A STUDY ON GRAPH OPERATORS AND COLORINGS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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To my family

DECLARATION

By the Ph.D. Research Scholar

I hereby *declare* that the Research Thesis entitled **A STUDY ON GRAPH OPERATORS AND COLORINGS** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **A STUDY ON GRAPH OPERATORS AND COLORINGS** submitted by **V. V. P. R. V. B. SURESH DARA**, (Reg. No.: 121196 MA12F04) as the record of the research work carried out by him, is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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ABSTRACT

In 1972, *Erdös - Faber - Lovász* conjectured that, if **H** is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the vertices with n colors so that no two vertices with the same color are in the same edge. In this research work we give a method of coloring of the linear hypergraph **H** satisfying the hypothesis of the conjecture and we partially prove the *Erdös - Faber - Lovász conjecture* theoretically.

Let G be a graph and $\mathscr{K}_G$ be the set of all cliques of G, then the clique graph of G denoted by $K(G)$ is the graph with vertex set $\mathscr{K}_G$ and two elements $Q_i, Q_j \in \mathscr{K}_G$ form an edge if and only if $Q_i \cap Q_j \neq \emptyset$.

We prove a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1 + G_2$, give a partial characterization for clique divergence of the join of graphs and prove that if G_1, G_2 are Clique-Helly graphs different from K_1 and $G = G_1 \square G_2$, then $K^2(G) = G$.

Let G be a labeled graph of order α, finite or infinite, and let $\mathfrak{N}(G)$ be the set of all labeled maximal forests of G. The forest graph of G, denoted by $\mathbf{F}(G)$, is the graph with vertex set $\mathfrak{N}(G)$ in which two maximal forests F_1, F_2 of G form an edge if and only if they differ exactly by one edge, i.e., $F_2 = F_1 - e + f$ for some edges $e \in F_1$ and $f \notin F_1$.

Using the theory of cardinal numbers, Zorn's lemma, transfinite induction, the axiom of choice and the well-ordering principle, we determine the **F**-convergence, **F**-divergence, **F**-depth and **F**-stability of any graph G.

Keywords: Chromatic number, Erdös - Faber - Lovász conjecture, Graph dynamics, Graph Operators, Forest graph operator, Maximal clique, Clique graph, Join of graphs, Cartesian product of graphs, Clique-Helly graphs and Infinite cardinals

Contents

List of Figures

Chapter 1

INTRODUCTION

In 1736, a Swiss Mathematician Leonhard Eular (1707-1783) solved the well known Konigsberg Bridge problem. The method he used to solve it is considered by many to be the birth of Graph Theory. Later in 19th century German Physicist Gustav Kirchhoff (1824-1887) investigated electrical circuits leads to the development of results on trees in graph. But the term tree was introduced by the British Mathematician Arthur Cayley (1821-1895) in 1857 while studying the enumeration of organic chemical isomers. In the early 20th century, a French Mathematician and a Theoretical Physicist, Poincare (1854-1912) defined in principle what is known as the incidence matrix of a graph. In 1936, the first book on graph theory was published by Denes Konig (1884-1944). After Second World War, further books appeared on graph theory (Ore, Behzad and Chartrand, Tutte, Berge, Harary, Gould, Wilson, West and Diestel among many others). Graph theory has found many applications in engineering and science, such as electrical, chemical, civil and mechanical, communication, operational research, computer science and other scientific and not-so-scientific areas.

1.1 Basic Definitions

A *graph* G consists of a set V of vertices (points, nodes) and a set E of edges(lines, connections) such that each edge $e \in E$ is associated with ordered or unordered pair of elements of V, i.e., there is a mapping from the set of edges E to set of ordered or unordered pairs of elements of V. The graph G with vertex set V and edge set E is written as $G = (V, E)$ or $G(V, E)$.

If an edge $e \in E$ is associated with an ordered pair (u,v) or an unordered pair (u,v), where $u,v \in V$, then e is said to connect u and v and u,v are called end points of e. An edge is said to be incident with vertices it joins. Thus, the edge e that joins the vertices u and v, is said to be incident on each of its end points u and v. Any pair of vertices that is connected by an edge in a graph are called adjacent vertices. In a graph a vertex that is not adjacent to any another vertex is called an isolated vertex.

A graph $G(V,E)$ is said to be *finite* if it has a finite number of vertices and finite number of edges. (A graph with finite number of vertices must also have finite number edges): otherwise, it is *infinite* graph, $|V(G)|$ denotes the number vertices in G and is called the *order* of G. Similarly, $|E(G)|$ denotes the number of edges in G and is called the *size* of G. If G is a (p,q) graph then G has p vertices and q edges.

Two or more edges joining the same pair of vertices are known as multiple edges, and an edge joining a vertex to itself is called a loop. A graph with no loops and multiple edges is called a *simple graph*. In a graph if multiple edges are allowed, but no loops, then the graph is known as a *multi graph*. If both the loops and the multiple edges are allowed in a graph, then the graph is considered to be a *pseudo graph*.

A *subgraph* of G is a graph having all of its vertices and edges in G. If G_1 is a subgraph of G, then G is a *super graph* of G_1. A *spanning subgraph* is a subgraph containing all the vertices of G. For any set S of vertices of G, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S. The *removal of a vertex* v_i from a graph G results in a maximal subgraph $G - v_i$, of G not containing the vertex v_i. Similarly, the removal of an edge x_i results in a maximal subgraph $G - x_i$, of G except x_i.

A *walk* of G is a finite sequence $\{v_0, e_1, v_1, e_2, v_2, e_3, \ldots, e_n, v_n\}$ whose terms are alternately vertices v_i and edges e_i of G for $1 \leq i \leq n$, and v_{i-1} and v_i are the two ends of e_i. A *trail* in G is a walk in which no edge of G appears more than once. A *path* P is a trail in which no vertex appears more than once.

Two vertices v_i and v_j are said to be *connected* in G if there exists a path between these vertices. A graph G is called *connected* if all pairs of its vertices are connected. A *component* of a graph G is a maximal connected subgraph, i.e., it is not a subgraph of any other connected subgraph of G.

A *tree* T is a connected acyclic graph. A tree of a graph G is an acyclic connected subgraph of G. A set of trees of G forms a *forest*. A *spanning tree* of G is a connected, acyclic, spanning subgraph of G. If G is disconnected, then the acyclic spanning subgraph is called the *forest* of G. A forest F of G is said to be *maximal* if there is no forest F' of G such that F is a proper subgraph of F'.

An *Euler trail* of a graph G is a trail that visits every edge once. A connected graph G is *Eulerian*, if it has a closed trail containing every edge of G. Such a trail is called an *Euler tour*. A path P of a graph G is a *Hamilton path*, if P visits every vertex of G once. Similarly, a cycle C is a *Hamilton cycle*, if it visits each vertex once. A graph is *Hamiltonian*, if it has a Hamilton cycle.

The set of vertices adjacent to a vertex v is called the *neighborhood* of v, denoted by $N(v)$. This is called the *open neighborhood* of v and the *closed neighborhood* of v is denoted by $N[v]$, defined by $N(v) \cup \{v\}$. The *degree* of a vertex v is the number of edges incident with v; it is denoted by $deg(v)$. The *minimum* degree among the vertices of G is denoted by $\delta(G)$ and the *maximum* degree by $\Delta(G)$. If $\delta(G) = \Delta(G) = r$, then G is called a *regular graph* of degree r. If $r = n-1$ then the graph is a *complete graph*. A vertex with degree 1 is called as a *pendant vertex*. The *degree sequence* of a graph is the list of vertex degrees, usually written in non-increasing order, as $d_1 \geq d_2 \geq \ldots.. \geq d_n$. A *graphic sequence* is a list of nonnegative numbers, that is, the degree sequence of some simple graph. A *clique* in G is a maximal complete subgraph in G.

A graph G is *labeled* if the n vertices of G are distinguished from each other by names, such as, $v_1, v_2, ..., v_n$. Two graphs G and H are *isomorphic*, written $G \cong H$, if there exists a one-to-one correspondence between their vertex sets, which preserves the adjacency.

A *bipartite graph* G is a graph whose vertex set can be partitioned into two sets V_1 and V_2 such that, every edge of G joins a vertex of V_1 with a vertex of V_2. If every vertex of V_1 is joined with every vertex of V_2 then G is said to be *complete bipartite graph* and is denoted by $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. In particular a complete bipartite graph $K_{1,n}$ is called a *star*. Every non-trivial tree is a bipartite graph.

A graph is said to be a *planar graph*, if it can be drawn on a plane so that no two edges intersect. A plane graph is the one which is already drawn in a plane so that no two edges intersect. The regions defined by the plane graph are the faces of the plane graph; the unbounded region is called the exterior face.

A graph can be associated with a matrix. Or in other words, a graph can be represented in terms of matrices. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and the edge set $E(G) = \{e_1, e_2, ..., e_m\}$. The *adjacency matrix* of G, written $A(G)$, is the $n \times n$ matrix in which entry a_{ij} is the number of edges in G with end vertices v_i, v_j. The *incidence matrix* $M(G)$ is the $n \times m$ matrix in which entry m_{ij} is 1 if v_i is an end vertex of e_j otherwise 0. Note that, every adjacency matrix is symmetric matrix. An adjacency matrix of a simple graph G has entries 0 or 1, with 0*s* on the diagonal. The degree of v is the sum of the entries in the rows for v in either $A(G)$ or $M(G)$. The study of the matrices associated with the graphs, created a branch of graph theory, called the spectral graph theory, which deals with the energy of graphs.

The *vertex coloring* of a graph $G = (V, E)$ is a map $c : V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of the set S are called the available colors. Such a coloring is often referred to as a proper coloring. If k distinct colors are used in coloring of G, it is referred to as a k-coloring of G and we say that G is k-colorable. The *chromatic number* $\chi(G)$ is the least k such that G is k-colorable. Note that a k-coloring is nothing but a vertex partition into k independent sets.

A *hypergraph* is a structure $H = (V, (E_i : i \in I))$ where the vertex set V is an arbitrary set, and every $E_i \subseteq V$. These sets E_i are called the hyperedges of the hypergraph.

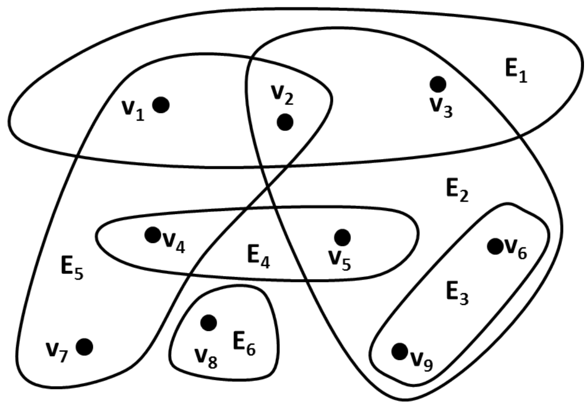


Figure 1.1 Hypergraph

A hypergraph is said to be *linear* if no two hyperedges have more than one vertex in common. A hypergraph is said to be *uniform* if all of its hyperedges have the same number of vertices as each other. The *degree* of a vertex v in H is the number of edges containing v. The *minimum* degree among the vertices is denoted by $\delta(H)$ and the *maximum* degree by $\Delta(H)$. A hypergraph H is said to be *dense* if $\delta(H)$ is greater than $\sqrt{n}$. A *coloring of a hypergraph* is an assignment of colors to the vertices so that no two vertices of an edge has the same color. A *k-coloring* of a hypergraph is a coloring of it where the number of used colors is at most k.

1.2 Erdös - Faber - Lovász conjecture

One of the famous conjectures in graph theory is Erdös - Faber - Lovász (EFL) conjecture. It states that any linear hypergraph H on n vertices has chromatic number at most n. Erdös, in 1975 offered 50 USD (Erdős, 1975) and in 1981, offered 500 USD (Erdős,

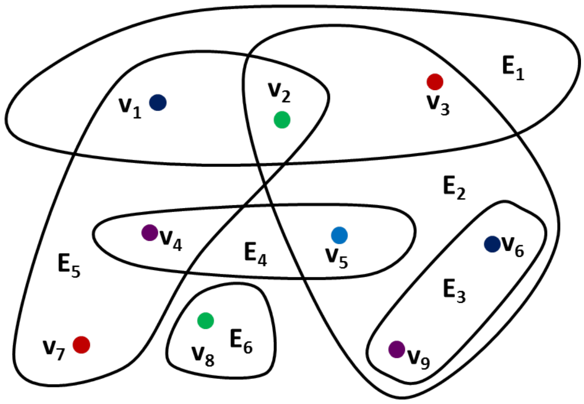


Figure 1.2 Colored Hypergraph

1981; Jensen and Toft, 2011) for the proof or the disproof of the conjecture.

Chang and Lawler (Chang and Lawler, 1988) presented a simple proof that the edges of a simple hypergraph on n vertices can be colored with at most [1.5n-2] colors. Kahn (Kahn, 1992) showed that the chromatic number of H is at most $n+o(n)$. Jackson et al., (Jackson et al., 2007) proved the conjecture is true when the partial hypergraph S of H determined by the edges of size at least three can be Δ_S-edge-colored and satisfies $\Delta_S \leq 3$. In particular, the conjecture holds when S is unimodular and $\Delta_S \leq 3$. Paul and Germina (Paul and Germina, 2012) established the truth of the conjecture for all linear hypergraphs on n vertices with $\Delta(H) \leq \sqrt{n+\sqrt{n}+1}$. Sanchez-Arroyo (Sánchez-Arroyo, 2008) proved that the conjecture is true for dense hypergraphs. Faber (Faber, 2010) proved that for fixed degree, there can be only finitely many counter examples to EFL on this class (both regular and uniform) of hypergraphs. Romero et.al., (Romero and Alonso-Pecina, 2014) proved that the conjecture is true for $n \leq 12$. We consider the equivalent version of the conjecture for simple graphs given by Deza et al., (Deza et al., 1978; Sánchez-Arroyo, 2008; Jensen and Toft, 2011; Mitchem and Schmidt, 2010), stated as below.

Conjecture: *Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $(A_1, A_2, \ldots, A_n)$, each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n.*

1.3 Graph Dynamics

The concept of a dynamical system has its origin in Newtonian mechanics. There, as in other natural sciences and in engineering disciplines, the evolution rule of dynamical systems is given implicitly by a relation that gives the state of system only a short time into the future (the relation is either a differential equation or difference equation or another time scale). To determine the state for all future times require the relation to be iterated many times-each advancing in time a small step. The iteration procedure is referred to as solving the system or integrating the system. Once the system is solved, it is possible to determine all its future positions, given an initial point. Linear dynamical systems can be solved in terms of simple functions.

A discrete dynamical system (or simply a dynamical system) is an ordered pair (X,ϕ), where X is nonempty set and ϕ is a mapping from $X \to X$. The set X is called as the underlying state space and ϕ as the rule of motion. Dynamics is introduced by the iterates of ϕ. For any $x \in X$, $\phi(x)$ is interpreted as the position to which x reaches after one unit of time. Similarly, $\phi^n(x)$ is interpreted as the position of x after n units of time, where $\phi^n(x) = \phi(\phi^{n-1}(x))$ for $n > 1$.

In mathematics, the concept of graph dynamical systems (GDS) can be used to capture a wide range of processes taking place on graphs or networks. A major theme in the mathematical and computational analysis of graph dynamical systems is to relate their structural properties (e.g. network connectivity).

A graph dynamical system is a discrete dynamical system where X is a set of graphs (see (Prisner, 1995)).

Examples: Line graph operator, Tree graph operator, Clique graph operator etc.

1.4 Graph Operators

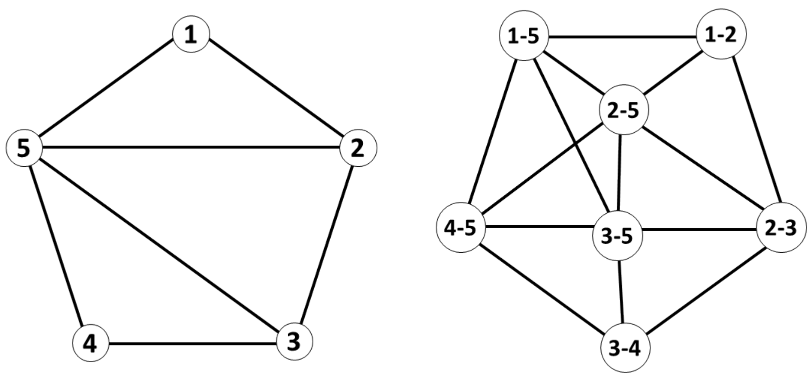
In the literature, line graph is the graph operator which started first and the term line graph appeared in the paper of Harary (Harary and Norman, 1960); but the construction of line graph is used by Whitney (Whitney, 1932) and Krausz (Krausz, 1943). Ore (Ore, 1962), used the definition of line graph in the name of interchange graphs and he posed some problems on it. In 1960's, several people worked on line graphs. In 1966, Cummins (Cummins, 1966) introduced tree graph operator. Hamelink (Hamelink, 1968) used the clique graph operator. In 1970's, number of graph operators were introduced.

Definition 1.4.1. *Let S be a set and $F = \{S_1, S_2, \ldots, S_p\}$ be a family of distinct nonempty subsets of S whose union is S. The intersection graph of F is denoted by $\Omega(F)$ and is defined by $V(\Omega(F)) = F$, with S_i and S_j adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. Then a graph G is an intersection graph on S if there exists a family F of subsets of S for which $G \cong \Omega(F)$.*

Theorem 1.4.2. *Every graph is an intersection graph*

Line graph of a graph $G = (V, E)$ is the intersection graph of E and clique graph of a graph G is the intersection graph of $\mathscr{K}_G$, where $\mathscr{K}_G$ is the set of all maximal cliques of G.

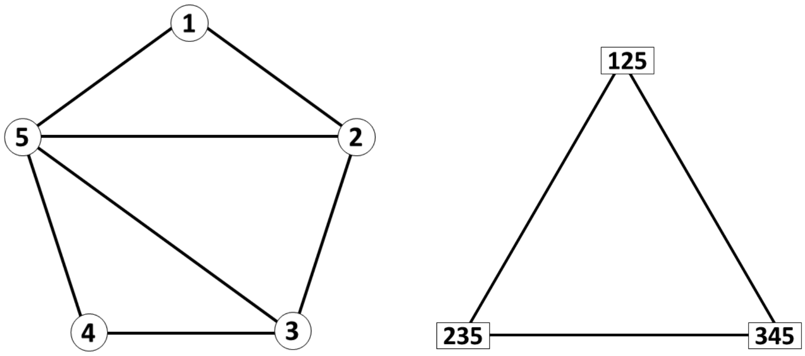
Definition 1.4.3. *The line graph of $G = (V, E)$, denoted by $L(G)$, is the intersection graph $\Omega(E)$. Thus the points of $L(G)$ are the lines of G, with two points of $L(G)$ adjacent whenever the corresponding lines of G are incident.*



(a) G (b) $L(G)$

Figure 1.3 L(G) is the line graph of the graph G

Definition 1.4.4. *Given a graph G of order finite or infinite, denote by $V = \mathscr{K}_G$ the set of all cliques of G. Define an adjacency relation in V as follows. The cliques Q_i, Q_j are* *said to be adjacent if $Q_i \cap Q_j \neq \emptyset$. The resultant graph is called the Clique Graph of G and is denoted by $K(G)$. The operator K is called the Clique Graph Operator.*



(a) G (b) $K(G)$

Figure 1.4 K(G) is the clique graph of the graph G

1.5 Properties of Graph Operators

The study of graphs and their iterated graphs using the graph operators is the dynamics of graphs and is called graph dynamics. A discrete dynamical system is any set X together with a mapping $\phi : X \to X$. The elements of X are called states. A graph dynamical system is a discrete dynamical system where X is a set of graphs (see (Prisner, 1995)).

There are some dynamical properties to study the graphs using graph operators. The following definitions are taken from (Prisner, 1995). Let (X, ϕ) be the discrete dynamical system, where ϕ is a mapping from $X \to X$.

Definition 1.5.1. *Let $x \in X$. Then x is said to be ϕ-convergent if the set $\{\phi^n(x) : n \in \mathbb{N}\}$ is finite, otherwise x is ϕ-Divergent.*

Definition 1.5.2. *For the given operator ϕ, a ϕ-root of an element $x \in X$ is any $y \in X$ with $\phi(y) = x$.*

Let $x \in X$. We say x has ϕ-root if there exists an element $y \in X$ such that $\phi(y) = x$.

Definition 1.5.3. *The ϕ-depth of an element $x \in X$ is defined as the supremum of the set of all natural numbers n for which there is an element $y \in X$ such that $\phi^n(y) = x$.*

The ϕ-depth of an element x is said to be zero if x has no ϕ-root.

Definition 1.5.4. *Let $x \in X$. Then x is said to be periodic if there is some natural number n with $x = \phi^n(x)$. The smallest such number is called the period of this periodic state x.*

If $n = 1$, x is called the stable(fixed).

Definition 1.5.5. *Let $x \in X$. If x is ϕ-convergent, then its ϕ-transition number $t_0^{\phi}(x)$ is defined as the least positive integer t such that $\phi^t(x)$ is ϕ-periodic.*

1.6 Clique Graph

Let G be a graph and $\mathscr{K}_G$ be the set of all cliques of G. The clique graph $K(G)$ of G is defined as the intersection graph $\Omega(\mathscr{K}_G)$ of the family of cliques of G, in the sense that the vertex set of $K(G)$ is the family $\mathscr{K}_G$ and two distinct vertices $Q_i, Q_j \in \mathscr{K}_G$ are adjacent in $K(G)$ if $Q_i \cap Q_j \neq \emptyset$. A given graph H is called a clique graph if there exists a graph G such that $H \cong K(G)$ and G is called a K-root of H. A graph which is not a clique graph in this sense is called a K-primitive graph. Further, the n^{th} iterated clique graph $K^n(G)$ of G is then defined by the following rule:

$$K^1(G) := K(G), K^n(G) := K(K^{n-1}(G)), \forall\, n \geq 2.$$

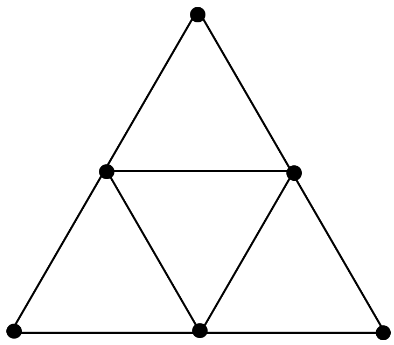
The sequence $\mathscr{O}_G^{\mathscr{K}} := (K^0 := G, K^1(G), K^2(G), \ldots,)$ is called the K-orbit of G and G is K-periodic (K-aperiodic) if there exists a (no) positive integer n such that $G \cong K^n(G)$ and the least such integer is called the K-periodicity of G, denoted by K-per(G). Further, G is said to be K-convergent if there are only a finite number of non isomorphic graphs in the K-orbit of G, otherwise it is said to be K-divergent. If G is a K-convergent graph then its K-transition number $t_0^{\mathscr{K}}(G)$ is defined as the least positive integer t such that $K^t(G)$ is K-periodic.

