. *Let $\mathcal{X}$ be the space of all triplets $(\alpha_1, \alpha_2, \alpha_3)$ with complex scalars and having 3/2-norm, denoted by $\ell_{3/2}(3)$. Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^* = \ell_3(3)$ be such that $g_i(e_j) = \delta_{ij}$, where e_j's are vectors in $\mathcal{X}$, having 1 in j^{th} place and 0 elsewhere, and $g_i = 0$ for all $i \geq 4$. Define $K : \mathcal{X}^* \to \mathcal{X}^*$ by*

$$Kg_1 = 0,\ \ Kg_2 = g_3,\ \ \text{and } Kg_3 = g_2.$$

For any $f \in \mathcal{X}$, we have $f = \sum_{i=1}^{3} \alpha_i e_i$ and

$$\|K^* f\|_{\mathcal{X}} = \|\alpha_2 e_3 + \alpha_3 e_2\|_{3/2} = \left(|\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} = \|\{g_i(f)\}_{i=2}^{\infty}\|_{\ell_{3/2}}.$$

Then $\{g_i\}_{i=2}^{\infty}$ is an $\mathcal{X}_d$-K frame for $\mathcal{X}$. But it is not an $\mathcal{X}_d$-frame because there is no constant λ such that for any scalar α_1,

$$\lambda\|f\|_{\mathcal{X}} = \left(|\alpha_1|^{3/2} + |\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} \leq \left(|\alpha_2|^{3/2} + |\alpha_3|^{3/2}\right)^{2/3} = \|\{g_i(f)\}_{i=2}^{\infty}\|_{\ell_{3/2}}.$$

4.3 Generating New $\mathcal{X}_d$-K-frames and Characterizations

We can generate new $\mathcal{X}_d$-K-frames for $\mathcal{X}$ from each $\mathcal{X}_d$-frame for $\mathcal{X}$ and each operator $K \in \mathcal{B}(\mathcal{X}^*)$, by the following proposition.

Proposition 4.3.1. *If $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-frame for $\mathcal{X}$ and $K \in \mathcal{B}(\mathcal{X}^*)$, then $\{Kg_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-K-frame for $\mathcal{X}$.*

Proof. Suppose $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-frame for $\mathcal{X}$. Then $\{g_i(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$, for all $f \in \mathcal{X}$ and there are constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in \mathcal{X}$

$$\lambda\|f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu\|f\|_{\mathcal{X}}.$$

Let $f \in \mathcal{X}$ be fixed. Since $(Kg_i)(f) = g_i(K^*f)$ and $K^*f \in \mathcal{X}$, we have $\{(Kg_i)(f)\}_{i=1}^{\infty} \in \mathcal{X}_d$. Also, $\|K^*f\|_{\mathcal{X}} \leq \|K\| \, \|f\|_{\mathcal{X}}$ gives that for each $f \in \mathcal{X}$,

$$\lambda \|K^*f\|_{\mathcal{X}} \leq \|\{(Kg_i)(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu \|K\| \, \|f\|_{\mathcal{X}}.$$

Thus $\{Kg_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-K-frame for $\mathcal{X}$. $\square$

The following example illustrates that an $\mathcal{X}_d$-Bessel sequence is an $\mathcal{X}_d$-K-frame but it is not the same for the other operator T.

Example 4.3.2. *Let $\mathcal{X} = \ell_{3/2}(3)$. Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^* = \ell_3(3)$ be such that for $i = 1, 2, 3$, $g_i(e_j) = \delta_{ij}$, and $g_i = 0$ for all $i \geq 4$. Define K and T from $\mathcal{X}^*$ to $\mathcal{X}^*$ as follows:*

$$Kg_1 = 0, \;\; Kg_2 = g_3, \;\; \textit{and} \; Kg_3 = g_2,$$

and

$$Tg_1 = g_1, \;\; Tg_2 = g_3, \;\; \textit{and} \; Tg_3 = g_2.$$

Then $\{g_i\}_{i=2}^{\infty}$ is an $\mathcal{X}_d$-K frame but it is not an $\mathcal{X}_d$-T-frame for $\mathcal{X}$.