Definition 1.6.1. *A graph G is said to have the Helly property if every set* $\{C_i : i \in I\}$ *of cliques of G, no two of which are disjoint (i.e.,* $C_i \cap C_j \neq \emptyset \ \forall\ i, j \in I$*), has nonempty total intersection (i.e.,* $\bigcap_{i \in I} C_i \neq \emptyset$*).*

Hamelink (Hamelink, 1968) gave a result that, every graph need not be a clique graph of some graph.

Theorem 1.6.2. *Any graph H containing a clique T on 3 vertices* $\{x,y,z\}$ *and 3 other cliques* A, B *and* C *so related that* $V(T) \cap V(A) = \{x,y\}$*,* $V(T) \cap V(B) = \{y,z\}$*, and* $V(T) \cap V(C) = \{z,x\}$ *is not the clique graph of any graph.*

Example 1.6.3. *The following six vertex graph is not the clique graph of any graph.*



Also he gave a partial characterization for clique graph,

Theorem 1.6.4. *If H satisfies Helly property then H is a clique graph.*

Using Helly property, Roberts and Spencer (Roberts and Spencer, 1971) gave a characterization of clique graph.

Theorem 1.6.5. *A graph H is a clique graph if and only if H satisfies Helly property.*

Definition 1.6.6. *We say that G has the* T_1 *property if for any distinct vertices* $x, y \in G^*$ *with* $deg(x, G^*) \geq 2$*,* $deg(y, G^*) \geq 2$*, there exists two cliques* C, D *in* $K(G)$ *with* $x \in C, y \notin C$ *and* $y \in D, x \notin D$*.*

Lim (Lim, 1982) generalized the result of S. T. Hedetniemi and P. J. Slater, on first iterated clique graph.

Theorem 1.6.7. *If G is a graph which satisfies the Helly property and the T_1 property, then $K^2(G) \cong G^* - \{x \in G^* : deg(x, G^*) = 1\}$.*

Hedman (Hedman, 1986) has given a polynomial algorithm for constructing the clique graph of a line graph $K(L(G))$.

A vertex v of a triangle C_3 with $deg(v) \geq 3$ is called an outlet of C_3.

Theorem 1.6.8. *Graph G satisfies $K(L(G)) = G$ if and only if G satisfies the following three conditions:*

1. *For all $v \in V(G)$, $deg(v) \geq 2$.*
2. *G has no adjacent triangles.*
3. *Every triangle of G has exactly two outlets.*

Gravier et al., (Gravier et al., 2004) proved the conjecture of Protti and Szwarcfiter (Protti and Szwarcfiter, 2000) on clique-inverse graphs of K_p-free graphs.

Theorem 1.6.9. *For every integer $p \geq 4$, the class of the clique-inverse graphs of the K_p-free graphs can be characterized by a finite family of forbidden induced subgraphs.*

Frias-Armenta et al., (Frías-Armenta et al., 2005) established a result on clique divergent

Theorem 1.6.10. *Every clique divergent graph is a spanning subgraph of a clique divergent graph with diameter 2.*

Alcon et al.,(Alcón et al., 2009) proved that the complexity of clique graph recognition is NP-complete.

1.7 Tree Graph

Linear graphs play an important role in the study of electrical networks and topological formulas are found to be convenient to study the effect of parameter variations in a network. Network functions such as the driving-point and transfer functions must be expressed in a symbolic form. These formulas require a list of trees of a given network. Many methods exist in the literature to find all the trees of a network. Among them, the method proposed by Mayeda and Seshu (Mayeda and Seshu, 1965) succeeded in generating all the trees without duplication by successive application of elementary tree transformations.

Cummins (Cummins, 1966) made an interesting investigation on trees. He defined T-graph (Tree graph) of a graph G as the graph whose vertex set is the set of all spanning trees of G, and two spanning trees T_1, T_2 of G form an edge if and only if T_1 and T_2 differ by exactly one edge. Hence the tree graph associated with a connected graph G is linear graph in which the vertices are in one-to-one correspondence with the spanning trees of G and the edges represent the adjacencies of trees. Cummins showed that a tree graph always contains a Hamilton circuit. This is then extended to directed graphs and generalized theorem for directed graphs is established by Chen (Chen, 1967). Genya Kishi and Yoji Kqajitani (Kishi and Kajitani, 1968) also worked on tree graphs. They proposed a decomposition of a tree graph into complete subgraphs. In 1968 Shank (Shank, 1968) gave a short proof for the Cummins result.

Let G be a labeled graph of order α, finite or infinite (all our graphs are labeled). A *spanning tree* of G is a connected, acyclic, spanning subgraph of G; it exists if and only if G is connected. Any acyclic subgraph of G, connected or not, is called a *forest* of G. A forest F of G is said to be *maximal*, if there is no forest F' of G such that F is a proper subgraph of F'. The *tree graph* $\mathbf{T}(G)$ of G has all the spanning trees of G as vertices, and distinct such trees are adjacent vertices if they differ in just one edge (Prisner, 1995; Suresh et al., 2010); i.e., two spanning trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The *iterated tree graphs* of G are defined by $\mathbf{T}^0(G) = G$ and $\mathbf{T}^n(G) = \mathbf{T}(\mathbf{T}^{n-1}(G))$ for $n > 0$. There are several results on tree graphs. See (Broersma and Xueliang, 1996; Zhang and Chen, 1986; Liu, 1988) for connectivity of the tree graph, (Grimmett, 1976; Rodriguez and Petingi, 1997; Teranishi, 2005; Das, 2007; Feng et al., 2008; Li et al., 2010; Das et al., 2013; Feng et al., 2014) for bounds on the order of $\mathbf{T}(G)$ (that is, on the number of spanning trees of G), (Cummins, 1966; Shank, 1968) for Hamilton circuits in a tree graph.

There is one difficulty with iterating the tree graph operator. The tree graph of an infinite connected graph need not be connected (Cummins, 1966; Shank, 1968), so $\mathbf{T}^2(G)$ may be undefined. For example, $\mathbf{T}(K_{\aleph_0})$ is disconnected (see Corollary 4.1.5 in this thesis; $\aleph_0$ denotes the cardinality of the set $\mathbb{N}$ of natural numbers); therefore $\mathbf{T}^2(K_{\aleph_0})$ is not defined. To obviate this difficulty with iterated tree graphs, and inspired by the tree graph operator $\mathbf{T}$, we define a forest graph operator. Let $\mathfrak{N}(G)$ be the set of all maximal forests of G. The *forest graph* of G, denoted by $\mathbf{F}(G)$, is the graph with vertex set $\mathfrak{N}(G)$ in which two maximal forests F_1, F_2 form an edge if and only if they differ by exactly one edge. The *forest graph operator* (or *maximal forest operator*) on graphs, $G \mapsto \mathbf{F}(G)$, is denoted by $\mathbf{F}$. Zorn's lemma implies that every connected graph contains a spanning tree (see (Diestel, 2005)); similarly, every graph has a maximal forest. Hence, the forest graph always exists. Since, when G is connected, maximal forests are the same as spanning trees, then $\mathbf{F}(G) = \mathbf{T}(G)$; that is, the tree graph is a special case of the forest graph. We write $\mathbf{F}^2(G)$ to denote $\mathbf{F}(\mathbf{F}(G))$, and in general $\mathbf{F}^n(G) = \mathbf{F}(\mathbf{F}^{n-1}(G))$ for $n \geq 1$, with $\mathbf{F}^0(G) = G$.

Definition 1.7.1. *A graph G is said to be* **F**-convergent *if* $\{\mathbf{F}^n(G) : n \in \mathbb{N}\}$ *is finite; otherwise it is* **F**-divergent.

Definition 1.7.2. *A graph H is said to be* **F**-root *of G if $\mathbf{F}(H)$ is isomorphic to G, $\mathbf{F}(H) \cong G$. The* **F**-depth *of G is*

$$\sup\{n \in \mathbb{N} : G \cong \mathbf{F}^n(H) \text{ for some graph } H\}.$$

The **F***-depth of a graph G that has no* **F***-root is said to be zero.*

Definition 1.7.3. *The graph G is said to be* **F**-periodic *if there exists a positive integer n such that $\mathbf{F}^n(G) = G$. The least such integer is called the* **F**-periodicity *of G. If $n = 1$, G is called* **F**-stable.

Chapter 2

ERDÖS - FABER - LOVÁSZ CONJECTURE

In 1972, Erdös - Faber - Lovász (EFL) conjectured that, if **H** is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the vertices with n colors so that no two vertices with the same color are in the same edge. In 1978, Deza, Erdös and Frankl had given an equivalent version of the same for graphs: Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $A_1, A_2, \ldots, A_n$, each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n.

The clique degree $d^K(v)$ of a vertex v in G is given by $d^K(v) = |\{A_i : v \in V(A_i), 1 \leq i \leq n\}|$. The maximum clique degree $\Delta^K(G)$ of the graph G is given by $\Delta^K(G) = max_{v \in V(G)} d^K(v)$. In this chapter, using Symmetric Latin Squares, we give an algorithmic proof of the above conjecture.

2.1 Introduction

One of the famous conjectures in graph theory is Erdös - Faber - Lovász conjecture. It states that, if **H** is a linear hypergraph consisting of n edges of cardinality n, then it is possible to color the vertices of **H** with n colors so that no two vertices with the same color are in the same edge (Berge, 1990). Erdös, in 1975, offered 50 USD (Erdős, 1975, 1981) and in 1981, offered 500USD (Erdős, 1981; Jensen and Toft, 2011) for the proof or disproof of the conjecture.

Vance Faber quoted: “In 1972, Paul Erdös, László Lovász and I got together at a tea party in Colorado. This was a continuation of the discussions we had a few weeks before in Columbus, Ohio, at a conference on hypergraphs. We talked about various conjectures for linear hypergraphs analogous to Vizing's theorem for graphs. Finding tight bounds in general seemed difficult, so we created an elementary conjecture that we thought that it would be easy to prove. We called this the n sets problem: given n sets, no two of which meet more than once and each with n elements, color the elements with n colors so that each set contains all the colors. In fact, we agreed to meet the next day to write down the solution. Thirty-Eight years later, this problem is still unsolved in general."

Chang and Lawler (Chang and Lawler, 1988) presented a simple proof that the edges of a simple hypergraph on n vertices can be colored with at most [1.5n-2] colors. Kahn (Kahn, 1992) showed that the chromatic number of $\mathbf{H}$ is at most $n+o(n)$. Jackson et al. (Jackson et al., 2007) proved that the conjecture is true when the partial hypergraph S of $\mathbf{H}$ determined by the edges of size at least three can be Δ_S-edge-colored and satisfies $\Delta_S \leq 3$. In particular, the conjecture holds when S is unimodular and $\Delta_S \leq 3$. Paul and Germina (Paul and Germina, 2012) established the truth of the conjecture for all linear hypergraphs on n vertices with $\Delta(\mathbf{H}) \leq \sqrt{n+\sqrt{n}+1}$. Sanchez - Arroyo (Sánchez-Arroyo, 2008) proved the conjecture for dense hypergraphs. We consider the equivalent version of the conjecture for graphs given by Deza, Erdös and Frankl in 1978 (Deza et al., 1978; Sánchez-Arroyo, 2008; Jensen and Toft, 2011; Mitchem and Schmidt, 2010).

Conjecture 2.1.1. *Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $(A_1, A_2, \ldots, A_n)$, each having exactly n vertices and have the property that every pair of complete graphs has at most one common vertex, then the chromatic number of G is n.*

Example 2.1.2. *Figure 2.1 shows all the graphs for $n = 3$ which are satisfying the hypothesis of the conjecture 2.1.1.*

Figure 2.2 shows the construction of the graph G from the hypergraph $\mathbf{H}$.

Haddad and Tardif (Haddad and Tardif, 2004) introduced the problem with a story about seating assignment in committees: suppose that, in a university department, there are k committees, each consisting of k faculty members, and that all committees meet

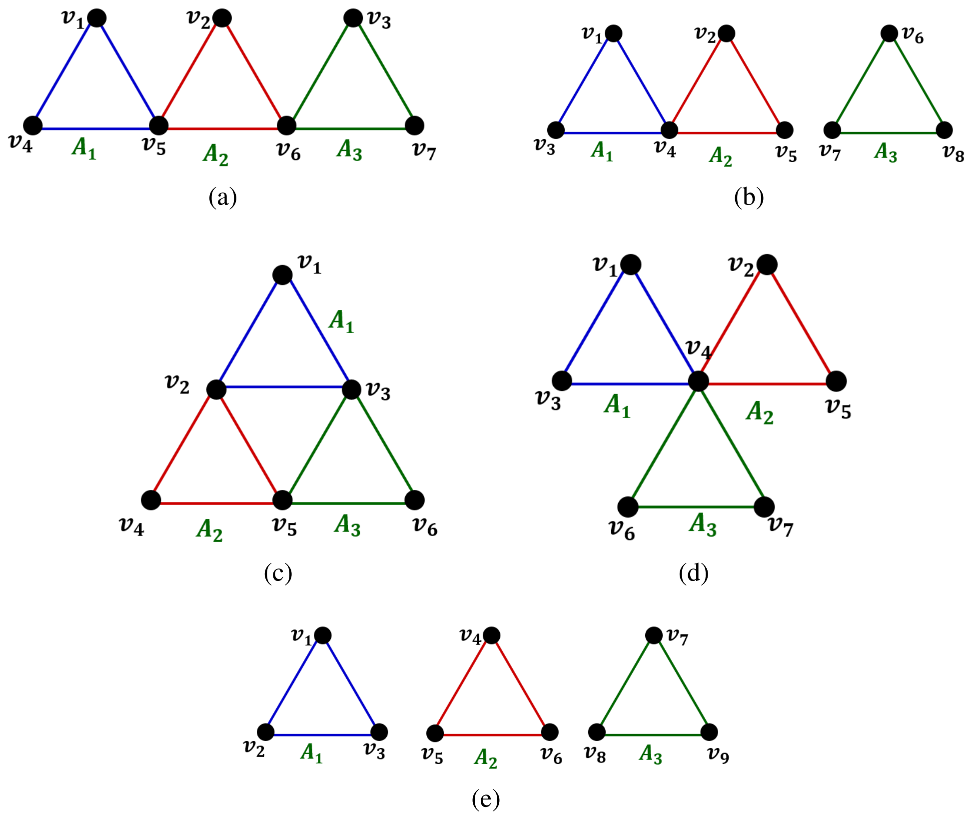
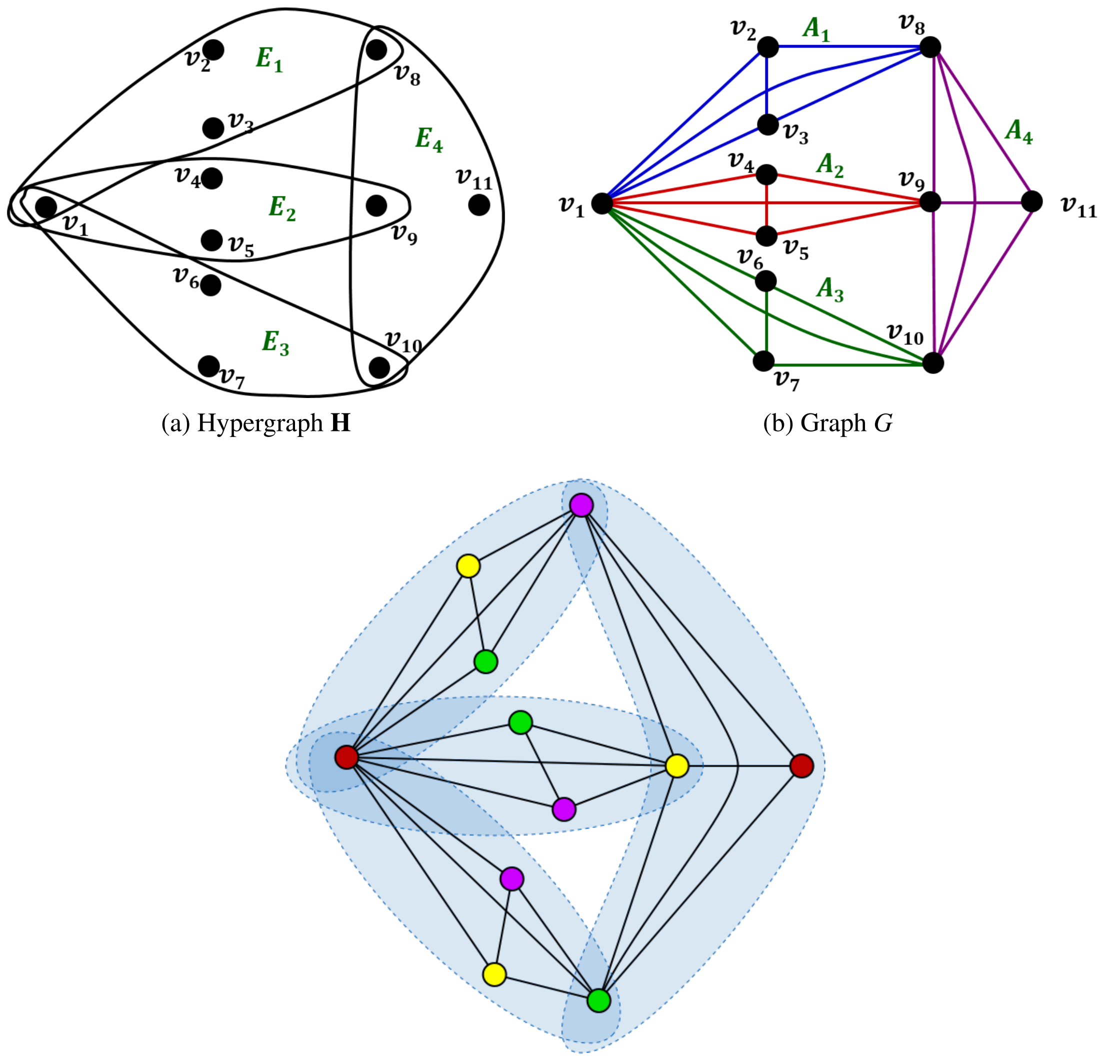


Figure 2.1 All graphs satisfying the hypothesis of the conjecture for n=3

in the same room, which has k chairs. Suppose also that at most one person belongs to the intersection of any two committees. Is it possible to assign the committee members to chairs in such a way that each member sits in the same chair for all the different committees to which he or she belongs? In this model of the problem, the faculty members correspond to graph vertices, committees correspond to complete graphs, and chairs correspond to vertex colors.

Definition 2.1.3. *Let $G = \bigcup_{i=1}^{n} A_i$ denote a graph with n complete graphs $A_1, A_2, \ldots, A_n$, each having exactly n vertices and the property that every pair of complete graphs has at most one common vertex. The clique degree $d^K(v)$ of a vertex v in G is given by $d^K(v) = |\{A_i : v \in V(A_i), 1 \leq i \leq n\}|$. The maximum clique degree $\Delta^K(G)$ of the graph G is given by $\Delta^K(G) = max_{v \in V(G)} d^K(v)$.*

From the above definition, one can observe that degree of a vertex in hypergraph is same as the clique degree of a vertex in a graph.



(a) Hypergraph **H** (b) Graph G

(c) Graph G and Hypergraph **H**

Figure 2.2 Graph G and Hypergraph **H**

Definition 2.1.4. *Let G_1 and G_2 be two vertex disjoint graphs, and let x_1, x_2 be two vertices of G_1, G_2 respectively. Then, the graph $G(x_1x_2)$ obtained by merging the vertices x_1 and x_2 into a single vertex is called the concatenation of G_1 and G_2 at the points x_1 and x_2 (see (Kundu et al., 1980)).*

Definition 2.1.5. *A Latin Square is an $n \times n$ array containing n different symbols such that each symbol appears exactly once in each row and once in each column. Moreover, a Latin Square of order n is an $n \times n$ matrix $M = [m_{ij}]$ with entries from an n-set $V = \{1, 2, \ldots, n\}$, where every row and every column is a permutation of V (see (Laywine and*

Mullen, 1998)). If the matrix M is symmetric, then the Latin Square is called Symmetric Latin Square.

We give below two methods of coloring to the graph G satisfying the hypothesis of the Conjecture. First one using symmetric latin squares and the second one using intersection matrix (the intersection matrix (color matrix) of the cliques $A_i's$ of G is the $n \times n$ matrix in which entry $c_{i,j}$ for $i \neq j$ is 0 if $A_i \cap A_j = \emptyset$ otherwise c, and $c_{i,i}$ is 0) and clique degrees of the vertices.

2.2 Construction of H_n

We know that a symmetric $n \times n$ matrix is determined by $\frac{n(n+1)}{2}$ scalars. Using symmetric latin squares we give an n-coloring of H_n constructed below. Then using the n-coloring of H_n we give an n-coloring of all the other graphs G satisfying the hypothesis of Conjecture 2.1.1.

Construction of H_n:

Let n be a positive integer and $B_1, B_2, \ldots, B_n$ be n copies of K_n. Let the vertex set $V(B_i) = \{a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,n}\}$, $1 \leq i \leq n$.

Step 1. Let $H^1 = B_1$.

Step 2. Consider the vertices $a_{1,2}$ of H^1 and $a_{2,1}$ of B_2. Let $b_{1,2}$ be the vertex obtained by the concatenation of the vertices $a_{1,2}$ and $a_{2,1}$. Let the resultant graph be H^2.

Step 3. Consider the vertices $a_{1,3}$, $a_{2,3}$ of H^2 and $a_{3,1}$, $a_{3,2}$ of B_3. Let $b_{1,3}$ be the vertex obtained by the concatenation of vertices $a_{1,3}$, $a_{3,1}$ and let $b_{2,3}$ be the vertex obtained by the concatenation of vertices $a_{2,3}$, $a_{3,2}$. Let the resultant graph be H^3.

Continuing in the similar way, at the n^{th} step we obtain the graph $H^n = H_n$ (for the sake of convenience we take H^n as H_n).

By the construction of H_n one can observe the following:

1. H_n is a connected graph and also it is satisfying the hypothesis of Conjecture 2.1.1.

2. H_n has exactly n vertices of clique degree one and $\frac{n(n-1)}{2}$ vertices of clique degree 2 (each B_i has exactly $(n-1)$ vertices of clique degree 2 and one vertex of clique degree one, $1 \leq i \leq n$).

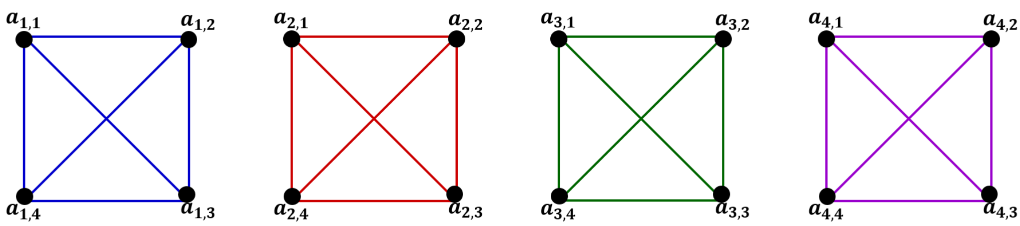
3. $H_n = \bigcup_{i=1}^{n} B_i$, where $B_i = A_i$ and B_i, B_j have exactly one common vertex for $1 \leq i < j \leq n$.

4. H_n has exactly $\frac{n(n+1)}{2}$ vertices.

5. One can observe that in a connected graph G if clique degree increases the number of vertices also increases. From this it follows that, H_n is the graph with minimum number of vertices satisfying the hypothesis of Conjecture 2.1.1. If all the vertices of G are of clique degree one, then G will have n^2 vertices. Thus, $\frac{n(n+1)}{2} \leq |V(G)| \leq n^2$.

Following example is an illustration of the graph H_n for $n = 4$

Example 2.2.1. *Let $n = 4$ and B_1, B_2, B_3, B_4 be the 4 copies of K_4. Let the vertex set $V(B_i) = \{a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}\}$, $1 \leq i \leq 4$.*

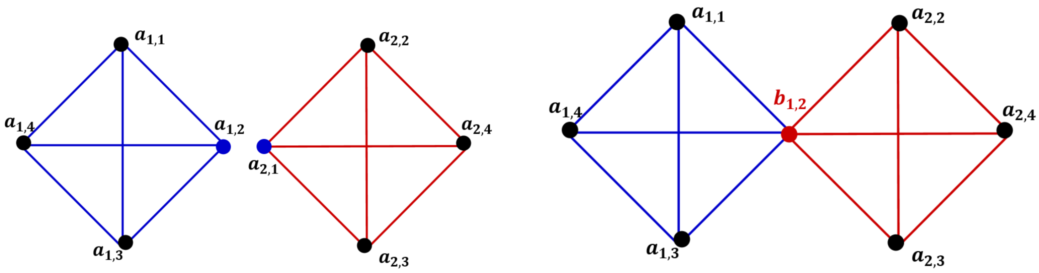


(a) B_1 (b) B_2 (c) B_3 (d) B_4

Figure 2.3 4 copies of K_4

Step 1: *Let $H^1 = B_1$. The graph H^1 is shown in Figure 2.3a.*

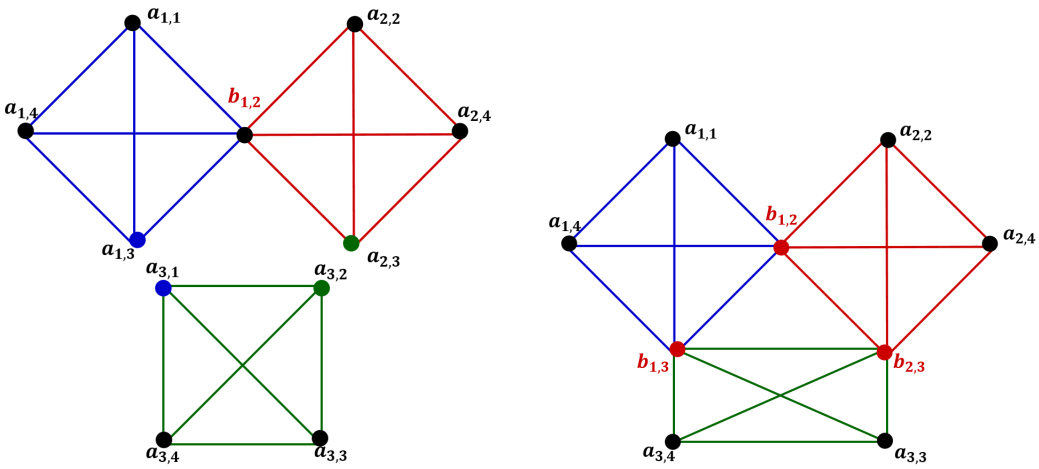
Step 2: *Consider the vertices $a_{1,2}$ of H^1 and $a_{2,1}$ of B_2. Let $b_{1,2}$ be the vertex obtained by the concatenation of vertices $a_{1,2}$, $a_{2,1}$. Let the resultant graph be H^2 as shown in Figure 2.4b.*



(a) H^1, B_2 (b) H^2

Figure 2.4 Construction of H^2 from H^1, B_2

Step 3: *Consider the vertices* $a_{1,3}$*,* $a_{2,3}$ *of* H^2 *and* $a_{3,1}$*,* $a_{3,2}$ *of* B_3*. Let* $b_{1,3}$ *be the vertex obtained by the concatenation of vertices* $a_{1,3}$*,* $a_{3,1}$ *and let* $b_{2,3}$ *be the vertex obtained by the concatenation of vertices* $a_{2,3}$*,* $a_{3,2}$*. Let the resultant graph be* H^3 *as shown in Figure 2.5b.*



(a) H^2, B_3 (b) H^3

Figure 2.5 Construction of H^3 from H^2, B_3

Step 4: *Consider the vertices* $a_{1,4}$*,* $a_{2,4}$*,* $a_{3,4}$ *of* H^3 *and* $a_{4,1}$*,* $a_{4,2}$*,* $a_{4,3}$ *of* B_4*. Let* $b_{1,4}$ *be the vertex obtained by the concatenation of vertices* $a_{1,4}$*,* $a_{4,1}$*, let* $b_{2,4}$ *be the vertex obtained by the concatenation of vertices* $a_{2,4}$*,* $a_{4,2}$ *and let* $b_{3,4}$ *be the vertex obtained by the concatenation of vertices* $a_{3,4}$*,* $a_{4,3}$*. Let the resultant graph be* H^4 *as shown in*

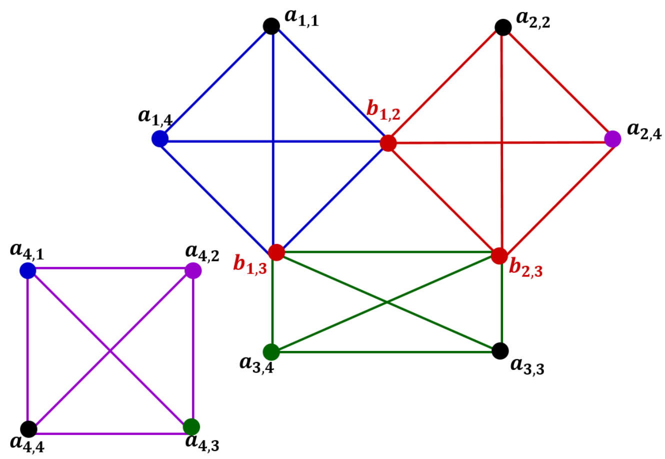


Figure 2.6 H^3, B_4

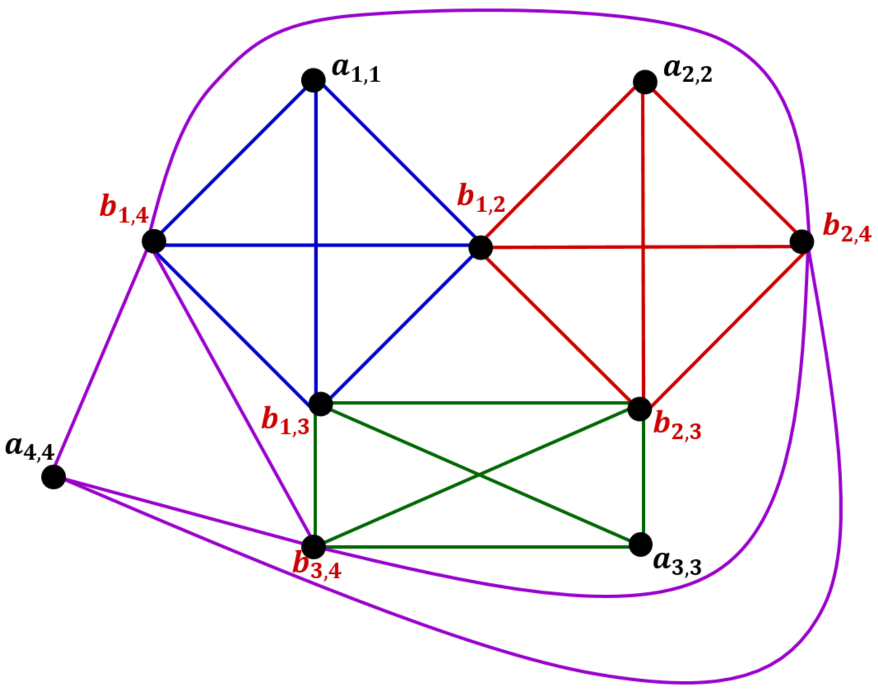


Figure 2.7 $H_4 = H^4$

Figure 2.7.

Example 2.2.2. *For $n = 6$, the graph H_6 is shown in Figure 2.8.*

Lemma 2.2.3. *If G is a graph satisfying the hypothesis of Conjecture 2.1.1, then G can be obtained from H_n, n in $\mathbb{N}$.*

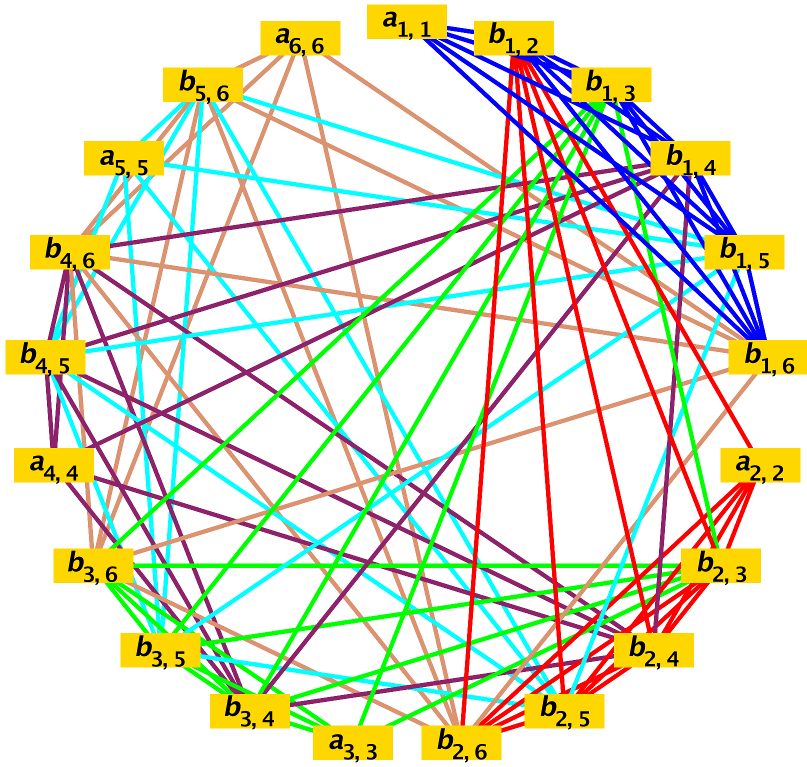


Figure 2.8 H_6

Proof. Let G be a graph satisfying the hypothesis of Conjecture 2.1.1. Let b_X be the new labeling to the vertices v of clique degree greater than 1 in G, where $X = \{i :$ vertex v is in $A_i\}$. Define $N_i = \{b_X : |X| = i\}$ for $i = 2, 3, \ldots, n$. Then the graph G is constructed from H_n as given below:

Step 1: For every common vertex $b_{i,j}$ in H_n which is not in N_2, split the vertex $b_{i,j}$ into two vertices $u_{i,j}, u_{j,i}$ such that vertex $u_{i,j}$ is adjacent only to the vertices of B_i and the vertex $u_{j,i}$ is adjacent only to the vertices of B_j in H_n.

Step 2: For every vertex b_X in N_i where $i = 3, 4, \ldots, n$, merge the vertices u_{l_1,l_2}, u_{l_2,l_3}, $\ldots$, u_{l_{m-1},l_m}, u_{l_m,l_1} into a single vertex u_X in H_n where $l_i \in X$ and $l_i < l_j$ for $i < j$.

Let G' be the graph obtained in Step 2. Let $V(B_i')$, $V(A_i')$ be the set of all clique degree 1 vertices of B_i of G', A_i of G respectively, $1 \leq i \leq n$. Thus, by splitting all the common vertices of H_n which are not in N_2 and merging the vertices of H_n corresponding to the vertices in $N_i, i \geq 3$, we get the graph G'. One can observe that $|V(A_i')| = |V(B_i')|$, $1 \leq i \leq n$. Define a function $f : V(G) \to V(G')$ by

$$
\begin{aligned}
f(b_{i,j}) &= b_{i,j} && \text{for } b_{i,j} \in N_2 \\
f(b_{i_1,i_2,\dots i_k}) &= u_{i_1,i_2,\dots i_k} && \text{for } b_{i_1,i_2,\dots i_k} \in \cup_{i=3}^{n} N_i \\
f|_{V(A_i')} &= g_i && (\text{any 1-1 map } g_i : V(A_i') \to V(B_i')), \text{ for } 1 \leq i \leq n
\end{aligned}
$$

One can observe that f is an isomorphism from G to G'. □

From Lemma 2.2.3, one can observe that in G there are at most $\frac{n(n-1)}{2}$ common vertices.

Following example is an illustration of the graph G obtained from H_n for $n = 4$.

Example 2.2.4. *Let G be the graph shown in Figure 2.2b.*

Relabel the vertices of clique degree greater than one in G by b_A where $A = \{i : v \in A_i$ for $1 \leq i \leq 4\}$. The labeled graph is shown in Figure 2.9.

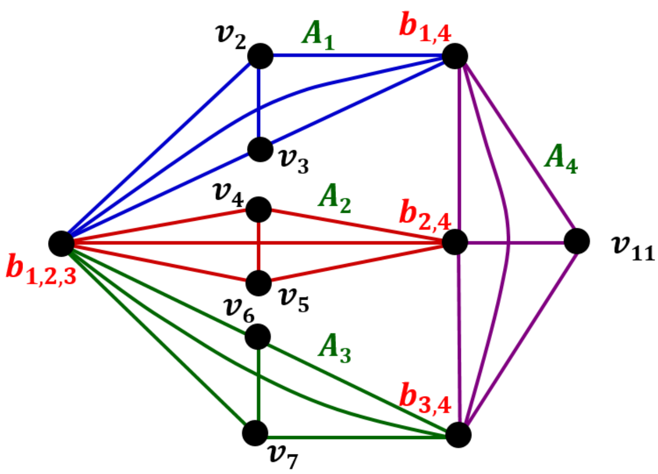


Figure 2.9 Graph G after relabeling the vertices

Let $N_i = \{b_X : |X| = i\}$ for $i = 2, 3, 4$, then $N_2 = \{b_{1,4}, b_{2,4}, b_{3,4}\}$, $N_3 = \{b_{1,2,3}\}$.

Consider the graph H_4 as shown in Figure 2.7, then $V(H_4) = \{a_{1,1}, a_{2,2}, a_{3,3}, a_{4,4}, b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}\}$ and common vertices of H_4 are $\{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}\} = A(say)$. Then $A \setminus N_2 = \{b_{1,2}, b_{1,3}, b_{2,3}\}$. By the construction given in the proof of Lemma 2.2.3 we get,

Step 1: *Since $A \setminus N_2 \neq \emptyset$, split the common vertices of H_4 which are not in N_2, as shown in Figure 2.10.*

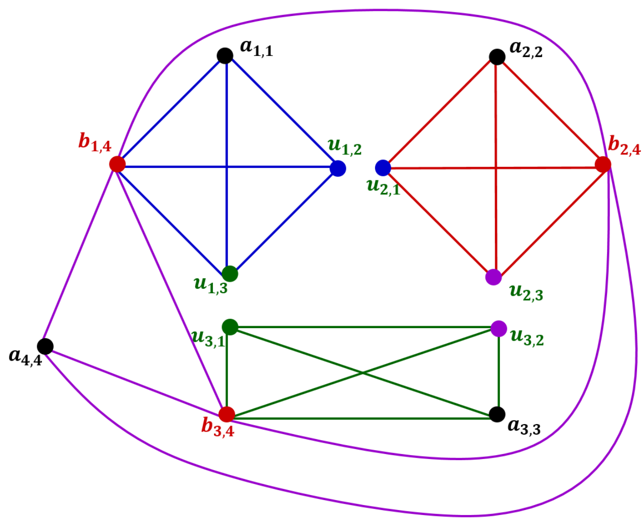


Figure 2.10 Splitting the common vertices of H_4 which are not in N_2.

Step 2: *Since $\cup_{i=2}^{4} N_i = \{b_{1,2,3}\} \neq \emptyset$, merge the vertices $u_{1,2}, u_{2,3}, u_{3,1}$ into a single vertex $u_{1,2,3}$, as shown in Figure 2.11. Let the resultant graph be G'.*

The isomorphism $f : V(G) \to V(G')$ is given below.

$$f(v_2) = a_{1,1} \quad f(v_3) = u_{1,3} \quad f(v_4) = u_{2,1}$$

$$f(v_5) = a_{2,2} \quad f(v_6) = u_{3,2} \quad f(v_7) = a_{3,3}$$

$$f(v_{11}) = a_{4,4} \quad f(b_{1,4}) = b_{1,4} \quad f(b_{2,4}) = b_{2,4}$$

$$f(b_{3,4}) = b_{3,4} \quad f(b_{1,2,3}) = u_{1,2,3}$$

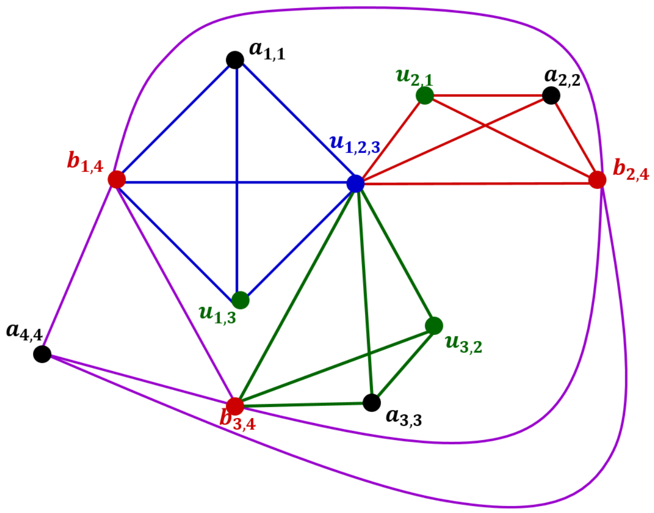


Figure 2.11 Graph G'.

2.3 Coloring of H_n

Lemma 2.3.1. *The chromatic number of H_n is n.*

Proof. Let H_n be the graph defined as above. Let M (given below) be an $n \times n$ matrix in which an entry $m_{ij} = b_{ij}$, be a vertex of H_n, belongs to both B_i, B_j for $i \neq j$ and $m_{ii} = a_{ii}$ be the vertex of H_n which belongs to B_i. i.e.,

$$\text{M=}\begin{pmatrix} a_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{12} & a_{22} & b_{23} & \dots & b_{2n} \\ b_{13} & b_{23} & a_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & b_{3n} & \dots & a_{nn} \end{pmatrix}.$$

Clearly M is a symmetric matrix. We know that, for every n in $\mathbb{N}$ there is a Symmetric Latin Square (see (Ye and Xu, 2011)) of order $n \times n$. Bryant and Rodger (Bryant and Rodger, 2004) gave a necessary and sufficient condition for the existence of an $(n-1)$-edge coloring of K_n (n even), and n-edge coloring of K_n (n odd) using Symmetric Latin Squares. Let $v_1, v_2, \dots, v_n$ be the vertices of K_n and e_{ij} be the edge joining the vertices v_i and v_j of K_n, where $i < j$, then arrange the edges of K_n in the matrix form $A = [a_{ij}]$ where $a_{ij} = e_{ij}$, $a_{ji} = e_{ij}$ for $i < j$ and $a_{ii} = 0$ for $1 \leq i \leq n$, we have $A =$

$$\begin{pmatrix} 0 & e_{12} & e_{13} & \dots & e_{1n} \\ e_{12} & 0 & e_{23} & \dots & e_{2n} \\ e_{13} & e_{23} & 0 & \dots & e_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{1n} & e_{2n} & e_{3n} & \dots & 0 \end{pmatrix}$$

and let V be a matrix given by

$$V = \begin{pmatrix} v_1 & 0 & 0 & \dots & 0 \\ 0 & v_2 & 0 & \dots & 0 \\ 0 & 0 & v_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & v_n \end{pmatrix}$$

. Then, define a matrix A' as

$$A' = A + V = \begin{pmatrix} v_1 & e_{12} & e_{13} & \dots & e_{1n} \\ e_{12} & v_2 & e_{23} & \dots & e_{2n} \\ e_{13} & e_{23} & v_3 & \dots & e_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{1n} & e_{2n} & e_{3n} & \dots & v_n \end{pmatrix}.$$

Let $C = [c_{ij}]$ be a matrix where c_{ij} $(i \neq j)$, is the color of e_{ij} (i.e., $c_{ij} = c(e_{ij})$) and c_{ii} is the color of v_i. We call C as the color matrix of A'. Then C is the Symmetric Latin Square (see(Bryant and Rodger, 2004)). As the elements of M are the vertices of H_n, one can assign the colors to the vertices of H_n from the color matrix C, by the color c_{ij}, for $i, j = 1, 2, \dots, n$ and $i \neq j$ to the vertex b_{ij} in H_n and the color c_{ii}, for $i = 1, 2, \dots n$ to the vertex a_{ii} in H_n. Hence H_n is n colorable. □

As H_n is the graph satisfying the hypothesis of Conjecture 2.1.1. By using the coloring of H_n which is the graph satisfying the hypothesis of Conjecture 2.1.1 we extend the n-coloring of all possible graphs G satisfying the hypothesis of Conjecture 2.1.1.

The following example is an illustration of assigning colors to the graph H_n for $n = 6$.