Theorem 4.3.3. *Let $\{g_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d$-K-frame for $\mathcal{X}$. Let $T \in \mathcal{B}(\mathcal{X}^*)$ be such that $R(T) \subseteq R(K)$. Then $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-T-frame for $\mathcal{X}$.*

Proof. Suppose $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-K-frame for $\mathcal{X}$. Then there are constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in \mathcal{X}$

$$\lambda \|K^*f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu \|f\|_{\mathcal{X}}. \tag{4.3.4}$$

Since $R(T) \subseteq R(K)$, by Theorem 4.1.13, there exists $c > 0$ such that $\|T^*f\|_{\mathcal{X}} \leq c\|K^*f\|_{\mathcal{X}}$. From the inequality (4.3.4), we have for each $f \in \mathcal{X}$

$$\frac{\lambda}{c} \|T^*f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu \|f\|_{\mathcal{X}}.$$

Hence $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-T-frame for $\mathcal{X}$. $\square$

Theorem 4.3.4. *Let $\mathcal{X}_d$ be a reflexive space and let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$. Let $\{e_i\}_{i=1}^{\infty}$ be the canonical unit vectors for $\mathcal{X}_d$ and ${\mathcal{X}_d}^*$. Then $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-K-frame for $\mathcal{X}$ if and only if there exists a bounded linear operator $L : \mathcal{X}_d^* \to \mathcal{X}^*$ such that $Le_i = g_i$ and $R(K) \subseteq R(L)$.*

Proof. Since $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-K-frame for $\mathcal{X}$, there exist constants $0 < \lambda \leq \mu < \infty$ such that for each $f \in \mathcal{X}$,

$$\lambda \|K^* f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu \|f\|_{\mathcal{X}}.$$

Hence the operator $U : \mathcal{X} \to \mathcal{X}_d$ defined by $Uf = \{g_i(f)\}_{i=1}^{\infty}$ is bounded and $\|U\| \leq \mu$. The adjoint of U, $U^* : \mathcal{X}_d^* \to \mathcal{X}^*$ satisfies

$$(U^* e_i)(f) = e_i(Uf) = g_i(f).$$

Since $\mathcal{X}_d$ is an RCB-space, $U^* = L$, hence we get $Le_i = g_i$. Also we have

$$\lambda \|K^* f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} = \|L^* f\|_{\mathcal{X}_d}, \text{ for each } f \in \mathcal{X}.$$

Thus by Theorem 4.1.13, $R(K) \subseteq R(L)$.

On the other hand, suppose there exists a bounded linear operator $L : \mathcal{X}_d^* \to \mathcal{X}^*$ such that $Le_i = g_i$ and $R(K) \subseteq R(L)$. Then by Theorem 4.1.13, there exists $\lambda > 0$ such that $\lambda \|K^* f\|_{\mathcal{X}} \leq \|L^* f\|_{\mathcal{X}_d}$. Thus for each $f \in \mathcal{X}$,

$$\lambda \|K^* f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} = \|L^* f\|_{\mathcal{X}_d} \leq \|L\| \, \|f\|_{\mathcal{X}}.$$

□

Corollary 4.3.5. *Let $\mathcal{X}_d$ be a reflexive space and let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$. Let $\{e_i\}_{i=1}^{\infty}$ be the canonical unit vectors for $\mathcal{X}_d$ and $\mathcal{X}_d^*$. Let $N(L)$ be complemented in $\mathcal{X}_d^*$. Then $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-K-frame for $\mathcal{X}$ if and only if $L = KV$ for some $V \in \mathcal{B}(\mathcal{X}_d^*, \mathcal{X}^*)$.*