Example 2.3.2. *Consider the graph H_6 as shown in Figure 2.8. The corresponding Symmetric Latin Square C of order 6×6 is given below:*

$$C = \begin{pmatrix} 6 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 6 & 2 & 4 \\ 2 & 5 & 4 & 1 & 6 & 3 \\ 3 & 6 & 1 & 4 & 5 & 2 \\ 4 & 2 & 6 & 5 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}.$$

Assign the six colors to the graph H_6 using the above Symmetric Latin Square as follows:

Assign the color $c_{i,j}$ (where $c_{i,j}$ denotes the value at the (i,j)-th entry in the color matrix C) for $i \neq j$ and $i,j = 1,2,\ldots,6$ to the vertex $b_{i,j}$ in H_6, and assign the color $c_{i,i}$ (where $c_{i,i}$ denotes the value at the (i,i)-th entry in the color matrix C) for $i = 1,2,\ldots,6$ to the vertex a_{ii} in H_6. The colors Red, Green, Cyan, Blue, Tan, Maroon in the Figure 2.12 corresponds to the numbers 1, 2, 3, 4, 5, 6 respectively in the matrix C.

Then one can verify that the resultant graph is 6 colorable as shown in Figure 2.12.

2.4 Coloring of G

Let G be the graph satisfying the hypothesis of Conjecture 2.1.1. Let $\hat{H}$ be the graph obtained by removing the vertices of clique degree one from graph G. i.e. $\hat{H}$ is the induced subgraph of G having all the common vertices of G.

Theorem 2.4.1. *If G is a graph satisfying the hypothesis of the Conjecture 2.1.1 and every A_i $(1 \leq i \leq n)$ has at most $\sqrt{n}$ vertices of clique degree greater than 1, then G is n-colorable.*

Proof. Let G be a graph satisfying the hypothesis of the Conjecture 2.1.1 and every A_i $(1 \leq i \leq n)$ has at most $\sqrt{n}$ vertices of clique degree greater than 1. Let $\hat{H}$ be the induced subgraph of G consisting of the vertices of clique degree greater than one in G. Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1,2,\ldots,n$.

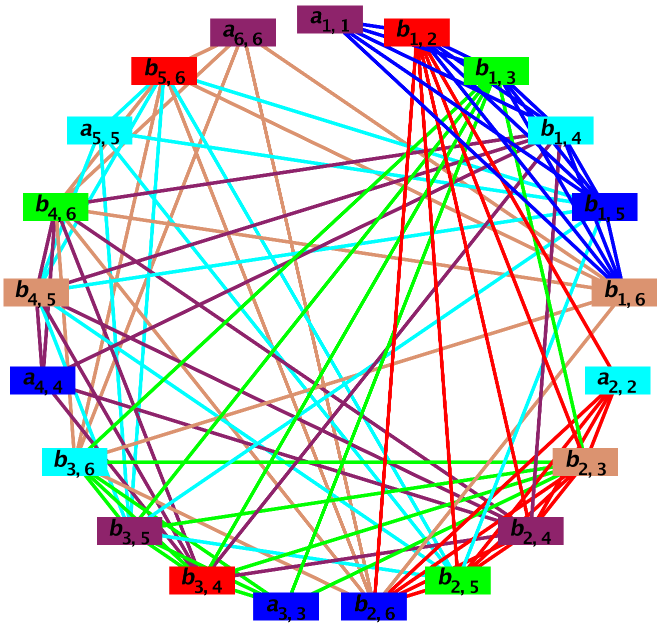


Figure 2.12 A coloring of H_6 with six colors

From (Sánchez-Arroyo, 2008), it is true that the vertices of clique degree greater than or equal to $\sqrt{n}$ can be assigned with at most n colors. Assign the colors to the vertices of clique degree in non increasing order. Assume we next color a vertex v of clique degree $1 < k < \sqrt{n}$. At this point only vertices of clique degree $\geq k$ have been assigned colors. By assumption every A_i $(1 \leq i \leq n)$ has at most $\sqrt{n}$ vertices of clique degree greater than 1 and clique degree of v is k $(k < \sqrt{n})$, then for these k $A_i's$ there are at most $k\sqrt{n} < n$ vertices have been colored. Therefore, there is an unused color from the set of n colors, then that color can be assigned to the vertex v. □

Theorem 2.4.2. *If G is a graph satisfying the hypothesis of the Conjecture 2.1.1 and every A_i $(1 \leq i \leq n)$ has at most $\left\lceil \frac{n+d-1}{d} \right\rceil$ vertices of clique degree greater than or equal to d $(2 \leq d \leq n)$, then G is n-colorable.*

Proof. Let G be a graph satisfying the hypothesis of the Conjecture 2.1.1 and every A_i $(1 \leq i \leq n)$ has at most $\left\lceil \frac{n+d-1}{d} \right\rceil$ vertices of clique degree greater than or equal to d $(2 \leq d \leq n)$. Let $\hat{H}$ be the induced subgraph of G consisting of the vertices of clique degree greater than one in G. Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1, 2, \ldots, m$.

Assign the colors to the vertices of clique degree in non increasing order. Assume we next color a vertex v of clique degree k. At this point only vertices of clique degree $\geq k$ have been assigned colors. By assumption every A_i $(1 \leq i \leq n)$ has at most $\lceil \frac{n+k-1}{k} \rceil$ vertices of clique degree greater than 1 and clique degree of v is k, then for these k $A_i's$ there are at most $k(\lceil \frac{n+k-1}{k} \rceil - 1) < n$ vertices have been colored. Therefore, there is an unused color from the set of n colors, then that color can be assigned to the vertex v. □

Theorem 2.4.3. *If G is a graph satisfying the hypothesis of Conjecture 2.1.1 and every A_i $(1 \leq i \leq n)$ has atmost $\frac{n}{2}$ vertices of clique degree greater than one, then G is n-colorable.*

Proof. Let G be a graph satisfying the hypothesis of Conjecture 2.1.1 and and every A_i $(1 \leq i \leq n)$ has atmost $\frac{n}{2}$ vertices of clique degree greater than one. Let $\hat{H}$ be the induced subgraph of G consisting of the vertices of clique degree greater than one in G. For every vertex v of clique degree greater than one in G, label the vertex v by u_A where $A = \{i : v \in A_i; i = 1, 2, \ldots, n\}$. Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1, 2, \ldots, m$.

Let $1, 2, \ldots, n$ be the n-colors and C be the color matrix(of size $n \times n$) as defined in the proof of Lemma 2.3.1. The following construction applied on the color matrix C, gives a modified color matrix C_M, using which we assign the colors to the graph $\hat{H}$. Then this coloring can be extended to the graph G. Construct a new color matrix C_1 by putting $c_{i,j} = 0, c_{j,i} = 0$ for every $b_{i,j}$ in X. Also, let $c_{i,i} = 0$ for each $i = 1, 2, \ldots, n$.

Construction:

Let $T = \cup_{i=3}^{n} X_i$, $P = \emptyset$, $T'' = X_2$ and $P'' = \emptyset$.

Step 1: If $T = \emptyset$, let C_m be the color matrix obtained in Step 4 and go to Step 5. Otherwise, choose a vertex $u_{i_1,i_2,\ldots,i_m}$ from T, where $i_1 < i_2 < \cdots < i_m$, and then choose

$\binom{m}{2}$ vertices b_{i_1,i_2}, b_{i_1,i_3}, $\ldots$, b_{i_1,i_m}, b_{i_2,i_3}, $\ldots$, b_{i_{m-1},i_m} from $V(H_n)$ corresponding to the set $\{i_1, i_2, \ldots, i_m\}$. Take $T' = \{b_{i_1,i_2}, b_{i_1,i_3}, \ldots, b_{i_1,i_m}, b_{i_2,i_3}, \ldots, b_{i_{m-1},i_m}\}$ and $P' = \emptyset$. Let $T'_1 = \{b_{i,j} : b_{i,j} \in T', c(b_{i,j})$ appear more than once in the i^{th} row or j^{th} column in $C\}$ and $T'_2 = \{b_{i,j} : b_{i,j} \in T', c(b_{i,j})$ appear exactly once in the i^{th} row and j^{th} column in $C\}$. If $T'_1 \neq \emptyset$, choose a vertex $b_{s,t}$ from T'_1, otherwise choose a vertex $b_{s,t}$ from T'_2. Then add the vertex $b_{s,t}$ to P' and remove it from T'. Go to Step 2.

Step 2: If $T'_2 \neq \emptyset$, go to Step 3. Otherwise, choose a vertex b_{i_{m-1},i_m} from T'_1. Let $A = \{c_{i,j} : c_{i,j} \neq 0; i = i_{m-1}, 1 \leq j \leq n\}$, $B = \{c_{i,j} : c_{i,j} \neq 0; j = i_m, 1 \leq i \leq n\}$. If $|A \cap B| < n$, then construct a new color matrix C_2, replacing c_{i_{m-1},i_m}, $c_{i_m,i_{m-1}}$ by x, where $x \in \{1, 2, \ldots, n\} \setminus A \cup B$. Then add the vertex b_{i_{m-1},i_m} to T'_2 and remove it from T'_1. Go to Step 3. Otherwise choose a color x which appears exactly once either in i^{th}_{m-1} row or in i^{th}_m column of the color matrix and construct a new color matrix C_2 replacing c_{i_{m-1},i_m}, $c_{i_m,i_{m-1}}$ by x. Then add the vertex b_{i_{m-1},i_m} to T'_2 and remove it from T'_1. Go to Step 3.

Step 3: If $T' = \emptyset$, then add the vertex $u_{i_1,i_2,\ldots,i_m}$ to P and remove it from T, go to Step 1. Otherwise, if $T' \cap T'_1 \neq \emptyset$ choose a vertex $b_{i,j}$ from $T' \cap T'_1$, if not choose a vertex $b_{i,j}$ from $T' \cap T'_2$. Go to Step 4.

Step 4: Let $c(b_{i,j}) = x$, $c(b_{s,t}) = y$. If $c(b_{i,j}) = c(b_{s,t})$, then add the vertex $b_{i,j}$ to P' and remove it from T'. Go to Step 3. Otherwise, let $A = \{c_{l,m} : c_{l,m} = x\}$, $B = \{c_{l,m} : c_{l,m} = y\} \setminus \{c_{l,m}, c_{m,l} : b_{l,m} \in P', l < m\}$. Construct a new color matrix C_3 by putting $c_{l,m} = y$ for every $c_{l,m}$ in A and $c_{l,m} = x$ for every $c_{l,m}$ in B. Then add the vertex $b_{i,j}$ to P' and remove it from T'. Go to Step 3.

Step 5: If $T'' = \emptyset$, consider $C_M = C_{m_1}$ stop the process. Otherwise, choose a vertex $u_{i,j}$ from T'' and go to Step 6.

Step 6: If $c_{i,j}$ appears exactly once in both i^{th} row and j^{th} column of the color matrix

C_m, then add the vertex $b_{i,j}$ to P'' and remove it from T'', go to Step 5. Otherwise, let $A = \{c_{i,j} : c_{i,j} \neq 0; 1 \leq j \leq n\}$, $B = \{c_{i,j} : c_{i,j} \neq 0; 1 \leq i \leq n\}$. Construct a new color matrix C_{m_1} by putting x in $c_{i,j}$, $c_{j,i}$ where $x \in \{1,2,\ldots,n\} \setminus A \cup B$ (Since every A_i $(1 \leq i \leq n)$ has atmost $\frac{n}{2}$ vertices of clique degree greater than one, $|A \cup B| < n$). Then add the vertex $u_{i,j}$ to P'' and remove it from T'', go to Step 5.

Thus, in step 6, we get the modified color matrix C_M. Then, color the vertex v of $\hat{H}$ by $c_{i,j}$ of C_M, whenever $v \in A_i \cap A_j$. Then, extend the coloring of $\hat{H}$ to G by assigning the remaining colors which are not used for A_i from the set of n-colors, to the vertices of clique degree one in A_i, $1 \leq i \leq n$. Thus G is n-colorable. □

Remark 2.4.4. *One can see that Theorem 2.4.3 covers the following cases:*

1. *G has no clique degree 2 vertices.*

2. *G has atmost $\frac{n}{2}$ vertices of clique degree greater than one in each A_i, $1 \leq i \leq n$.*

Corollary 2.4.5. *Sánchez-Arroyo (2008) Consider a linear hypergraph **H** consisting of n edges each of size at most n and $\delta(\boldsymbol{H}) \geq 2$. If H is dense, then $\chi(\boldsymbol{H}) \leq n$.*

Following is an example illustrating the construction given in the proof of Theorem 2.4.3.

Example 2.4.6. *Let G be the graph shown in Figure 2.13.*

Let $V(A_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V(A_2) = \{v_1, v_7, v_8, v_9, v_{10}, v_{11}\}$, $V(A_3) = \{v_1, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$, $V(A_4) = \{v_1, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}\}$, $V(A_5) = \{v_6, v_7, v_{16}, v_{22}, v_{23}, v_{24}\}$, $V(A_6) = \{v_9, v_{16}, v_{19}, v_{25}, v_{26}, v_{27}\}$.

Relabel the vertices of clique degree greater than one in G by u_A where $A = \{i : v \in A_i$ for $1 \leq i \leq 6\}$. The labeled graph is shown in Figure 2.14. Figure 2.15 is the graph $\hat{H}$, where $\hat{H}$ is obtained by removing the vertices of clique degree 1 from G.

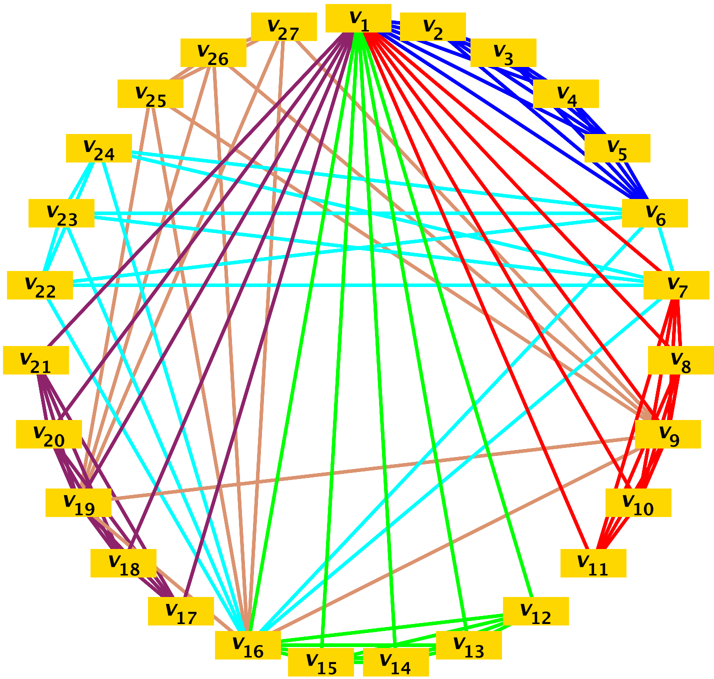


Figure 2.13 Graph *G*

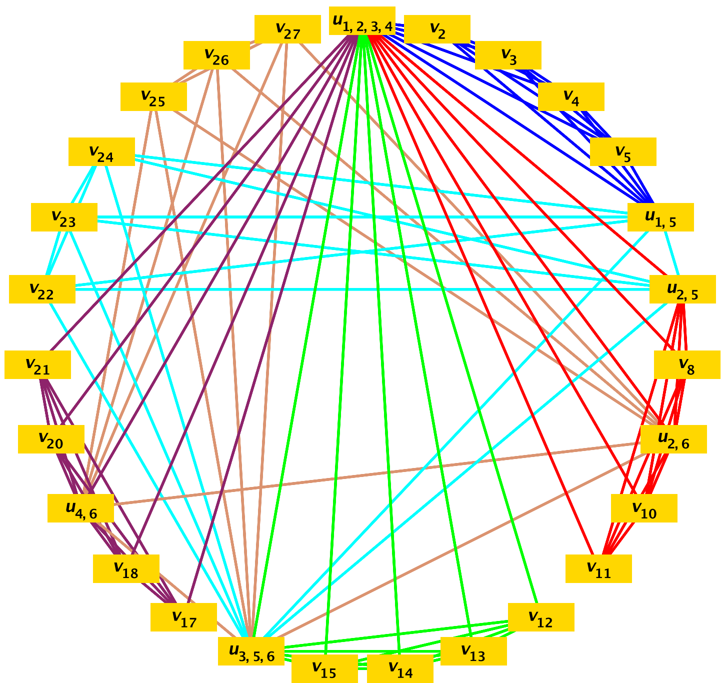


Figure 2.14 Graph *G* after relabeling the vertices of clique degree greater than one

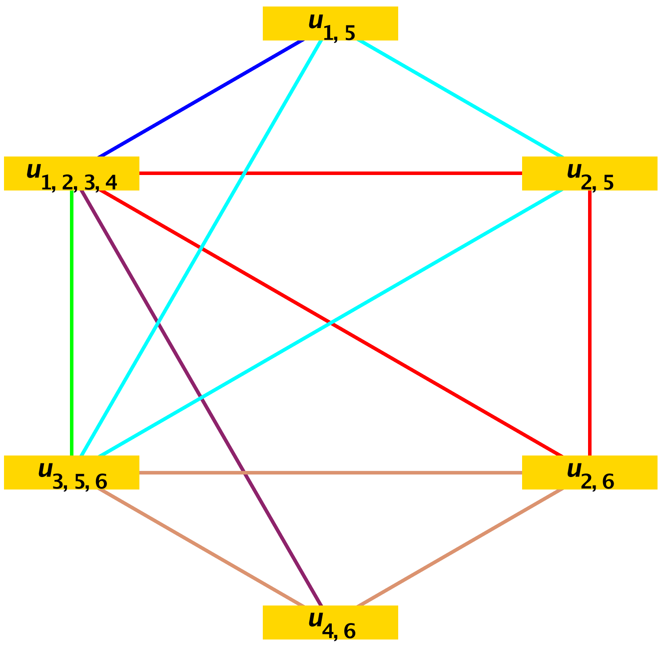


Figure 2.15 Graph $\hat{H}$

Let $X = \{b_{ij} : A_i \cap A_j = \emptyset\} = \{b_{1,6}, b_{4,5}\}$,

$$X_1 = \{v \in G : d^K(v) = 1\} = \{v_2, v_3, v_5, v_8, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{17}, v_{18}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\},$$

$X_2 = \{v \in G : d^K(v) = 2\} = \{v_6, v_7, v_9, v_{19}\} = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$,
$X_3 = \{v \in G : d^K(v) = 3\} = \{v_{16}\} = \{u_{3,5,6}\}$,
and $X_4 = \{v \in G : d^K(v) = 4\} = \{v_1\} = \{u_{1,2,3,4}\}$.

Let 1, 2, …, 6 be the six colors and $C = \begin{pmatrix} 6 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 6 & 2 & 4 \\ 2 & 5 & 4 & 1 & 6 & 3 \\ 3 & 6 & 1 & 4 & 5 & 2 \\ 4 & 2 & 6 & 5 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$

be the color matrix (as well as symmetric latin square) of order 6×6.

Consider the sets $T = X_3 \cup X_4 = \{u_{3,5,6}, u_{1,2,3,4}\}$, $T'' = X_2 = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$,

$P = \emptyset$ and $P'' = \emptyset$. Then, by applying the construction given in the proof of Theorem 2.4.3 we get a new color matrix C_1 by putting $c_{i,j} = 0$, $c_{j,i} = 0$ for every $b_{i,j}$ in X and $c_{i,i} = 0$ for each $i = 1, 2, \ldots, 6$ and go to Step 1.

$$C_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 0 \\ 1 & 0 & 5 & 6 & 2 & 4 \\ 2 & 5 & 0 & 1 & 6 & 3 \\ 3 & 6 & 1 & 0 & 0 & 2 \\ 4 & 2 & 6 & 0 & 0 & 1 \\ 0 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Step 1: *Since $T \neq \emptyset$, choose the vertex $u_{1,2,3,4}$ from T. Let $T' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}\}$ and $P' = \emptyset$, then $T'_1 = \emptyset$ and $T'_2 = T'$. Since $T'_1 = \emptyset$, choose the vertex $b_{2,4}$ from T'_2, add it to P' and remove it from T'. Then $T' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{3,4}\}$ and $P' = \{b_{2,4}\}$. Go to step 2.*

Step 2: *Since $T'_2 \neq \emptyset$, go to step 3.*

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{1,2}$ from $T' \cap T'_2$ and go to step 4.*

Step 4: *Since $c(b_{1,2}) = 1$, $c(b_{2,4}) = 6$ and $c(b_{1,2}) \neq c(b_{2,4})$, interchange 1, 6 in the matrix C_1 except the color of $b_{2,4}$. Add the vertex $b_{1,2}$ to P' and remove it from T'. Then*

$$C_2 = \begin{pmatrix} 0 & 6 & 2 & 3 & 4 & 0 \\ 6 & 0 & 5 & 6 & 2 & 4 \\ 2 & 5 & 0 & 6 & 1 & 3 \\ 3 & 6 & 6 & 0 & 0 & 2 \\ 4 & 2 & 1 & 0 & 0 & 6 \\ 0 & 4 & 3 & 2 & 6 & 0 \end{pmatrix},$$

$T' = \{b_{1,3}, b_{1,4}, b_{2,3}, b_{3,4}\}$ and $P' = \{b_{1,2}, b_{2,4}\}$. Go to step 3.

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{1,3}$ from $T' \cap T'_2$ and go to step 4.*

Step 4: *Since $c(b_{1,3}) = 2$, $c(b_{2,4}) = 6$ and $c(b_{1,3}) \neq c(b_{2,4})$, interchange 2, 6 in the*

matrix C_2 except the color of $b_{1,2}, b_{2,4}$. Add the vertex $b_{1,3}$ to P' and remove it from T'. Then

$$C_3 = \begin{pmatrix} 0 & 6 & 6 & 3 & 4 & 0 \\ 6 & 0 & 5 & 6 & 6 & 4 \\ 6 & 5 & 0 & 2 & 1 & 3 \\ 3 & 6 & 2 & 0 & 0 & 6 \\ 4 & 6 & 1 & 0 & 0 & 2 \\ 0 & 4 & 3 & 6 & 2 & 0 \end{pmatrix},$$

$T' = \{b_{1,4}, b_{2,3}, b_{3,4}\}$ *and* $P' = \{b_{1,2}, b_{1,3}, b_{2,4}\}$. *Go to step 3.*

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{1,4}$ from $T' \cap T'_2$ and go to step 4.*

Step 4: *Since $c(b_{1,4}) = 3$, $c(b_{2,4}) = 6$ and $c(b_{1,4}) \neq c(b_{2,4})$, interchange 3, 6 in the matrix C_3 except the color of $b_{1,2}, b_{1,3}, b_{2,4}$. Add the vertex $b_{1,4}$ to P' and remove it from T'. Then*

$$C_4 = \begin{pmatrix} 0 & 6 & 6 & 6 & 4 & 0 \\ 6 & 0 & 5 & 6 & 3 & 4 \\ 6 & 5 & 0 & 2 & 1 & 6 \\ 6 & 6 & 2 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 0 & 2 \\ 0 & 4 & 6 & 3 & 2 & 0 \end{pmatrix},$$

$T' = \{b_{2,3}, b_{3,4}\}$ *and* $P' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,4}\}$. *Go to step 3.*

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{2,3}$ from $T' \cap T'_2$ and go to step 4.*

Step 4: *Since $c(b_{2,3}) = 5$, $c(b_{2,4}) = 6$ and $c(b_{2,3}) \neq c(b_{2,4})$, interchange 5, 6 in the matrix C_4 except the color of $b_{1,2}, b_{1,3}, b_{1,4}, b_{2,4}$. Add the vertex $b_{2,3}$ to P' and remove it from T'. Then*

$$C_5 = \begin{pmatrix} 0 & 6 & 6 & 6 & 4 & 0 \\ 6 & 0 & 6 & 6 & 3 & 4 \\ 6 & 6 & 0 & 2 & 1 & 5 \\ 6 & 6 & 2 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 0 & 2 \\ 0 & 4 & 5 & 3 & 2 & 0 \end{pmatrix},$$

$T' = \{b_{3,4}\}$ and $P' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}\}$. Go to step 3.

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{3,4}$ from $T' \cap T'_2$ and go to step 4.*

Step 4: *Since $c(b_{3,4}) = 2$, $c(b_{2,4}) = 6$ and $c(b_{3,4}) \neq c(b_{2,4})$, interchange 2, 6 in the matrix C_5 except the color of $b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}$. Add the vertex $b_{3,4}$ to P' and remove it from T'. Then*

$$C_6 = \begin{pmatrix} 0 & 6 & 6 & 6 & 4 & 0 \\ 6 & 0 & 6 & 6 & 3 & 4 \\ 6 & 6 & 0 & 6 & 1 & 5 \\ 6 & 6 & 6 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 0 & 6 \\ 0 & 4 & 5 & 3 & 6 & 0 \end{pmatrix},$$

$T' = \emptyset$ and $P' = \{b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}\}$. Go to step 3.

Step 3: *Since $T' = \emptyset$, add the vertex $u_{1,2,3,4}$ to P and remove it from T, then $T = \{u_{3,5,6}\}$ and $P = \{u_{1,2,3,4}\}$. Go to step 1.*

Step 1: *Since $T \neq \emptyset$, choose the vertex $u_{3,5,6}$ from T. Let $T' = \{b_{3,5}, b_{3,6}, b_{5,6}\}$ and $P' = \emptyset$, then $T'_1 = \emptyset$ and $T'_2 = T'$. Since $T'_1 = \emptyset$, choose the vertex $b_{5,6}$ from T'_2, add it to P' and remove it from T'. Then $T' = \{b_{3,5}, b_{3,6}\}$ and $P' = \{b_{5,6}\}$. Go to step 2.*

Step 2: *Since $T'_2 \neq \emptyset$, go to step 3.*

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{3,6}$ from $T' \cap T'_2$ and go to step 4.*

Step 4: *Since $c(b_{3,6}) = 5$, $c(b_{5,6}) = 6$ and $c(b_{3,6}) \neq c(b_{5,6})$, interchange 5, 6 in the*

matrix C_6 except the color of $b_{5,6}$. Add the vertex $b_{3,6}$ to P' and remove it from T'. Then

$$C_7 = \begin{pmatrix} 0 & 5 & 5 & 5 & 4 & 0 \\ 5 & 0 & 5 & 5 & 3 & 4 \\ 5 & 5 & 0 & 5 & 1 & 6 \\ 5 & 5 & 5 & 0 & 0 & 3 \\ 4 & 3 & 1 & 0 & 0 & 6 \\ 0 & 4 & 6 & 3 & 6 & 0 \end{pmatrix},$$

$T' = \{b_{3,5}\}$ and $P' = \{b_{3,6}, b_{5,6}\}$. Go to step 3.

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T_1' = \emptyset$, choose the vertex $b_{3,5}$ from $T' \cap T_2'$ and go to step 4.*

Step 4: *Since $c(b_{3,5}) = 1$, $c(b_{5,6}) = 6$ and $c(b_{3,5}) \neq c(b_{5,6})$, interchange 1, 6 in the matrix C_7 except the color of $b_{3,6}, b_{5,6}$. Add the vertex $b_{3,5}$ to P' and remove it from T'. Then*

$$C_8 = \begin{pmatrix} 0 & 5 & 5 & 5 & 4 & 0 \\ 5 & 0 & 5 & 5 & 3 & 4 \\ 5 & 5 & 0 & 5 & 6 & 6 \\ 5 & 5 & 5 & 0 & 0 & 3 \\ 4 & 3 & 6 & 0 & 0 & 6 \\ 0 & 4 & 6 & 3 & 6 & 0 \end{pmatrix},$$

$T' = \emptyset$ and $P' = \{b_{3,5}, b_{3,6}, b_{5,6}\}$. Go to step 3.

Step 3: *Since $T' = \emptyset$, add the vertex $u_{3,5,6}$ to P and remove it from T, then $T = \emptyset$ and $P = \{u_{3,5,6}, u_{1,2,3,4}\}$. Go to step 1.*

Step 1: *Since $T = \emptyset$ consider $C_m = C_8$, go to step 5.*

Step 5: *Since $T'' \neq \emptyset$, choose the vertex $u_{1,5}$ from T''. Go to step 6.*

Step 6: *Since $c_{1,5} = 4$ appears exactly once in both 1^{st} row and 5^{th} column of the color matrix C_m. Add the vertex $u_{1,5}$ to P'' and remove it from T''. Then $T'' = \{u_{2,5}, u_{2,6}, u_{4,6}\}$ and $P'' = \{u_{1,5}\}$. Go to Step 5.*

Step 5: *Since $T'' \neq \emptyset$, choose the vertex $u_{2,5}$ from T''. Go to step 6.*

Step 6: *Since $c_{2,5} = 3$ appears exactly once in both 2^{nd} row and 5^{th} column of the* *color matrix C_m. Add the vertex $u_{2,5}$ to P'' and remove it from T''. Then $T'' = \{u_{2,6}, u_{4,6}\}$ and $P'' = \{u_{1,5}, u_{2,5}\}$. Go to Step 5.*

Step 5: *Since $T'' \neq \emptyset$, choose the vertex $u_{2,6}$ from T''. Go to step 6.*

Step 6: *Since $c_{2,6} = 4$ appears exactly once in both 2^{nd} row and 6^{th} column of the color matrix C_m. Add the vertex $u_{2,6}$ to P'' and remove it from T''. Then $T'' = \{u_{4,6}\}$ and $P'' = \{u_{1,5}, u_{2,5,u_{2,6}}\}$. Go to Step 5.*

Step 5: *Since $T'' \neq \emptyset$, choose the vertex $u_{4,6}$ from T''. Go to step 6.*

Step 6: *Since $c_{4,6} = 3$ appears exactly once in both 4^{th} row and 6^{th} column of the color matrix C_m. Add the vertex $u_{4,6}$ to P'' and remove it from T''. Then $T'' = \emptyset$ and $P'' = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$. Go to Step 5.*

Step 5: *Since $T'' = \emptyset$, consider $C_M = C_m$.*

Stop the process.

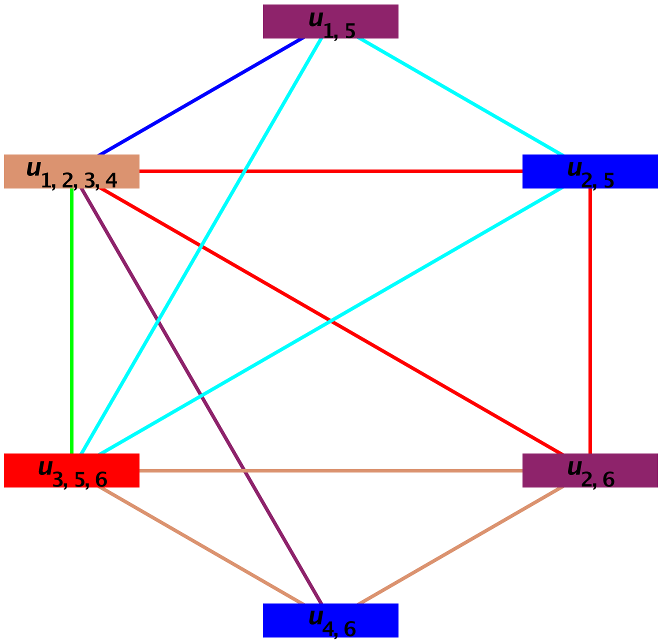
Assign the colors to the graph $\hat{H}$ using the matrix C_M, i.e., color the vertex v by the (i, j)-th entry $c_{i,j}$ of the matrix C_M, whenever $A_i \cap A_j \neq \emptyset$ (see Figure 2.16a), where the numbers 1, 2, 3, 4, 5, 6 corresponds to the colors Green, Cyan, Blue, Maroon, Tan, Red respectively. Extend the coloring of $\hat{H}$ to G by assigning the remaining colors which are not used for A_i from the set of 6-colors to the vertices of clique degree one in each A_i, $1 \leq i \leq 6$. The colored graph G is shown in Figure 2.16b.

Following example shows that the construction mentioned in the proof of Theorem 2.4.3 does not work, if the graph G has more than $\frac{n}{2}$ vertices of clique degree greater than one in some A_i, $1 \leq i \leq n$.

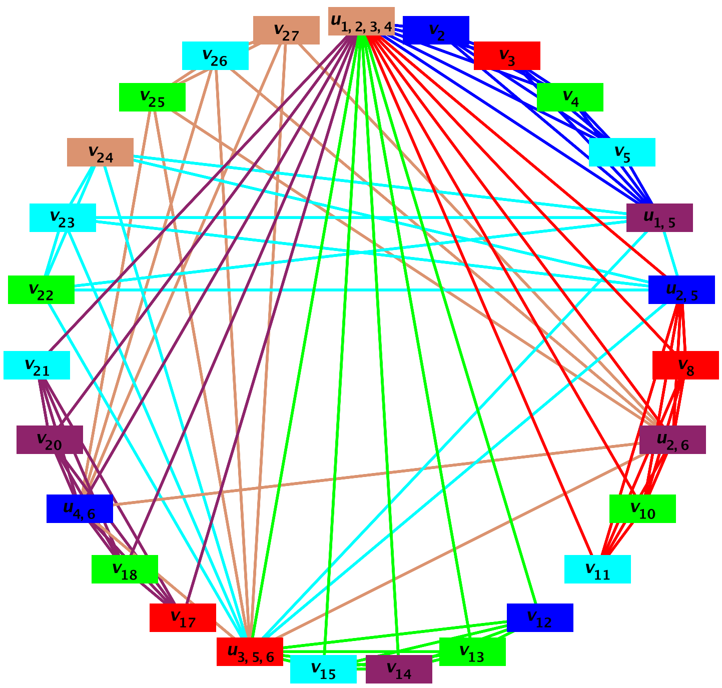
Example 2.4.7. *Let G be the graph shown in Figure 2.17a.*

Let $V(A_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V(A_2) = \{v_2, v_7, v_8, v_9, v_{10}, v_{11}\}$, $V(A_3) = \{v_3, v_8, v_{12}, v_{13}, v_{14}, v_{15}\}$, $V(A_4) = \{v_4, v_9, v_{16}, v_{17}, v_{18}, v_{20}, v_{21}\}$, $V(A_5) = \{v_5, v_{10}, v_{14}, v_{18}, v_{20}, v_{21}\}$, $V(A_6) = \{v_6, v_{10}, v_{15}, v_{19}, v_{22}, v_{23}\}$.

Relabel the vertices of clique degree greater than one in G by u_A where $A = \{i : v \in A_i$ for $1 \leq i \leq 6\}$. The labeled graph is shown in Figure 2.17b. Figure 2.18 is the graph



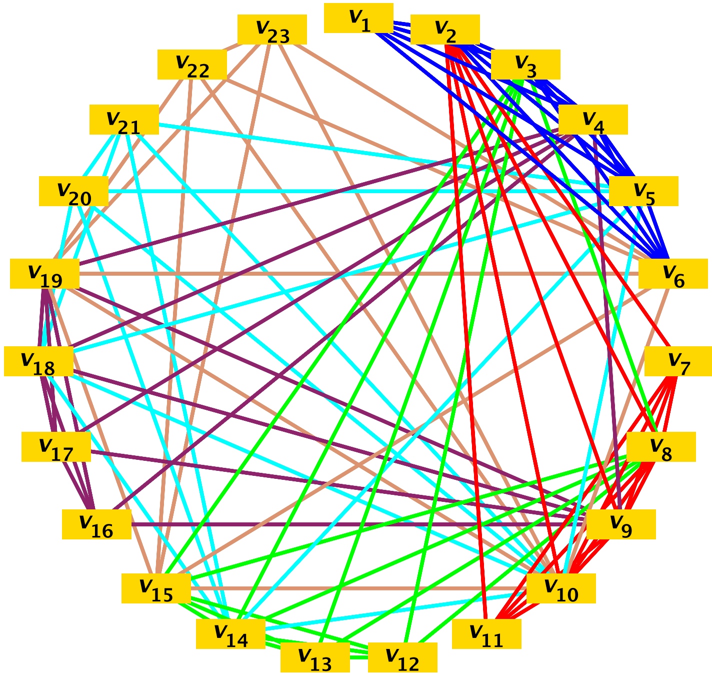
(a) Graph $\hat{H}$



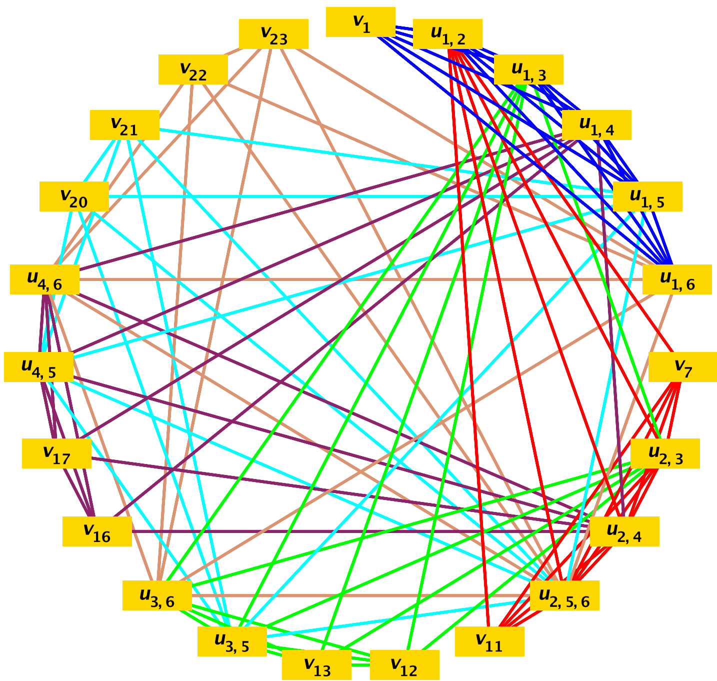
(b) A 6 coloring of graph G

Figure 2.16 The graphs $\hat{H}$ and G, after colors have been assigned to their vertices.

$\hat{H}$, where $\hat{H}$ is obtained by removing the vertices of clique degree 1 from G.



(a) Graph G



(b) Graph G after relabeling the vertices of clique degree greater than one

Figure 2.17 Graph G: before and after relabeling the vertices

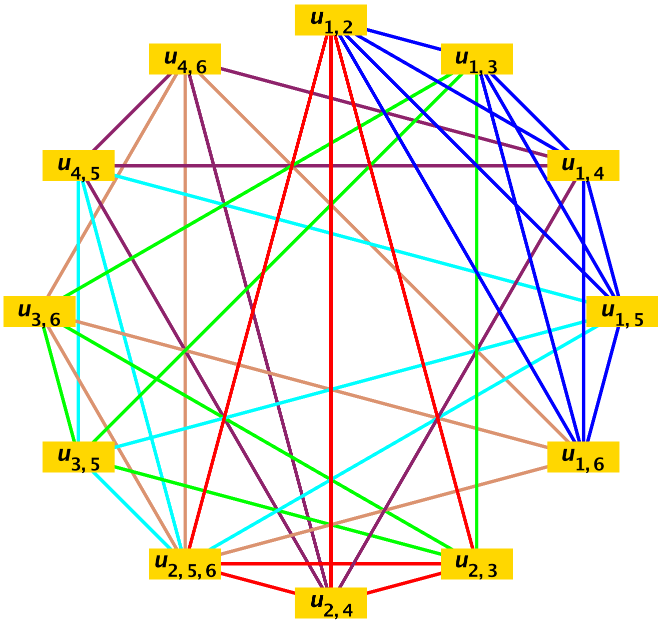


Figure 2.18 Graph $\hat{H}$

Let $X = \{b_{ij} : A_i \cap A_j = \emptyset\} = \{b_{3,4}\}$,

$X_1 = \{v \in G : d^K(v) = 1\} = \{v_1, v_7, v_{11}, v_{12}, v_{13}, v_{16}, v_{17}, v_{20}, v_{21}, v_{22}, v_{23}\}$,

$$\begin{aligned} X_2 &= \{v \in G : d^K(v) = 2\} = \{v_2, v_3, v_4, v_5, v_6, v_8, v_9, v_{14}, v_{15}, v_{18}, v_{19}\} \\ &= \{u_{1,2}, u_{1,3}, u_{1,4}, u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}, \end{aligned}$$

and $X_3 = \{v \in G : d^K(v) = 3\} = \{v_{10}\} = \{u_{2,5,6}\}$,

Let 1, 2, ..., 6 be the six colors and $C = \begin{pmatrix} 6 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 6 & 2 & 4 \\ 2 & 5 & 4 & 1 & 6 & 3 \\ 3 & 6 & 1 & 4 & 5 & 2 \\ 4 & 2 & 6 & 5 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$

be the color matrix (as well as symmetric latin square) of order 6×6.

Consider the sets $T = X_3 = \{u_{2,5,6}\}$,

$T'' = X_2 = \{u_{1,2}, u_{1,3}, u_{1,4}, u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}$, $P = \emptyset$ and $P'' = \emptyset$. Then by applying the construction given in the proof of Theorem 2.4.3 we get a new color matrix C_1 by putting $c_{i,j} = 0$, $c_{j,i} = 0$ for every $b_{i,j}$ in X and $c_{i,i} = 0$ for each $i = 1, 2, \ldots, 6$ and go to Step 1.

$$C_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 5 & 6 & 2 & 4 \\ 2 & 5 & 0 & 0 & 6 & 3 \\ 3 & 6 & 0 & 0 & 5 & 2 \\ 4 & 2 & 6 & 5 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Step 1: *Since $T \neq \emptyset$, choose the vertex $u_{2,5,6}$ from T. Let $T' = \{b_{2,5}, b_{2,6}, b_{5,6}\}$ and $P' = \emptyset$, then $T'_1 = \emptyset$ and $T'_2 = T'$. Since $T'_1 = \emptyset$, choose the vertex $b_{5,6}$ from T'_2, add it to P' and remove it from T'. Then $T' = \{b_{2,5}, b_{2,6}\}$ and $P' = \{b_{5,6}\}$. Go to step 2.*

Step 2: *Since $T'_2 \neq \emptyset$, go to step 3.*

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{2,5}$ from $T' \cap T'_2$ and go to step 4.*

Step 4: *Since $c(b_{2,5}) = 2$, $c(b_{5,6}) = 1$ and $c(b_{2,5}) \neq c(b_{5,6})$, interchange 2, 1 in the matrix C_1 except the color of $b_{5,6}$. Add the vertex $b_{2,5}$ to P' and remove it from T'. Then*

$$C_2 = \begin{pmatrix} 0 & 2 & 1 & 3 & 4 & 5 \\ 2 & 0 & 5 & 6 & 1 & 4 \\ 1 & 5 & 0 & 0 & 6 & 3 \\ 3 & 6 & 0 & 0 & 5 & 2 \\ 4 & 1 & 6 & 5 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix},$$

$T' = \{b_{2,6}\}$ and $P' = \{b_{2,5}, b_{5,6}\}$. Go to step 3.

Step 3: *Since $T' \neq \emptyset$ and $T' \cap T'_1 = \emptyset$, choose the vertex $b_{2,6}$ from $T' \cap T'_2$ and go to step 4.*

Step 4: *Since $c(b_{2,6}) = 4$, $c(b_{5,6}) = 1$ and $c(b_{2,6}) \neq c(b_{5,6})$, interchange 4, 1 in the*

matrix C_2 except the color of $b_{2,5}, b_{5,6}$. Add the vertex $b_{2,6}$ to P' and remove it from T'. Then

$$C_3 = \begin{pmatrix} 0 & 2 & 4 & 3 & 1 & 5 \\ 2 & 0 & 5 & 6 & 1 & 1 \\ 4 & 5 & 0 & 0 & 6 & 3 \\ 3 & 6 & 0 & 0 & 5 & 2 \\ 1 & 1 & 6 & 5 & 0 & 1 \\ 5 & 1 & 3 & 2 & 1 & 0 \end{pmatrix},$$

$T' = \emptyset$ and $P' = \{b_{2,5}, b_{2,6}, b_{5,6}\}$. Go to step 3.

***Step 3:** Since $T' = \emptyset$, add the vertex $u_{2,5,6}$ to P and remove it from T, then $T = \emptyset$ and $P = \{u_{2,5,6}\}$. Go to step 1.*

***Step 1:** Since $T = \emptyset$ consider $C_m = C_3$, go to step 5.*

***Step 5:** Since $T'' \neq \emptyset$, choose the vertex $u_{1,2}$ from T''. Go to step 6.*

***Step 6:** Since $c_{1,2} = 2$ appears exactly once in both 1^{st} row and 2^{nd} column of the color matrix C_m. Add the vertex $u_{1,2}$ to P'' and remove it from T''. Then*

$T'' = \{u_{1,3}, u_{1,4}, u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}$ and $P'' = \{u_{1,2}\}$. Go to Step 5.

***Step 5:** Since $T'' \neq \emptyset$, choose the vertex $u_{1,3}$ from T''. Go to step 6.*

***Step 6:** Since $c_{1,3} = 4$ appears exactly once in both 1^{st} row and 3^{rd} column of the color matrix C_m. Add the vertex $u_{1,3}$ to P'' and remove it from T''. Then*

$T'' = \{u_{1,4}, u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}$ and $P'' = \{u_{1,2}, u_{1,3}\}$. Go to Step 5.

***Step 5:** Since $T'' \neq \emptyset$, choose the vertex $u_{1,4}$ from T''. Go to step 6.*

***Step 6:** Since $c_{1,4} = 3$ appears exactly once in both 1^{st} row and 4^{th} column of the color matrix C_m. Add the vertex $u_{1,4}$ to P'' and remove it from T''. Then*

$T'' = \{u_{1,5}, u_{1,6}, u_{2,3}, u_{2,4}, u_{3,5}, u_{3,6}, u_{4,5}, u_{4,6}\}$ and $P'' = \{u_{1,2}, u_{1,3}, u_{1,4}\}$. Go to Step 5.