Zhang and Zhang defined frames in Banach spaces via a compatible semi-inner product, which is a natural substitute for inner products on Hilbert spaces. As assumed in the paper (Zhang and Zhang, 2011), we assume that $\mathcal{X}_d$ is reflexive, the canonical unit vectors $\{e_i\}_{i=1}^{\infty}$ form a Schauder basis for $\mathcal{X}_d$ and $\mathcal{X}_d^*$; the following crucial requirement is also imposed as in (Zhang and Zhang, 2011): If $d = \{d_i\}_{i=1}^{\infty}$ is a sequence of scalars satisfying $\sum_{i=1}^{\infty} c_i d_i$ converges for every $c = \{c_i\}_{i=1}^{\infty} \in \mathcal{X}_d$, then $d \in \mathcal{X}_d^*$, and if the above series converges for all $d \in \mathcal{X}_d^*$, then $c \in \mathcal{X}_d$.

For instance, if $\mathcal{X}_p = \ell_p, 1 < p < \infty$, then $\mathcal{X}_d^* = \ell_q$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, it satisfies all of our requirements on $\mathcal{X}_d$ and $\mathcal{X}_d^*$. The above requirements about the spaces $\mathcal{X}$ and $\mathcal{X}_d$ are assumed in the rest of the chapter. We now prove that the converse of the Theorem 4.2.2 with the above assumptions.

Theorem 4.3.6. *Let $\mathcal{X}$ be a Banach space and $\mathcal{X}_d$ be a BK-space. Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ be an $\mathcal{X}_d$-Bessel sequence for $\mathcal{X}$, and $K \in \mathcal{B}(\mathcal{X}^*)$. If $N(L)$ is complemented, and if there exists a constant $\lambda > 0$ such that for each $f \in \mathcal{X}$,*

$$\|K^* f\|_{\mathcal{X}} \leq \lambda \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d},$$

then $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-atomic system for $\mathcal{X}$ with respect to K.

Proof. Using the synthesis operator L, the given inequality in hypothesis can be written as

$$\|K^* f\|_{\mathcal{X}} \leq \lambda \|L^* f\|_{\mathcal{X}_d} \text{ for all } f \in \mathcal{X}.$$

By Theorem 4.1.13, $K = LT$ for some $T \in \mathcal{B}(\mathcal{X}^*, \mathcal{X}_d^*)$. Let $g \in \mathcal{X}^*$ be fixed. Then $Tg \in \mathcal{X}_d^*$. Since $\mathcal{X}_d$ has the canonical unit vectors $\{e_i\}_{i=1}^{\infty}$ as a Schauder basis, the continuous linear functional Tg on $\mathcal{X}_d$ has the form $Tg(c) = \sum_{i=1}^{\infty} c_i d_i$, where $\{d_i\}_{i=1}^{\infty} \in \mathcal{X}_d$ is uniquely determined $d_i = F(e_i)$, and

$$\|Tg\|_{\mathcal{X}_d^*} = \|\{d_i\}_{i=1}^{\infty}\|_{\mathcal{X}_d}.$$

Since T is bounded, the sequence $\{d_i\}_{i=1}^{\infty}$ associated for $g \in \mathcal{X}^*$ satisfies

$$\|\{d_i\}_{i=1}^{\infty}\|_{\mathcal{X}_d} = \|Tg\|_{\mathcal{X}_d^*} \leq \|T\| \, \|g\|_{\mathcal{X}^*}.$$

Also, we have

$$Kg = LTg = L(\{d_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} d_i g_i.$$

Thus $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-atomic system for $\mathcal{X}$ with respect to K. □

Theorem 4.3.7. *Let $K_1, K_2 \in \mathcal{B}(\mathcal{X}^*)$. Let $\{g_i\}_{i=1}^{\infty}$ be an $\mathcal{X}_d$-atomic system for $\mathcal{X}$ with respect to K_1, K_2, and α, β are scalars. If $N(L)$ is complemented, then $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-atomic system for $\alpha K_1 + \beta K_2$.*