***Step 5:** Since $T'' \neq \emptyset$, choose the vertex $u_{1,5}$ from T''. Go to step 6.*

***Step 6:** Since $c_{1,5} = 1$ and it appears more than once in the 5^{th} column of the color matrix C_m. Let $A = \{c_{1,j} : c_{1,j} \neq 0; 1 \leq j \leq 6\} = \{1,2,3,4,5\}$, $B = \{c_{i,5} : c_{i,5} \neq 0; 1 \leq i \leq 6\} = \{1,5,6\}$, then $A \cup B$ = {1, 2, 3, 4, 5, 6} and $\{1,2,3,4,5,6\} \setminus A \cup B = \emptyset$.*

It can't be go further.

In the illustration of Example 2.4.7, if we choose the color matrix (symmetric latin square) given below, then exists an n-coloring of G.

$$\text{Let } C' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

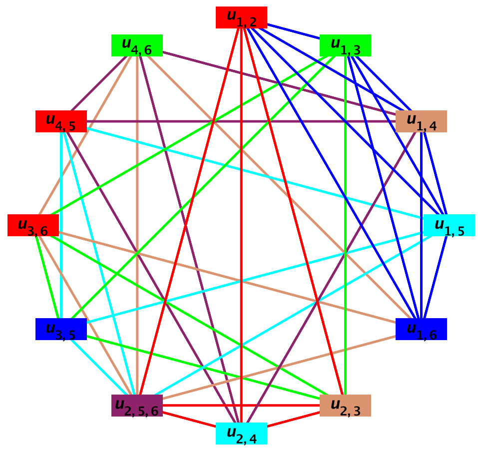
Appying the method of construction as in Example 2.4.7, we get

$$C'_M = \begin{pmatrix} 0 & 2 & 3 & 6 & 5 & 1 \\ 2 & 0 & 6 & 5 & 4 & 4 \\ 3 & 6 & 0 & 0 & 1 & 2 \\ 6 & 5 & 0 & 0 & 2 & 3 \\ 5 & 4 & 1 & 2 & 0 & 4 \\ 1 & 4 & 2 & 3 & 4 & 0 \end{pmatrix}.$$

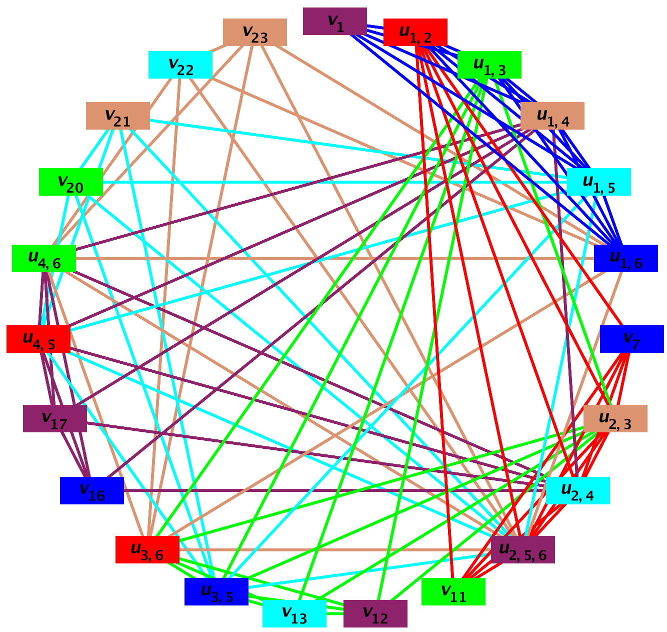
Color the vertex v by the (i,j)-th entry $c_{i,j}$ of the matrix C'_M, whenever $A_i \cap A_j \neq \emptyset$ (see Figure 2.19a), where the numbers 1, 2, 3, 4, 5, 6 corresponds to the colors Blue, Red, Green, Maroon, Tan, Cyan respectively. Extend the coloring of $\hat{H}$ to G by assigning the remaining colors which are not used for A_i from the set of 6-colors to the vertices of clique degree one in each A_i, $1 \leq i \leq 6$. The colored graph G is shown in Figure 2.19b.

Remark 2.4.8. *From the above example, one can see that the construction will work for some symmetric latin squares and will not work for some other, for the graphs having more than $\frac{n}{2}$ vertices of clique degree greater than one in some A_i ($1 \leq i \leq n$) in G.*

Theorem 2.4.9. *If G is a graph satisfying the hypothesis of Conjecture 2.1.1, then G is n-colorable.*



(a) Graph $\hat{H}$



(b) A 6 coloring of graph G

Figure 2.19 The graphs $\hat{H}$ and G, after colors have been assigned to their vertices.

Proof. Let G be a graph satisfying the hypothesis of Conjecture 2.1.1. Let $\hat{H}$ be the induced subgraph of G consisting of the vertices of clique degree greater than 1 in G. Relable the vertex v of clique degree greater than 1 in G by u_x, where $x = k_1, k_2, \ldots, k_j$; vertex v is in $A_{k_i}, 1 \leq i \leq j$. Define $X = \{b_{i,j} : A_i \cap A_j = \emptyset\}$, $X_i = \{v \in G : d^K(v) = i\}$ for $i = 1, 2, \ldots, n$.

Let C be the intersection matrix (color matrix) of the cliques $A_i's$ of G is the $n \times n$

matrix in which entry $c_{i,j}$ for $i \neq j$ is 0 if $A_i \cap A_j = \emptyset$ otherwise c, and $c_{i,i}$ is 0.

Let $1,2,\dots,n$ be the n-colors. The following construction applied on the color matrix C, gives a modified color matrix C_M, using which we assign the colors to the graph $\hat{H}$. Then this coloring can be extended to the graph G.

Construction:

Let $T_i = X_i$, $P_i = \emptyset$ and $S = \{j : T_j \neq \emptyset, 2 \leq j \leq n\}$.

If $S = \emptyset$, then the graph G has no vertex of clique degree greater than one, which implies G has exactly n^2(maximum number) vertices. i.e., G is n components of K_n. Otherwise follow the steps.

Step 1: If $S = \emptyset$, stop the process. Otherwise, let $\max(S) = k$, for some $k, 2 \leq k \leq n$. Then consider the sets T_k and P_k, go to step 2.

Step 2: If $T_k = \emptyset$, go to step 1. Otherwise, choose a vertex $u_{i_1,i_2,\dots,i_k}$ from T_k, where $i_1 < i_2 < \cdots < i_k$ and go to Step 3.

Step 3: Let $Y_i = \{y :$ color y appears atleast $k-1$ times in the i^{th} row of the color matrix $\}$, $i = 1,2,\dots,n$. If $|\bigcup_{i=i_1}^{i_k} Y_i| = n$, let $B_T = \bigcup_{i=2}^{n} P_i$, $B_P = \emptyset$ and go to Step 4. Otherwise, construct a new color matrix C_1 by putting least x in $c_{i,j}$, where $x \in \{1,2,3,\dots n\} \setminus \bigcup_{i=i_1}^{i_k} Y_i$, $i \neq j$, $i_1 \leq i, j \leq i_k$. Then add the vertex $u_{i_1,i_2,\dots,i_k}$ to P_k and remove it from T_k, go to Step 2.

Step 4: Choose a vertex v from B_T such that $v \in A_i$, for some i, $i_1 \leq i \leq i_k$. Let $B = \{i : v \in A_i, 1 \leq i \leq n\}$ and go to Step 5.

Step 5: Let $Y_i = \{y :$ color y appears atleast $k-1$ times in the i^{th} row of the color matrix$\}$, for every $i \in B$. If $|\bigcup_{i \in B} Y_i| = n$ add the vertex v to B_P and remove it from B_T, go to Step 4. Otherwise construct a new color matrix C_2 by putting x in $c_{i,j}$, where $x \in \{1,2,3,\dots n\} \setminus \bigcup_{i \in B} Y_i$, $i \neq j$, $i,j \in B$. Go to Step 3.

Thus, we get the modified color matrix C_M. Then, color the vertex v of $\hat{H}$ by $c_{i,j}$

of C_M, whenever $v \in A_i \cap A_j$. Then, extend the coloring of $\hat{H}$ to G. Thus G is n-colorable. $\square$

Following is an example illustrating the algorithm given in the proof of Theorem 2.4.9.

Example 2.4.10. *Let G be the graph shown in Figure 2.20a.*

Let $V(A_1) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V(A_2) = \{v_1, v_7, v_8, v_9, v_{10}, v_{11}\}$, $V(A_3) = \{v_1, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$, $V(A_4) = \{v_1, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}\}$, $V(A_5) = \{v_6, v_7, v_{16}, v_{22}, v_{23}, v_{24}\}$, $V(A_6) = \{v_9, v_{16}, v_{19}, v_{25}, v_{26}, v_{27}\}$.

Relabel the vertices of clique degree greater than one in G by u_A *where* $A = \{i : v \in A_i$ *for* $1 \leq i \leq 6\}$. *The labeled graph is shown in Figure 2.20b. Figure 2.21 is the graph* $\hat{H}$, *where* $\hat{H}$ *is obtained by removing the vertices of clique degree 1 from G.*

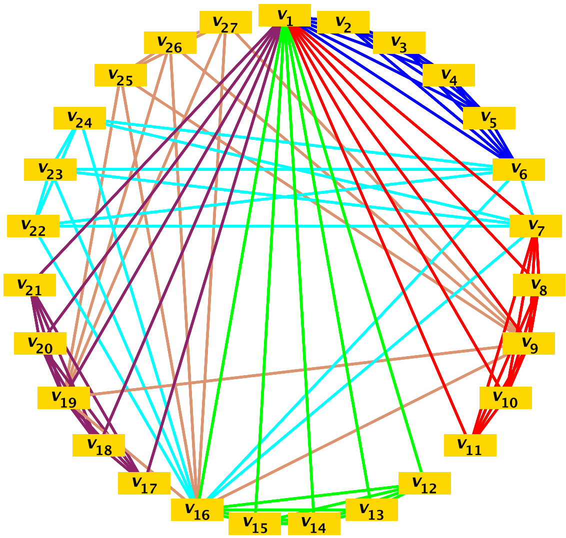
Let $X = \{b_{i,j} : A_i \cap A_j = \emptyset\} = \{b_{1,6}, b_{4,5}\}$,

$$\begin{aligned} X_1 &= \{v \in G : d^K(v) = 1\} = \{v_2, v_3, v_5, v_8, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, \\ &\quad v_{17}, v_{18}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}\}, \end{aligned}$$

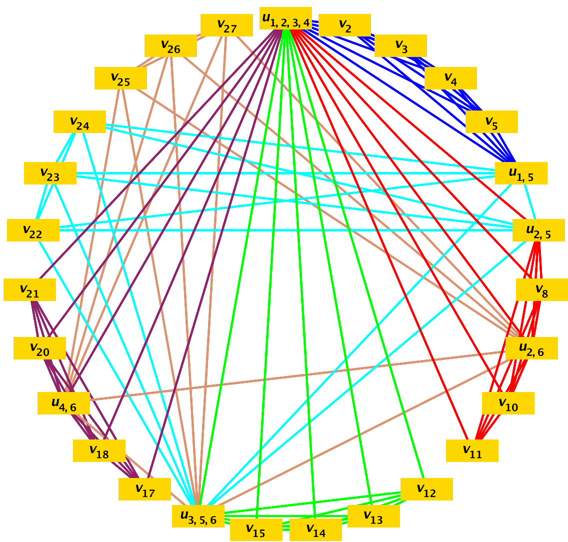
$X_2 = \{v \in G : d^K(v) = 2\} = \{v_6, v_7, v_9, v_{19}\} = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$,
$X_3 = \{v \in G : d^K(v) = 3\} = \{v_{16}\} = \{u_{3,5,6}\}$,
$X_4 = \{v \in G : d^K(v) = 4\} = \{v_1\} = \{u_{1,2,3,4}\}$,
$X_5 = \emptyset$ *and* $X_6 = \emptyset$.

Let 1, 2, …, 6 be the six colors and $C = \begin{pmatrix} 0 & c & c & c & c & 0 \\ c & 0 & c & c & c & c \\ c & c & 0 & c & c & c \\ c & c & c & 0 & 0 & c \\ c & c & c & 0 & 0 & c \\ 0 & c & c & c & c & 0 \end{pmatrix}$

be the color matrix (intersection matrix) of order 6×6.



(a) Graph G



(b) Graph G after relabeling the vertices of clique degree greater than one

Figure 2.20 Graph G: before and after relabeling the vertices

Consider the sets $T_i = X_i$, $P_i = \emptyset$ *for* $i = 1, 2, \ldots 6$ *and* $S = \{j : T_j \neq \emptyset, 2 \leq j \leq n\} = \{2, 3, 4\}$. *Then by applying the algorithm given in the proof of Theorem 2.4.9 we get the following,*

Step 1: *Since* $S \neq \emptyset$ *and* $\max(S) = 4$, *then choose the sets* $T_4 = \{u_{1,2,3,4}\}$ *and* $P_4 = \emptyset$. *Go to step 2.*

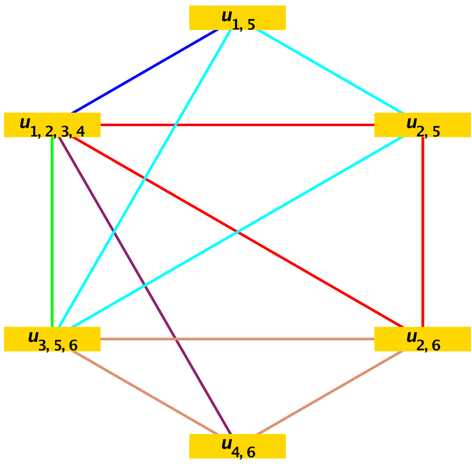


Figure 2.21 Graph $\hat{H}$

Step 2: *Since $T_4 \neq \emptyset$, choose the vertex $u_{1,2,3,4}$ from T_4, go to step 3.*

Step 3: *Since $Y_1 = \emptyset$, $Y_2 = \emptyset$, $Y_3 = \emptyset$, $Y_4 = \emptyset$ and $|Y_1 \cup Y_2 \cup Y_3 \cup Y_4| < 6$, choose the minimum color from the set $\{1,2,\ldots,6\} \setminus \cup_{i=1,2,3,4} Y_i$ and construct a new color matrix C_1 by putting 1 in $c_{i,j}$, $i \neq j$, $i,j = 1,2,3,4$. Add the vertex $u_{1,2,3,4}$ to P_4 and remove it from T_4. Then*

$$C_1 = \begin{pmatrix} 0 & 1 & 1 & 1 & c & 0 \\ 1 & 0 & 1 & 1 & c & c \\ 1 & 1 & 0 & 1 & c & c \\ 1 & 1 & 1 & 0 & 0 & c \\ c & c & c & 0 & 0 & c \\ 0 & c & c & c & c & 0 \end{pmatrix},$$

$T_4 = \emptyset$, $P_4 = \{u_{1,2,3,4}\}$. Go to step 2.

Step 2: *Since $T_4 = \emptyset$, go to step 1.*

Step 1: *Since $S \neq \emptyset$ and $\max(S) = 3$, then choose the sets $T_3 = \{u_{3,5,6}\}$ and $P_3 = \emptyset$. Go to step 2.*

Step 2: *Since $T_3 \neq \emptyset$, choose the vertex $u_{3,5,6}$ from T_3, go to step 3.*

Step 3: *Since $Y_3 = \{1\}$, $Y_5 = \emptyset$, $Y_6 = \emptyset$, and $|Y_3 \cup Y_5 \cup Y_6| < 6$, choose the minimum color from the set $\{1,2,\ldots,6\} \setminus \cup_{i=3,5,6} Y_i$ and construct a new color matrix C_2 by* *putting* 2 *in* $c_{i,j}$, $i \neq j$, $i,j = 3,5,6$. *Add the vertex* $u_{3,5,6}$ *to* P_3 *and remove it from* T_3. *Then*

$$C_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & c & 0 \\ 1 & 0 & 1 & 1 & c & c \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & c \\ c & c & 2 & 0 & 0 & 2 \\ 0 & c & 2 & c & 2 & 0 \end{pmatrix},$$

$T_3 = \emptyset$, $P_3 = \{u_{3,5,6}\}$. *Go to step 2.*

Step 2: *Since* $T_3 = \emptyset$, *go to step 1.*

Step 1: *Since* $S \neq \emptyset$ *and* $\max(S) = 2$, *then choose the sets* $T_2 = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$ *and* $P_2 = \emptyset$. *Go to step 2.*

Step 2: *Since* $T_2 \neq \emptyset$, *choose the vertex* $u_{1,5}$ *from* T_2, *go to step 3.*

Step 3: *Since* $Y_1 = \{1\}$, $Y_5 = \{2\}$ *and* $|Y_1 \cup Y_5| < 6$, *choose the minimum color from the set* $\{1,2,\ldots,6\} \setminus \cup_{i=1,5} Y_i$ *and construct a new color matrix* C_3 *by putting* 3 *in* $c_{i,j}$, $i \neq j$, $i,j = 1,5$. *Add the vertex* $u_{1,5}$ *to* P_2 *and remove it from* T_2. *Then*

$$C_3 = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 1 & c & c \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & c \\ 3 & c & 2 & 0 & 0 & 2 \\ 0 & c & 2 & c & 2 & 0 \end{pmatrix},$$

$T_2 = \{u_{2,5}, u_{2,6}, u_{4,6}\}$, $P_2 = \{u_{1,5}\}$. *Go to step 2.*

Step 2: *Since* $T_2 \neq \emptyset$, *choose the vertex* $u_{2,5}$ *from* T_2, *go to step 3.*

Step 3: *Since* $Y_2 = \{1\}$, $Y_5 = \{2,3\}$ *and* $|Y_2 \cup Y_5| < 6$, *choose the minimum color from the set* $\{1,2,\ldots,6\} \setminus \cup_{i=2,5} Y_i$ *and construct a new color matrix* C_4 *by putting* 4 *in* $c_{i,j}$, $i \neq j$, $i,j = 2,5$. *Add the vertex* $u_{2,5}$ *to* P_2 *and remove it from* T_2. *Then*

$$C_4 = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 1 & 4 & c \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & c \\ 3 & 4 & 2 & 0 & 0 & 2 \\ 0 & c & 2 & c & 2 & 0 \end{pmatrix},$$

$T_2 = \{u_{2,6}, u_{4,6}\}$, $P_2 = \{u_{1,5}, u_{2,5}\}$. Go to step 2.

Step 2: *Since $T_2 \neq \emptyset$, choose the vertex $u_{2,6}$ from T_2, go to step 3.*

Step 3: *Since $Y_2 = \{1,4\}$, $Y_6 = \{2\}$ and $|Y_2 \cup Y_6| < 6$, choose the minimum color from the set $\{1,2,\ldots,6\} \setminus \cup_{i=2,6} Y_i$ and construct a new color matrix C_5 by putting 3 in $c_{i,j}$, $i \neq j$, $i,j = 2,6$. Add the vertex $u_{2,6}$ to P_2 and remove it from T_2. Then*

$$C_5 = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 1 & 4 & 3 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & c \\ 3 & 4 & 2 & 0 & 0 & 2 \\ 0 & 3 & 2 & c & 2 & 0 \end{pmatrix},$$

$T_2 = \{u_{4,6}\}$, $P_2 = \{u_{1,5}, u_{2,5}, u_{2,6}\}$. Go to step 2.

Step 2: *Since $T_2 \neq \emptyset$, choose the vertex $u_{4,6}$ from T_2, go to step 3.*

Step 3: *Since $Y_4 = \{1\}$, $Y_6 = \{2,3\}$ and $|Y_4 \cup Y_6| < 6$, choose the minimum color from the set $\{1,2,\ldots,6\} \setminus \cup_{i=4,6} Y_i$ and construct a new color matrix C_6 by putting 4 in $c_{i,j}$, $i \neq j$, $i,j = 4,6$. Add the vertex $u_{4,6}$ to P_2 and remove it from T_2. Then*

$$C_6 = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 0 \\ 1 & 0 & 1 & 1 & 4 & 3 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 & 4 \\ 3 & 4 & 2 & 0 & 0 & 2 \\ 0 & 3 & 2 & 4 & 2 & 0 \end{pmatrix},$$

$T_2 = \emptyset$, $P_2 = \{u_{1,5}, u_{2,5}, u_{2,6}, u_{4,6}\}$. Go to step 2.

Step 2: *Since $T_2 = \emptyset$, go to step 1.*

Step 1: *Since $S = \emptyset$, stop the process.*

Assign the colors to the graph $\hat{H}$ using the matrix $C_M = C_6$, i.e., color the vertex v by the (i, j)-th entry $c_{i,j}$ of the matrix C_M, whenever $A_i \cap A_j \neq \emptyset$ (see Figure 2.22a), where the numbers 1, 2, 3, 4, 5, 6 corresponds to the colors Maroon, Tan, Green, Red, Blue, Cyan respectively. Extend the coloring of $\hat{H}$ to G by assigning the remaining colors which are not used for A_i from the set of 6-colors to the vertices of clique degree one in each A_i, $1 \leq i \leq 6$. The colored graph G is shown in Figure 2.22b.

Here we give the construction for assigning colors to the linear hypergraph **H** with n edges each with at most n vertices.

Coloring of **H**:

Let **H** be a linear hypergraph with n edges each with at most n vertices. By using the following method one can color the linear hypergraph **H** with at most n colors.

Let $E_1, E_2, \dots, E_n$ be the edges of **H**. Let $1, 2, \dots, n$ be the n-colors. Define $X_i = \{v \in H :$ degree of v is $i\}$ for $i = 1, 2, \dots, n$.

Construction:

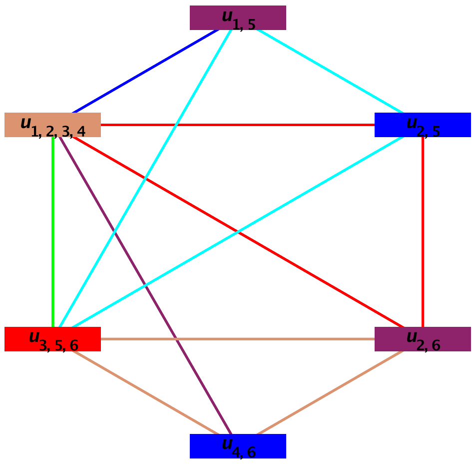
Let $T_i = X_i$, $P_i = \emptyset$ and $S = \{j : T_j \neq \emptyset, 2 \leq j \leq n\}$.

Step 1: If $S = \emptyset$, stop the process. Otherwise, let $\max(S) = k$, for some $k, 2 \leq k \leq n$. Then consider the sets T_k and P_k, go to step 2.

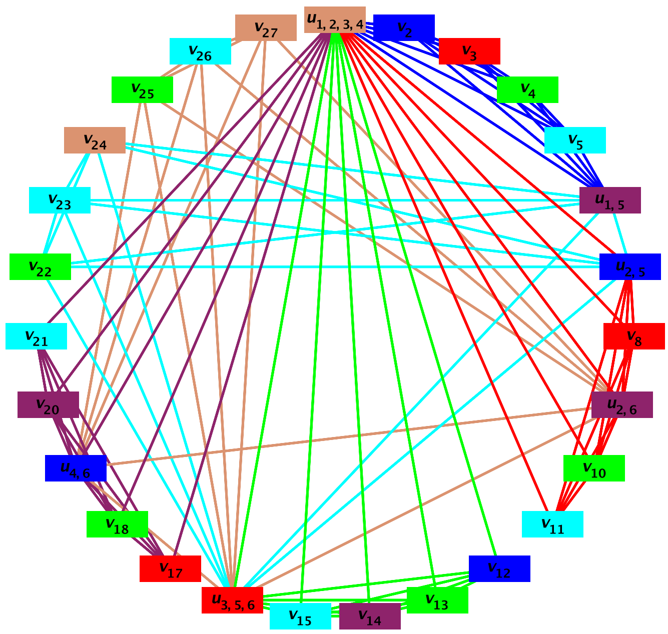
Step 2: If $T_k = \emptyset$, go to Step 1. Otherwise choose a vertex v from T_k and go to Step 3.

Step 3: Let Y_v be the set of all used colors of the edges which contains the vertex v. If $|Y_v| = n$, let $B_T = \bigcup_{i=2}^{n} P_i$, $B_P = \emptyset$ and go to Step 4. Otherwise, assign the minimum value color from the set of unused colors to the vertex v. Then add the vertex v to P_k and remove it from T_k, go to Step 2.

Step 4: Choose a vertex u from B_T such that v and u belong to same edge E_i for some $1 \leq i \leq n$. Go to Step 5.



(a) Graph $\hat{H}$



(b) A 6 coloring of graph G

Figure 2.22 The graphs $\hat{H}$ and G, after colors have been assigned to their vertices.

Step 5: Let Y_u be the set of all used colors of the edges which contains the vertex u. If $|Y_u| = n$ add the vertex u to B_P and remove it from B_T, go to Step 4. Otherwise, assign the minimum value color from the set of unused colors to the vertex u. Go to Step 3.

Thus, we get a proper coloring of the linear hypergraph **H** using at most n colors.

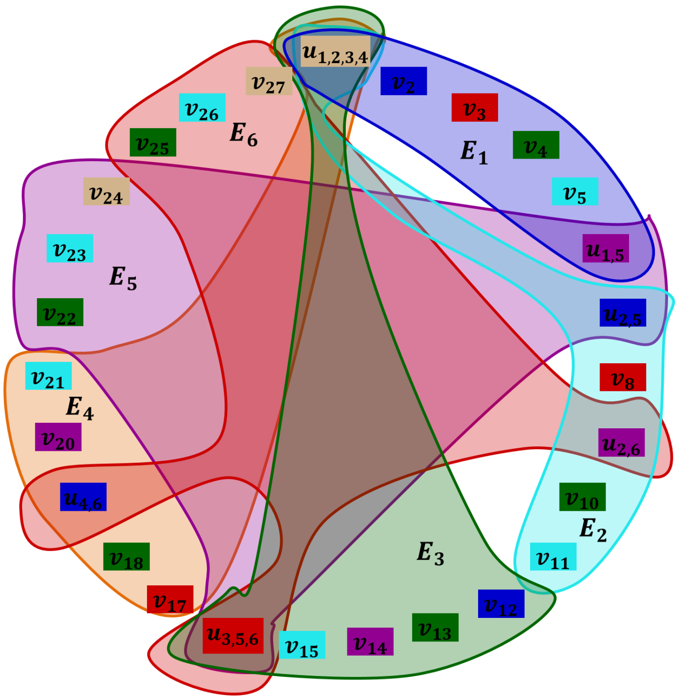


Figure 2.23 A 6 coloring of hypergraph **H** corresponding to the graph G shown in Figure 2.22b

2.4.1 Fano plane

A projective plane has the same number of lines as it has points (infinite or finite). Thus, for every finite projective plane there is an integer $N \geq 2$ such that the plane has

- $N^2 + N + 1$ points,
- $N^2 + N + 1$ lines,
- $N + 1$ points on each line, and
- $N + 1$ lines through each point.

The number N is called the order of the projective plane.

In finite geometry, the **Fano plane** is the finite projective plane of order 2, having the smallest possible number of points and lines, 7 each, with 3 points on every line and 3 lines through every point.

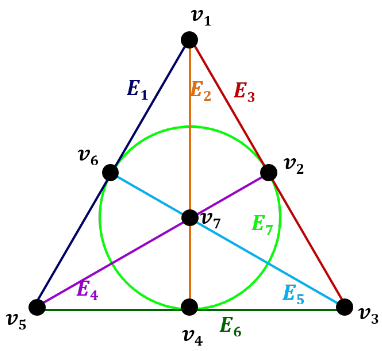


Figure 2.24 Fano Plane

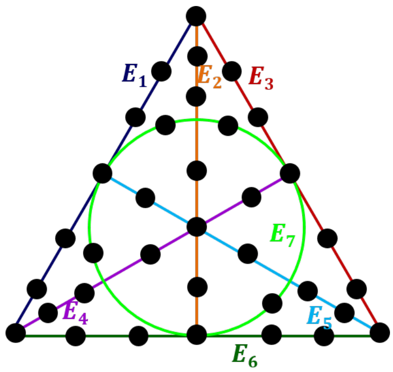


Figure 2.25 Graph G

Example 2.4.11. *Let G be the graph shown in Figure 2.25.*

Figure 2.26 is the graph $\hat{H}$ (Fano plane) of G, where $\hat{H}$ is obtained by removing the vertices of clique degree 1 from G.

Let $E_1 = \{v_1, v_5, v_6\}$, $E_2 = \{v_1, v_4, v_7\}$, $E_3 = \{v_1, v_2, v_3\}$, $E_4 = \{v_2, v_5, v_7\}$, $E_5 = \{v_3, v_6, v_7\}$, $E_6 = \{v_3, v_4, v_5\}$, $E_7 = \{v_2, v_4, v_6\}$ *and* $1, 2, 3 \ldots 7$ *be the 7 colors.*

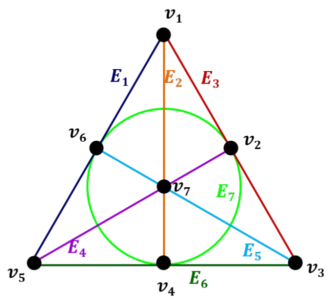


Figure 2.26 Fano Plane ($\hat{H}$)

Let $X_1 = \emptyset$*,* $X_2 = \{v \in H : d(v) = 2\} = \emptyset$*,*

$X_3 = \{v \in H : d(v) = 3\} = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$*,* $X_4 = \emptyset$*,* $X_5 = \emptyset$*,* $X_6 = \emptyset$ *and* $X_7 = \emptyset$*.*

Consider the sets $T_i = X_i$*,* $P_i = \emptyset$*, for* $i = 1, 2, \ldots 7$ *and* $S = \{j : T_j \neq \emptyset, 1 \leq j \leq 7\} = \{3\}$*. Then by applying the above construction we get,*

Step 1: *Since* $S \neq \emptyset$ *and* $\max(S) = 3$*, then choose the sets* T_3 *and* P_3*. Go to step 2.*

Step 2: *Since* $T_3 \neq \emptyset$*, choose the vertex* v_1 *from* T_3*, go to step 3.*

Step 3: *Since* $Y_{v_1} = \emptyset$*, choose the minimum color from the set* $\{1, 2, \ldots, 7\} \setminus Y_{v_1}$*. Add the vertex* v_1 *to* P_3 *and remove it from* T_3*. Then color of* v_1 *is 1,* $T_3 = \{v_2, v_3, v_4, v_5, v_6, v_7\}$ *and* $P_3 = \{v_1\}$*. Go to step 2.*

Step 2: *Since* $T_3 \neq \emptyset$*, choose the vertex* v_2 *from* T_3*, go to step 3.*

Step 3: *Since* $Y_{v_2} = \{1\}$*, choose the minimum color from the set* $\{1, 2, \ldots, 7\} \setminus Y_{v_2}$*. Add the vertex* v_2 *to* P_3 *and remove it from* T_3*. Then color of* v_2 *is 2,* $T_3 = \{v_3, v_4, v_5, v_6, v_7\}$ *and* $P_3 = \{v_1, v_2\}$*. Go to step 2.*

Step 2: *Since* $T_3 \neq \emptyset$*, choose the vertex* v_3 *from* T_3*, go to step 3.*

Step 3: *Since* $Y_{v_3} = \{1, 2\}$*, choose the minimum color from the set* $\{1, 2, \ldots, 7\} \setminus Y_{v_3}$*. Add the vertex* v_3 *to* P_3 *and remove it from* T_3*. Then color of* v_3 *is 3,* $T_3 = \{v_4, v_5, v_6, v_7\}$ *and* $P_3 = \{v_1, v_2, v_3\}$*. Go to step 2.*

Step 2: *Since* $T_3 \neq \emptyset$*, choose the vertex* v_4 *from* T_3*, go to step 3.*

Step 3: *Since* $Y_{v_4} = \{1, 2, 3\}$*, choose the minimum color from the set* $\{1, 2, \ldots, 7\} \setminus Y_{v_4}$*. Add the vertex* v_4 *to* P_3 *and remove it from* T_3*. Then color of* v_4 *is 4,* $T_3 = \{v_5, v_6, v_7\}$ *and* $P_3 = \{v_1, v_2, v_3, v_4\}$*. Go to step 2.*

Step 2: *Since* $T_3 \neq \emptyset$*, choose the vertex* v_5 *from* T_3*, go to step 3.*

Step 3: *Since* $Y_{v_5} = \{1, 2, 3, 4\}$*, choose the minimum color from the set* $\{1, 2, \ldots, 7\} \setminus Y_{v_5}$*. Add the vertex* v_5 *to* P_3 *and remove it from* T_3*. Then color of* v_5 *is 5,* $T_3 = \{v_6, v_7\}$ *and* $P_3 = \{v_1, v_2, v_3, v_4, v_5\}$*. Go to step 2.*

Step 2: *Since* $T_3 \neq \emptyset$*, choose the vertex* v_6 *from* T_3*, go to step 3.*

Step 3: *Since* $Y_{v_6} = \{1, 2, 3, 4, 5\}$*, choose the minimum color from the set* $\{1, 2, \ldots, 7\} \setminus$

Y_{v_6}. *Add the vertex* v_6 *to* P_3 *and remove it from* T_3. *Then color of* v_6 *is 6,* $T_3 = \{v_7\}$ *and* $P_3 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. *Go to step 2.*

Step 2: *Since* $T_3 \neq \emptyset$, *choose the vertex* v_7 *from* T_3, *go to step 3.*

Step 3: *Since* $Y_{v_7} = \{1, 2, 3, 4, 5, 6\}$, *choose the minimum color from the set* $\{1, 2, \ldots, 7\} \setminus Y_{v_7}$. *Add the vertex* v_7 *to* P_3 *and remove it from* T_3. *Then color of* v_7 *is 7,* $T_3 = \emptyset$ *and* $P_3 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. *Go to step 2.*

Step 2: *Since* $T_3 = \emptyset$, *go to step 1.*

Step 1: *Since* $S = \emptyset$, *stop the process.*

Assign the colors to the graph $\hat{H}$ *(see Figure 2.27).*

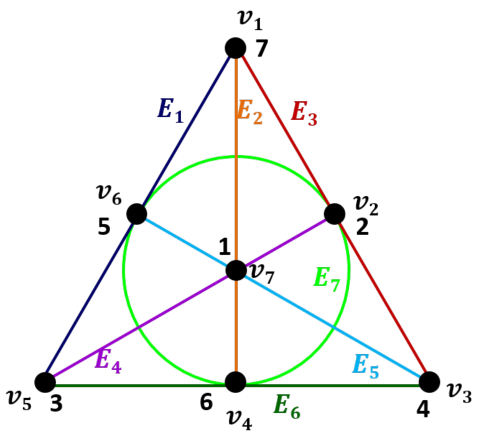


Figure 2.27 A 7 coloring of Fano Plane

2.4.2 Steiner Triple Systems

Definition 2.4.12 (Grannell et al. (2000))**.** *A* **Steiner triple system (STS)** $S = (V; B)$ *of order v, denoted by STS(v), is a collection B of triples (3-element subsets) of the set V, where* $|V| = v$, *such that each unordered pair of elements (points) of V is contained in precisely one triple from B. It is well known that an STS(v) exists if and only if* $v \equiv 1$ *or* $3(mod 6)$*; such values of v are called admissible.*

Definition 2.4.13. *A* **Pasch configuration***, also known as a quadrilateral, consists of four triples of a Steiner triple system whose union is a set of six points, that is to say, four* *triples which must be of the form $\{a,b,c\}$, $\{a,y,z\}$, $\{x,b,z\}$ and $\{x,y,c\}$. An STS(v) is* **anti-Pasch** *or* **quadrilateral-free** *if it does not contain a Pasch configuration. We will denote such a system by QFSTS(v).*

Definition 2.4.14. *BQFSTS(u,-m) designs (m-bipartite quadrilateral-free STS(u,-m)) The points of the system are $1,2,\ldots,u$. These comprise the points of the hole labelled M $(1,2,\ldots,m)$, points labelled A $(m+1,m+2,\ldots,\frac{m+u}{2})$ and points labelled B $(\frac{m+u+2}{2},\frac{m+u+4}{2},\ldots,u)$. The systems are STS(u,-m)s,*

i.e. Steiner triple systems of order u with a hole of size m. No pairs labelled M,M appear in a triple, but all other pairs do appear in a triple.

Each system is m-bipartite, i.e. there are no M,A,A or M,B,B triples. Each system is quadrilateral-free (i.e. anti-Pasch) Grannell et al. (2000).

A full listing of the triples of a BQFSTS(19;-3) is given below. For clarity, we list blocks omitting set brackets and commas. A specimen of each of the BQFSTS(u;-m) designs used in this paper is available from the JCD website (JCD). $1,2,\ldots 19$ are the 19 edges of BQFSTS(19;-3) and each block is a triplet a,b,c and it represents a vertex. Triplet a,b,c means it is the common vertex to the edges a,b and c.

Example 2.4.15. *BQFSTS(19 , -3)*

v_1 : 1 4 12	v_2 : 1 5 13	v_3 : 1 6 14	v_4 : 1 7 15
v_5 : 1 8 16	v_6 : 1 9 17	v_7 : 1 10 18	v_8 : 1 11 19
v_9 : 2 4 13	v_{10} : 2 5 14	v_{11} : 2 6 15	v_{12} : 2 7 16
v_{13} : 2 8 17	v_{14} : 2 9 18	v_{15} : 2 10 19	v_{16} : 2 11 12
v_{17} : 3 4 14	v_{18} : 3 5 15	v_{19} : 3 6 16	v_{20} : 3 7 17
v_{21} : 3 8 18	v_{22} : 3 9 19	v_{23} : 3 10 12	v_{24} : 3 11 13
v_{25} : 4 5 17	v_{26} : 4 6 10	v_{27} : 4 7 9	v_{28} : 4 8 15
v_{29} : 4 11 16	v_{30} : 4 18 19	v_{31} : 5 6 19	v_{32} : 5 7 10
v_{33} : 5 8 9	v_{34} : 5 11 18	v_{35} : 5 12 16	v_{36} : 6 7 18
v_{37} : 6 8 12	v_{38} : 6 9 11	v_{39} : 6 13 17	v_{40} : 7 8 19

$v_{41}: 7$	11	14	$v_{42}: 7$	12	13	$v_{43}: 8$	10	11	$v_{44}: 8$	13	14
$v_{45}: 9$	10	13	$v_{46}: 9$	12	14	$v_{47}: 9$	15	16	$v_{48}: 10$	14	15
$v_{49}: 10$	16	17	$v_{50}: 11$	15	17	$v_{51}: 12$	15	18	$v_{52}: 12$	17	19
$v_{53}: 13$	15	19	$v_{54}: 13$	16	18	$v_{55}: 14$	16	19	$v_{56}: 14$	17	18

$E_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ $E_2 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$

$E_3 = \{v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}\}$ $E_4 = \{v_1, v_9, v_{17}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}\}$

$E_5 = \{v_2, v_{10}, v_{18}, v_{25}, v_{31}, v_{32}, v_{33}, v_{34}, v_{35}\}$ $E_6 = \{v_3, v_{11}, v_{19}, v_{26}, v_{31}, v_{36}, v_{37}, v_{38}, v_{39}\}$

$E_7 = \{v_4, v_{12}, v_{20}, v_{27}, v_{32}, v_{36}, v_{40}, v_{41}, v_{42}\}$ $E_8 = \{v_5, v_{13}, v_{21}, v_{28}, v_{33}, v_{37}, v_{40}, v_{43}, v_{44}\}$

$E_9 = \{v_6, v_{14}, v_{22}, v_{27}, v_{33}, v_{38}, v_{45}, v_{46}, v_{47}\}$ $E_{10} = \{v_7, v_{15}, v_{23}, v_{26}, v_{32}, v_{43}, v_{45}, v_{48}, v_{49}\}$

$E_{11} = \{v_8, v_{16}, v_{24}, v_{29}, v_{34}, v_{38}, v_{41}, v_{43}, v_{50}\}$ $E_{12} = \{v_1, v_{16}, v_{23}, v_{35}, v_{37}, v_{42}, v_{46}, v_{51}, v_{52}\}$

$E_{13} = \{v_2, v_9, v_{24}, v_{39}, v_{42}, v_{44}, v_{45}, v_{53}, v_{54}\}$ $E_{14} = \{v_3, v_{10}, v_{17}, v_{41}, v_{44}, v_{46}, v_{48}.v_{55}, v_{56}\}$

$E_{15} = \{v_4, v_{11}, v_{18}, v_{28}, v_{47}, v_{48}, v_{50}, v_{51}, v_{53}\}$ $E_{16} = \{v_5, v_{12}, v_{19}, v_{29}, v_{35}, v_{47}, v_{49}, v_{54}, v_{55}\}$

$E_{17} = \{v_6, v_{13}, v_{20}, v_{25}, v_{39}, v_{49}, v_{50}, v_{52}, v_{56}\}$ $E_{18} = \{v_7, v_{14}, v_{21}, v_{30}, v_{34}, v_{36}, v_{51}, v_{54}, v_{56}\}$

$E_{19} = \{v_8, v_{15}, v_{22}, v_{30}, v_{31}, v_{40}, v_{52}, v_{53}, v_{55}\}$

BQFSTS(19 , -3) is the hypergraph $\boldsymbol{H}$ *with* 19 *edges and every vertex of degree is exactly* 3. *Let* $X_3 = \{v \in H : d(v) = 3\} = \{v_1, v_2, v_3, \ldots v_{56}\}$, $X_i = \emptyset$ *for* $1 \le i \le 19, i \ne 3$.

Consider the sets $T_i = X_i$, $P_i = \emptyset$ *for* $i = 1, 2, \ldots 19$ *and* $S = \{j : T_j \ne \emptyset, 1 \le j \le 19\} = \{3\}$. *Then by applying the above construction we get,*

Step 1: *Since* $S \ne \emptyset$ *and* $\max(S) = 3$, *then choose the sets* T_3 *and* P_3. *Go to step 2.*

Step 2: *Since* $T_3 \ne \emptyset$, *choose the vertex* v_1 *from* T_3, *go to step 3.*

Step 3: *Since* $Y_{v_1} = \emptyset$, *choose the minimum color from the set* $\{1, 2, \ldots, 19\} \setminus Y_{v_1}$. *Add the vertex* v_1 *to* P_3 *and remove it from* T_3. *Then color of* v_1 *is 1,* $T_3 = \{v_2, v_3, \ldots, v_{56}\}$ *and* $P_3 = \{v_1\}$. *Go to step 2.*

Step 2: *Since* $T_3 \ne \emptyset$, *choose the vertex* v_2 *from* T_3, *go to step 3.*

Step 3: *Since* $Y_{v_2} = \{1\}$, *choose the minimum color from the set* $\{1, 2, \ldots, 19\} \setminus Y_{v_2}$. *Add the vertex* v_2 *to* P_3 *and remove it from* T_3. *Then color of* v_2 *is 2,* $T_3 = \{v_3, v_4, \ldots, v_{56}\}$ *and* $P_3 = \{v_1, v_2\}$. *Go to step 2.*

Continuing like this we get

v_1 : 1	4	12(1)	v_2 : 1	5	13(2)	v_3 : 1	6	14(3)	v_4 : 1	7	15(4)
v_5 : 1	8	16(5)	v_6 : 1	9	17(6)	v_7 : 1	10	18(7)	v_8 : 1	11	19(8)
v_9 : 2	4	13(3)	v_{10} : 2	5	14(1)	v_{11} : 2	6	15(2)	v_{12} : 2	7	16(6)
v_{13} : 2	8	17(4)	v_{14} : 2	9	18(5)	v_{15} : 2	10	19(9)	v_{16} : 2	11	12(7)
v_{17} : 3	4	14(2)	v_{18} : 3	5	15(3)	v_{19} : 3	6	16(1)	v_{20} : 3	7	17(5)
v_{21} : 3	8	18(6)	v_{22} : 3	9	19(4)	v_{23} : 3	10	12(8)	v_{24} : 3	11	13(9)
v_{25} : 4	5	17(7)	v_{26} : 4	6	10(4)	v_{27} : 4	7	9(8)	v_{28} : 4	8	15(9)
v_{29} : 4	11	16(10)	v_{30} : 4	18	19(11)	v_{31} : 5	6	19(5)	v_{32} : 5	7	10(10)
v_{33} : 5	8	9(11)	v_{34} : 5	11	18(4)	v_{35} : 5	12	16(9)	v_{36} : 6	7	18(9)
v_{37} : 6	8	12(10)	v_{38} : 6	9	11(12)	v_{39} : 6	13	17(8)	v_{40} : 7	8	19(1)
v_{41} : 7	11	14(11)	v_{42} : 7	12	13(12)	v_{43} : 8	10	11(2)	v_{44} : 8	13	14(7)
v_{45} : 9	10	13(1)	v_{46} : 9	12	14(13)	v_{47} : 9	15	16(7)	v_{48} : 10	14	15(5)
v_{49} : 10	16	17(3)	v_{50} : 11	15	17(1)	v_{51} : 12	15	18(14)	v_{52} : 12	17	19(2)
v_{53} : 13	15	19(6)	v_{54} : 13	16	18(13)	v_{55} : 14	16	19(12)	v_{56} : 14	17	18(10)

Triplet $a, b, c(x)$ means, it is the common vertex to the edges a, b and c and x is the color assigned to that vertex. In this example it takes only 14 colors.