Proof. Suppose $\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-atomic system for $\mathcal{X}$ with respect to K_1, K_2 and α, β be any scalars. Then there are constants $0 < \lambda_i \leq \mu_i < \infty$ $(i = 1, 2)$ such that for each $f \in \mathcal{X}$

$$\lambda_i \|K_i^* f\|_{\mathcal{X}} \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \leq \mu_i \|f\|_{\mathcal{X}}.$$

By simple calculations, we get

$$\Big(\frac{|\alpha|}{\lambda_1}+\frac{|\beta|}{\lambda_2}\Big)^{-1}\|(\alpha K_1+\beta K_2)^* f\|_{\mathcal{X}} \le \|\{g_i(f)\}_{i=1}^{\infty}\|_{\mathcal{X}_d} \le \Big(\frac{\mu_1+\mu_2}{2}\Big)\|f\|_{\mathcal{X}}.$$

Therefore by Theorem 4.3.6, $\{g_i\}_{i=1}^{\infty}$ is an atomic system for $\alpha K_1+\beta K_2$. $\square$

We now prove that the notions, "atomic systems" and "frames for operators" are equivalent under the crucial assumptions. The proof of the result given below follows from Theorem 4.2.2 and Theorem 4.3.6.

Theorem 4.3.8. *Let $\mathcal{X}_d$ be a sequence space satisfying the inequality (4.2.3) and $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{X}^*$ be an $\mathcal{X}_d$-Bessel sequence for $\mathcal{X}$. Let $N(L)$ be complemented and $K \in \mathcal{B}(\mathcal{X}^*)$. Then the following statements are equivalent :*

1. *$\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-atomic system for $\mathcal{X}$ with respect to K.*

2. *$\{g_i\}_{i=1}^{\infty}$ is an $\mathcal{X}_d$-K-frame for $\mathcal{X}$.*

Corollary 4.3.9. *(Găvruţa, 2012) Let $\{f_i\}_{i=1}^{\infty}$ be a sequence in a Hilbert space $\mathcal{H}$ and let $K \in \mathcal{B}(\mathcal{H})$. Then the following statements are equivalent :*

1. *$\{f_i\}_{i=1}^{\infty}$ is an atomic system for K.*

2. *$\{f_i\}_{i=1}^{\infty}$ is a K-frame for $\mathcal{H}$.*

Proof. The proof follows from the Theorem 4.3.8 because the assumptions are "redundant" if $\mathcal{X}$ is considered to be a Hilbert space with the sequence space $\mathcal{X}_d = \ell_2$ in Theorem 4.3.8. $\square$

4.4 Conclusion

The notion of frames for operators is introduced for a sequence of continuous linear functionals defined on a Banach space. It is proved that the new notion is a natural extension of Banach frames defined by Casazza et al., in 2005. Necessary and sufficient conditions are derived and results on generating frames for operators are given. Moreover, it is shown that "atomic systems" and "frame for operators" defined in the thesis are not equivalent in general unless some additional requirements are not met by the associated sequence spaces.

Bibliography

Aldroubi, A., Sun, Q., and Tang, W.-S. (2001). "p-frames and shift invariant subspaces of L^p". *J. Fourier Anal. Appl.*, 7(1):1–21.

Antoine, J.-P. and Balazs, P. (2012). "Frames, semi-frames, and Hilbert scales". *Numer. Funct. Anal. Optim.*, 33(7-9):736–769.

Balazs, P. (2008). "Frames and finite dimensionality: frame transformation, classification and algorithms". *Appl. Math. Sci. (Ruse)*, 2(41-44):2131–2144.

Barnes, B. A. (2005). "Majorization, range inclusion, and factorization for bounded linear operators". *Proc. Amer. Math. Soc.*, 133(1):155–162 (electronic).

Casazza, P., Christensen, O., and Stoeva, D. T. (2005). "Frame expansions in separable Banach spaces". *J. Math. Anal. Appl.*, 307(2):710–723.

Casazza, P. G., Han, D., and Larson, D. R. (1999). "Frames for Banach spaces". In *The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999)*, volume 247 of *Contemp. Math.*, pages 149–182. Amer. Math. Soc., Providence, RI.

Casazza, P. G. and Kovačević, J. (2003). "Equal-norm tight frames with erasures". *Adv. Comput. Math.*, 18(2-4):387–430.

Casazza, P. G. and Kutyniok, G. (2004). "Frames of subspaces". In *Wavelets, frames and operator theory*, volume 345 of *Contemp. Math.*, pages 87–113. Amer. Math. Soc., Providence, RI.

Christensen, O. (2003). *"An introduction to frames and Riesz bases"*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA.

Christensen, O. and Stoeva, D. T. (2003). "p-frames in separable Banach spaces". *Adv. Comput. Math.*, 18(2-4):117–126.

Dastourian, B. and Janfada, M. (2016). "Frames for operators in Banach spaces via semi-inner products". *Int. J. Wavelets Multiresolut. Inf. Process.*, 14(3):1650011, 17.

Daubechies, I. (1992). *"Ten lectures on wavelets"*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.

Daubechies, I., Grossmann, A., and Meyer, Y. (1986). "Painless nonorthogonal expansions". *J. Math. Phys.*, 27(5):1271–1283.

Debnath, L. and Mikusiński, P. (1999). *"Introduction to Hilbert spaces with applications"*. Academic Press, Inc., San Diego, CA, second edition.

Ding, M. L., Xiao, X. C., and Zeng, X. M. (2013). "Tight K-frames in Hilbert spaces". *Acta Math. Sinica*, 56(1):105–112.

Donoho, D. L. and Elad, M. (2003). "Optimally sparse representation in general (nonorthogonal) dictionaries via l^1 minimization". *Proc. Natl. Acad. Sci. USA*, 100(5):2197–2202 (electronic).

Douglas, R. G. (1966). "On majorization, factorization, and range inclusion of operators on Hilbert space". *Proc. Amer. Math. Soc.*, 17:413–415.

Duffin, R. J. and Schaeffer, A. C. (1952). "A class of nonharmonic Fourier series". *Trans. Amer. Math. Soc.*, 72:341–366.

Easwaran Nambudiri, T. C. and Parthasarathy, K. (2012). "Generalised Weyl-Heisenberg frame operators". *Bull. Sci. Math.*, 136(1):44–53.

Eldar, Y. C. and Forney, Jr., G. D. (2002). “Optimal tight frames and quantum measurement”. *IEEE Trans. Inform. Theory*, 48(3):599–610.

Faroughi, M. H. and Ahmadi, R. (2010). “Some properties of C-fusion frames”. *Turkish J. Math.*, 34(3):393–415.

Fillmore, P. A. and Williams, J. P. (1971). “On operator ranges”. *Advances in Math.*, 7:254–281.

Fornasier, M. and Rauhut, H. (2005). “Continuous frames, function spaces, and the discretization problem”. *J. Fourier Anal. Appl.*, 11(3):245–287.

Gabor, D. (1946). “Theory of Communication”. *J. IEE*, 93(26):429–457.

Găvruţa, L. (2012). “Frames for operators”. *Appl. Comput. Harmon. Anal.*, 32(1):139–144.

Gröchenig, K. (1991). “Describing functions: atomic decompositions versus frames”. *Monatsh. Math.*, 112(1):1–42.

Gröchenig, K. (2001). *“Foundations of time-frequency analysis”*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA.

Gröchenig, K. and Heil, C. (1999). “Modulation spaces and pseudodifferential operators”. *Integral Equations Operator Theory*, 34(4):439–457.

Han, D., Jing, W., Larson, D., and Mohapatra, R. N. (2008). “Riesz bases and their dual modular frames in Hilbert C^*-modules”. *J. Math. Anal. Appl.*, 343(1):246–256.

Han, D. and Larson, D. R. (2000). “Frames, bases and group representations”. *Mem. Amer. Math. Soc.*, 147(697):x+94.

Jr., R. W. H. and Paulraj, A. (2002). “Linear dispersion codes for MIMO systems based on frame theory”. *IEEE Transactions on Signal Processing*, 50(10):2429–2441.

Kaufman, W. E. (1979). "Semiclosed operators in Hilbert space". *Proc. Amer. Math. Soc.*, 76(1):67–73.

Limaye, B. V. (1996). *"Functional analysis"*. New Age International Publishers Limited, New Delhi, second edition.

Mallat, S. (2009). *"A wavelet tour of signal processing"*. Elsevier/Academic Press, Amsterdam, third edition. The sparse way, With contributions from Gabriel Peyré.

Obeidat, S., Samarah, S., Casazza, P. G., and Tremain, J. C. (2009). "Sums of Hilbert space frames". *J. Math. Anal. Appl.*, 351(2):579–585.

Pełczyński, A. and Singer, I. (1964/1965). "On non-equivalent bases and conditional bases in Banach spaces". *Studia Math.*, 25:5–25.

Shokrollahi, A., Hassibi, B., Hochwald, B. M., and Sweldens, W. (2001). "Representation theory for high-rate multiple-antenna code design". *IEEE Trans. Inform. Theory*, 47(6):2335–2367.

Singer, I. (1970). *"Bases in Banach spaces. I"*. Springer-Verlag, New York-Berlin. Die Grundlehren der mathematischen Wissenschaften, Band 154.

Singer, I. (1981). *"Bases in Banach spaces. II"*. Editura Academiei Republicii Socialiste România, Bucharest; Springer-Verlag, Berlin-New York.

Stoeva, D. T. (2006). "On p-frames and reconstruction series in separable Banach spaces". *Integral Transforms Spec. Funct.*, 17(2-3):127–133.

Stoeva, D. T. (2008). "Generalization of the frame operator and the canonical dual frame to Banach spaces". *Asian-Eur. J. Math.*, 1(4):631–643.

Sun, W. (2006). "G-frames and g-Riesz bases". *J. Math. Anal. Appl.*, 322(1):437–452.

Xiao, X., Zhu, Y., and Găvruţa, L. (2013). "Some properties of K-frames in Hilbert spaces". *Results Math.*, 63(3-4):1243–1255.

Xiao, X.-c., Zhu, Y.-c., Shu, Z.-b., and Ding, M.-l. (2015). "G-frames with bounded linear operators". *Rocky Mountain J. Math.*, 45(2):675–693.

Zhang, H. and Zhang, J. (2011). "Frames, Riesz bases, and sampling expansions in Banach spaces via semi-inner products". *Appl. Comput. Harmon. Anal.*, 31(1):1–25.

Zhong, X. and Yong, M. (2016). "Frame sequences and dual frames for operators". *ScienceAsia*, 42(3):222–230.

PUBLICATIONS

1. P. Sam Johnson and G. Ramu, Class of bounded operators associated with an atomic system, *Tamkang J. Math*, Vol.46, No.1 (2015), 85-90.

2. G. Ramu and P. Sam Johnson, Frame operators of K-frames, *S$\vec{e}$MA J.*, Vol.73 (2016), No.2, 171-181.

3. G. Ramu and P. Sam Johnson, Frames for operators in Banach spaces, to appear in Acta Mathematica Vietnamica.

BIODATA

Name	:	Ramu Geddavalasa
Email	:	ramug09@gmail.com
Date of Birth	:	01 July 1985.
Permanent address	:	G. Ramu, *S/o* G. Appala Naidu, G.N Puram Village, Santhakavati Mandal, Andhra Pradesh-532 168.

Educational Qualifications :

Degree	**Year**	**Institution / University**
B.Sc. Mathematics	2005	D.N.R College, Bhimavaram.
M.Sc. Mathematics	2007	Andhra University, Visakhapatnam.
M.Tech. Computer Applications	2011	NITK Surathkal, Mangalore.