Here is the example BQFSTS (31, -7). Using the above algorithm it takes only 23 colors.

BQFSTS (31, -7)

1	8	20	1	9	30	1	10	22	1	11	26	1	12	24	1	13	28	1	14	21
1	15	31	1	16	23	1	17	27	1	18	25	1	19	29	2	8	24	2	9	26
2	10	20	2	11	28	2	12	22	2	13	30	2	14	25	2	15	27	2	16	21
2	17	29	2	18	23	2	19	31	3	8	22	3	9	28	3	10	24	3	11	30
3	12	20	3	13	26	3	14	23	3	15	29	3	16	25	3	17	31	3	18	21
3	19	27	4	8	21	4	9	31	4	10	23	4	11	27	4	12	25	4	13	29
4	14	26	4	15	24	4	16	28	4	17	20	4	18	30	4	19	22	5	8	25
5	9	27	5	10	21	5	11	29	5	12	23	5	13	31	5	14	30	5	15	20
5	16	26	5	17	22	5	18	28	5	19	24	6	8	23	6	9	29	6	10	25
6	11	31	6	12	21	6	13	27	6	14	28	6	15	22	6	16	30	6	17	24
6	18	26	6	19	20	7	8	31	7	9	25	7	10	27	7	11	21	7	12	29
7	13	23	7	14	24	7	15	30	7	16	20	7	17	26	7	18	22	7	19	28
8	9	16	8	10	13	8	11	14	8	12	30	8	15	18	8	17	19	8	26	29
8	27	28	9	10	17	9	11	24	9	12	13	9	14	18	9	15	23	9	19	21
9	20	22	10	11	12	10	14	19	10	15	28	10	16	31	10	18	29	10	26	30
11	13	20	11	15	16	11	17	23	11	18	19	11	22	25	12	14	15	12	16	19
12	17	28	12	18	31	12	26	27	13	14	22	13	15	21	13	16	24	13	17	18
13	19	25	14	16	17	14	20	27	14	29	31	15	17	25	15	19	26	16	18	27
16	22	29	17	21	30	18	20	24	19	23	30	20	21	26	20	23	31	20	25	30
20	28	29	21	22	24	21	23	28	21	25	31	21	27	29	22	23	27	22	26	31
22	28	30	23	24	26	23	25	29	24	25	27	24	28	31	24	29	30	25	26	28
27	30	31																		

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}	E_{13}	E_{14}	E_{15}	E_{16}	E_{17}	E_{18}	e_{19}	E_{20}	E_{21}	E_{22}	E_{23}	E_{24}	E_{25}	E_{26}	E_{27}	E_{28}	E_{29}	E_{30}	E_{31}
E_1	0	0	0	0	0	0	0	1	2	3	4	5	6	7	8	9	10	11	12	1	7	3	9	5	11	4	10	6	12	2	8
E_2	0	0	0	0	0	0	0	2	1	4	3	6	5	8	7	10	9	12	11	4	10	6	12	2	8	1	7	3	9	5	11
E_3	0	0	0	0	0	0	0	4	5	1	6	2	3	10	11	7	12	8	9	2	8	4	10	1	7	3	9	5	11	6	12
E_4	0	0	0	0	0	0	0	3	4	2	5	1	7	6	9	8	11	10	13	11	3	13	2	9	1	6	5	8	7	10	4
E_5	0	0	0	0	0	0	0	5	3	6	1	4	2	9	10	11	7	13	8	10	6	7	4	8	5	11	3	13	1	9	2
E_6	0	0	0	0	0	0	0	6	8	9	7	11	1	2	5	3	4	14	15	15	11	5	6	4	9	14	1	2	8	3	7
E_7	0	0	0	0	0	0	0	9	6	8	2	3	11	12	1	13	5	15	4	13	2	15	11	12	6	5	8	4	3	1	9
E_8	1	2	4	3	5	6	9	0	12	10	11	7	10	11	16	12	14	16	14	1	3	4	6	2	5	13	15	15	13	7	9
E_9	2	1	5	4	3	8	6	12	0	13	10	9	9	17	14	12	13	17	16	18	16	18	14	10	6	1	3	5	8	2	4
E_{10}	3	4	1	2	6	9	8	10	13	0	12	12	10	5	17	14	13	18	5	4	6	3	2	1	9	15	8	17	18	15	14
E_{11}	4	3	6	5	1	7	2	11	10	12	0	12	8	11	15	15	16	19	19	8	2	14	16	10	14	4	5	3	1	6	7
E_{12}	5	6	2	1	4	11	3	7	9	12	12	0	9	13	13	17	18	20	17	2	11	6	4	5	1	16	16	18	3	7	20
E_{13}	6	5	3	7	2	1	11	10	9	10	8	9	0	16	4	18	21	21	20	8	4	16	11	18	20	3	1	6	7	5	2
E_{14}	7	8	10	6	9	2	12	11	17	5	11	13	16	0	13	1	1	17	5	14	7	16	10	12	8	6	14	2	15	9	15
E_{15}	8	7	11	9	10	5	1	16	14	17	15	13	4	13	0	15	2	16	18	10	4	5	14	9	2	18	7	17	11	1	8
E_{16}	9	10	7	8	11	3	13	12	12	14	15	17	18	1	15	0	1	2	17	13	10	19	9	18	7	11	2	8	19	3	14
E_{17}	10	9	12	11	7	4	5	14	13	13	16	18	21	1	2	1	0	21	14	11	17	7	16	4	2	5	10	18	9	17	12
E_{18}	11	12	8	10	13	14	15	16	17	18	19	20	21	17	16	2	21	0	19	3	8	15	12	3	11	14	2	13	18	10	20
E_{19}	12	11	9	13	8	15	4	14	16	5	19	17	20	5	18	17	14	19	0	15	16	13	21	8	20	18	9	4	12	21	11
E_{20}	1	4	2	11	10	15	13	1	18	4	8	2	8	14	10	13	11	3	15	0	9	18	5	3	12	9	14	16	16	12	5
E_{21}	7	10	8	3	6	11	2	3	16	6	2	11	4	7	4	10	17	8	16	9	0	20	1	20	13	9	21	1	21	17	13
E_{22}	3	6	4	13	7	5	15	4	18	3	14	6	16	16	5	19	7	15	13	18	20	0	17	20	14	10	17	11	19	11	10
E_{23}	9	12	10	2	4	6	11	6	14	2	16	4	11	10	14	9	16	12	21	5	1	17	0	7	22	7	17	1	22	21	5
E_{24}	5	2	1	9	8	4	12	2	10	1	10	5	18	12	9	18	4	3	8	3	20	20	7	0	19	7	19	21	14	14	21
E_{25}	11	8	7	1	5	9	6	5	6	9	14	1	20	8	2	7	2	11	20	12	13	14	22	19	0	23	19	23	22	12	13
E_{26}	4	1	3	6	11	14	5	13	1	15	4	16	3	6	18	11	5	14	18	9	9	10	7	7	23	0	16	23	13	15	10
e_{27}	10	7	9	5	3	1	8	15	3	8	5	16	1	14	7	2	10	2	9	14	21	17	17	19	19	16	0	15	21	18	18
E_{28}	6	3	5	8	13	2	4	15	5	17	3	18	6	2	17	8	18	13	4	16	1	11	1	21	23	23	15	0	16	11	21
e_{29}	12	9	11	7	1	8	3	13	8	18	1	3	7	15	11	19	9	18	12	16	21	19	22	14	22	13	21	16	0	14	15
E_{30}	2	5	6	10	9	3	1	7	2	15	6	7	5	9	1	3	17	10	21	12	17	11	21	14	12	15	18	11	14	0	18
E_{31}	8	11	12	4	2	7	9	9	4	14	7	20	2	15	8	14	12	20	11	5	13	10	5	21	13	10	18	21	15	18	0

Symmetric Color Matrix of BQFSTS (31, -7)

The following results give a relation between the number of complete graphs and clique degrees of a graph.

Theorem 2.4.16. *Let G be a graph satisfying the hypothesis of Conjecture 2.1.1. If the intersection of any two A_i's is non empty, then*

$$\binom{d^K(v_1)}{2} + \binom{d^K(v_2)}{2} + \cdots + \binom{d^K(v_l)}{2} = \frac{n(n-1)}{2},$$

where $\{v_1, v_2, \ldots, v_l\}$ is the set of all vertices of clique degree greater than 1 in G.

Proof. If G is isomorphic to the graph H_n for some n, then the result is obvious. If not there exists at least one vertex v of clique degree greater than 2. Define $I_v = \{i : v \in A_i\}$ then $d^K(v) = |I_v| = p$. For every unordered pair of elements (i, j) of I_v there is a vertex $b_{ij}($ where $i < j)$ in H_n. Therefore corresponding to the elements of I_v there are $\binom{p}{2}$ vertices in H_n. Since G satisfies the hypothesis of Conjecture 2.1.1, there is no vertex v' different from v in G such that $v' \in A_i \cap A_j$ where $i, j \in I_v$. Therefore for every vertex v of clique degree greater than 1 in G, there are $\binom{d^K(v)}{2}$ vertices of clique degree greater than 1 in H_n. As there are $\frac{n(n-1)}{2}$ vertices of clique degree greater than 1 in H_n, $\frac{n(n-1)}{2} = \binom{d^K(v_1)}{2} + \binom{d^K(v_2)}{2} + \cdots + \binom{d^K(v_l)}{2}$ where $\{v_1, v_2, \ldots, v_l\}$ is the set of all vertices of clique degree greater than 1 in G. $\square$

Corollary 2.4.17. *If G is a graph satisfying the hypothesis of conjecture 2.1.1, then G has at most $\frac{\binom{n}{2}}{\binom{m}{2}}$ vertices of clique degree m where $m \geq 2$.*

Proof. Let $A = \{v_1, v_2, \ldots, v_l\}$ be the set of vertices of clique degree greater than 1 in G and $p = \frac{\binom{n}{2}}{\binom{m}{2}}$. We have to prove that G has at most p vertices of clique degree m. Suppose G has $q > p$ vertices of clique degree m. Then by the definition of A, it follows that, q vertices are in A. Let those vertices be $v_1, v_2, \ldots, v_q$. By Theorem 2.4.16 we get,

$$
\begin{aligned}
\frac{n(n-1)}{2} &= \binom{d^K(v_1)}{2} + \binom{d^K(v_2)}{2} + \cdots + \binom{d^K(v_l)}{2} \\
&\geq \binom{d^K(v_1)}{2} + \binom{d^K(v_2)}{2} + \cdots + \binom{d^K(v_q)}{2} \\
&= q\binom{m}{2} \\
&\geq (p+1)\binom{m}{2} \\
\frac{\binom{n}{2}}{\binom{m}{2}} &\geq p+1 \\
p &\geq p+1,
\end{aligned}
$$

which is a contradiction. Hence there are at most $\frac{\binom{n}{2}}{\binom{m}{2}}$ vertices of clique degree m in G, where $m \geq 2$. □

Chapter 3

CLIQUE GRAPH

Let G be a graph and $\mathscr{K}_G$ be the set of all cliques of G, then the clique graph of G denoted by $K(G)$ is the graph with vertex set $\mathscr{K}_G$ and two elements $Q_i, Q_j \in \mathscr{K}_G$ form an edge if and only if $Q_i \cap Q_j \neq \emptyset$. Iterated clique graphs are defined by $K^0(G) = G$, and $K^n(G) = K(K^{n-1}(G))$ for $n > 0$.

In this chapter, we prove a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1 + G_2$, give a partial characterization for clique divergence of the join of graphs and prove that if G_1, G_2 are Clique-Helly graphs different from K_1 and $G = G_1 \square G_2$, then $K^2(G) = G$.

3.1 Introduction

Given a simple graph $G = (V, E)$, not necessarily finite, a clique in G is a maximal complete subgraph in G. Let G be a graph and $\mathscr{K}_G$ be the set of all cliques of G, then the clique graph operator is denoted by K and the clique graph of G is denoted by $K(G)$, where $K(G)$ is the graph with vertex set $\mathscr{K}_G$ and two elements $Q_i, Q_j \in \mathscr{K}_G$ form an edge if and only if $Q_i \cap Q_j \neq \emptyset$. Clique graph was introduced by Hamelink in 1968 (Hamelink, 1968). Iterated clique graphs are defined by $K^0(G) = G$, and $K^n(G) = K(K^{n-1}(G))$ for $n > 0$ (see (Hedetniemi and Slater, 1972; Prisner, 1995; Szwarcfiter, 2003)).

Definition 3.1.1. *A graph G is said to be K-periodic if there exists a positive integer n such that $G \cong K^n(G)$ and the least such integer is called the K-periodicity of G, denoted K-per (G).*

Definition 3.1.2. *A graph G is said to be K-Convergent if $\{K^n(G) : n \in \mathbb{N}\}$ is finite, otherwise it is K-Divergent (see (Neumann-Lara, 1978)).*

Definition 3.1.3. *A graph H is said to be K-root of a graph G if $K(H) = G$.*

If G is a clique graph, then one can observe that, the set of all K- roots of G is either empty or infinite.

Definition 3.1.4. *(Prisner, 1995) A graph G is a Clique-Helly Graph if the set of cliques has the Helly-Property. That is, for every family of pairwise intersecting cliques of the graph, the total intersection of all these cliques should be non-empty also.*

Definition 3.1.5. *Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be the two graphs. Then their join $G_1 + G_2$ is obtained by adding all possible edges between the vertices of G_1 and G_2.*

Definition 3.1.6. *The Cartesian product of two graphs G and H, denoted $G \square H$, is a graph with vertex set $V(G \square H) = V(G) \times V(H)$, i.e., the set $\{(g,h) | g \in G, h \in H\}$. The edge set of $G \square H$ consists of all pairs $[(g_1,h_1),(g_2,h_2)]$ of vertices with $[g_1,g_2] \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $[h_1,h_2] \in E(H)$ (see (Imrich et al., 2008) page no 3).*

3.2 Results

One can observe that the clique graph of a complete graph and star graph are always complete. Let G be a graph with n vertices and having a vertex of degree $n-1$, then the clique graph of G is also complete.

Theorem 3.2.1. *Let G_1, G_2 be two graphs and $G = G_1 + G_2$, then X is a clique in G_1 and Y is clique in G_2 if and only if $X + Y$ is a clique in $G_1 + G_2$.*

Proof. Let $G = G_1 + G_2$ and X be a clique in G_1 and Y be a clique in G_2. Suppose that $X + Y$ is not a maximal complete subgraph in $G_1 + G_2$, then there is a maximal complete subgraph (clique) Q in $G_1 + G_2$ such that $X + Y$ is a proper subgraph of Q. Since $X + Y$ is a proper subgraph of Q, there is a vertex v in Q which is not in $X + Y$ and v is adjacent to every vertex of $X + Y$, then by the definition of $G_1 + G_2$, v should be in either G_1 or G_2. Suppose v is in G_1, then the induced subgraph of $V(X)+\{v\}$ is complete in G_1, which is a contradiction as X is maximal. Therefore $X+Y$ is the maximal complete subgraph (clique) in G_1+G_2.

Conversely, let Q be a clique in G_1+G_2. Suppose that $Q \neq X+Y$, where X is a clique in G_1 and Y is a clique in G_2. If $Q \cap G_1 = \emptyset$, then Q is a subgraph of G_2. This implies that Q is a clique in G_2 as Q is a clique in G. Let v be a vertex of G_1. Then by the definition of G_1+G_2, one can observe that the induced subgraph of $V(Q) \cup \{v\}$ is complete in G, which is a contradiction as Q is a maximal complete subgraph. Therefore $Q \cap G_1 \neq \emptyset$. Similarly we can prove that $Q \cap G_2 \neq \emptyset$. Let X be the induced subgraph of G with vertex set $V(Q) \cap V(G_1)$ and Y be the induced subgraph of G with vertex set $V(Q) \cap V(G_2)$, then $Q = X+Y$. Since Q is a maximal complete subgraph of G, X and Y should be maximal complete subgraphs in G_1 and G_2 respectively. Otherwise, if X is not a maximal complete subgraph in G_1 then there is a maximal complete subgraph X' in G_1 such that X is subgraph of X' and this implies that $X+Y$ is a subgraph of $X'+Y$ and $X'+Y$ is complete, which is a contradiction. Therefore X and Y are maximal complete subgraphs (cliques) in G_1 and G_2 respectively. □

Corollary 3.2.2. *Let G_1, G_2 be two graphs and $G = G_1+G_2$. If n, m are the number of cliques in G_1, G_2 respectively, then G has nm cliques.*

Proof. Let $G = G_1+G_2$, $\mathscr{K}_{G_1} = \{X_1, X_2, \ldots, X_n\}$ be the set of all cliques of G_1 and $\mathscr{K}_{G_2} = \{Y_1, Y_2, \ldots, Y_m\}$ be the set of all cliques of G_2. Then by Theorem 3.2.1 it follows that $\mathscr{K}_G = \{X_i + Y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$ is the set of all cliques of G. Since G_1 has n and G_2 has m number of cliques, G_1+G_2 has nm number of cliques. □

In the following result we give a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1+G_2$.

Theorem 3.2.3. *Let G_1, G_2 be two graphs. If $G = G_1+G_2$, then $K(G)$ is complete if and only if either $K(G_1)$ is complete or $K(G_2)$ is complete.*

Proof. Let $G = G_1 + G_2$ and $K(G)$ be complete. Suppose that neither $K(G_1)$ nor $K(G_2)$ are complete, then there exist two cliques X, X' in G_1 and two cliques Y, Y' in G_2 such that $X \cap X' = \emptyset$ and $Y \cap Y' = \emptyset$. By Theorem 3.2.1 it follows that $X + Y, X' + Y'$ are cliques in G. Since $X \cap X'$ and $Y \cap Y'$ are empty, it follows that $\{X + Y\} \cap \{X' + Y'\} = \emptyset$, which is a contradiction as $K(G)$ is complete.

Conversely, suppose that $K(G_1)$ is complete and $\mathscr{K}_{G_1} = \{X_1, X_2, \ldots, X_n\}$, $\mathscr{K}_{G_2} = \{Y_1, Y_2, \ldots, Y_m\}$. By Corollary 3.2.2, it follows that G has exactly nm number of cliques. Let $\mathscr{K}_G = \{Q_{ij} : Q_{ij} = X_i + Y_j, \text{ for } i = 1, 2, \ldots, n; j = 1, 2, \ldots, m\}$ be the set of all cliques of G. Then Q is the vertex set of $K(G)$. Arranging the elements of $\mathscr{K}_G$ in the matrix form $M = [m_{ij}]$ where $m_{ij} = Q_{ij}$, we have

$$M = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & \cdots & Q_{1m} \\ Q_{21} & Q_{22} & Q_{23} & \cdots & Q_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{n1} & Q_{n2} & Q_{n3} & \cdots & Q_{nm} \end{pmatrix}.$$

Let Q_{ij}, Q_{kl} be any two elements in M. Since $Q_{ij} = X_i + Y_j$, $Q_{kl} = X_k + Y_l$, it follows that X_i, X_k are cliques in G_1. Since $K(G_1)$ is complete, $X_i \cap X_k \neq \emptyset$ and then $Q_{ij} \cap Q_{kl} \neq \emptyset$. Therefore Q_{ij}, Q_{kl} are adjacent in $K(G)$. Hence $K(G)$ is complete. □

Lemma 3.2.4. *Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If $K(G_1)$, $K(G_2)$ are not complete, then for every clique in $K(G_1)$ there is a clique in $K(G)$ and for every clique in $K(G_2)$ there is a clique in $K(G)$.*

Proof. Let $G = G_1 + G_2$ be a graph such that $K(G_1)$ and $K(G_2)$ are not complete. Let $V(K(G_1)) = \{X_i : X_i \text{ is a clique in } G_1, 1 \leq i \leq n\}$ and $V(K(G_2)) = \{Y_j : Y_j \text{ is a clique in } G_2, 1 \leq j \leq m\}$, then by Theorem 3.2.1 it follows that $V(K(G)) = \{X_i + Y_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Let Q be a clique of size l in $K(G_1)$ and $V(Q) = \{X_{Q_1}, X_{Q_2}, \ldots, X_{Q_l}\}$ where X_{Q_i} is a clique in G_1 for $1 \leq i \leq l$. Let $A_Q = \{X_{Q_i} + Y_j : 1 \leq i \leq l, 1 \leq j \leq m\}$. Then clearly A_Q is subset of $V(K(G))$.

Let $X_{Q_1} + Y_1, X_{Q_2} + Y_2$ be two elements in A_Q. Since X_{Q_1}, X_{Q_2} are the vertices of the clique Q of $K(G_1)$, we have $X_{Q_1} \cap X_{Q_2} \neq \emptyset$. Therefore $\{X_{Q_1} + Y_1\} \cap \{X_{Q_2} + Y_2\} \neq \emptyset$.

Hence the intersection of any two elements in A_Q is non-empty. Then, it follows that the elements of A_Q form a complete subgraph in $K(G)$. Suppose that it is not a maximal complete subgraph in $K(G)$. Then there is a vertex, say X_1+Y_1 in $K(G)$ which is not in A_Q and X_1+Y_1 is adjacent with every vertex of A_Q. Since $K(G_2)$ is not complete, there exists a vertex say Y_2 in $K(G_2)$ such that Y_2 is not adjacent to Y_1 in $K(G_2)$. Since Q is a clique in $K(G_1)$ and $K(G_1)$ is not complete, there is a vertex say X_{Q_1} in $V(Q)$ which is not adjacent to X_1 in $K(G_1)$. By the definition of A_Q one can see that $X_{Q_1}+Y_2$ is an element of A_Q. Therefore $\{X_{Q_1}+Y_2\} \cap \{X_1+Y_1\} = \emptyset$, which is a contradiction. Thus A_Q is a maximal complete subgraph in $K(G)$. Hence for every clique in $K(G_1)$ there is a clique in $K(G)$.

On similar lines we can also prove that for every clique in $K(G_2)$, there is a clique in $K(G)$. □

Corollary 3.2.5. *Let G_1, G_2 be two graphs and $G = G_1+G_2$. If $K(G_1)$, $K(G_2)$ are not complete, then the number of cliques in $K(G)$ is at least the sum of the number of cliques in $K(G_1)$ and $K(G_2)$.*

Theorem 3.2.6. *Let G_1, G_2 be two graphs and $G = G_1+G_2$. If $K(G_1)$, $K(G_2)$ are not complete, then $K^2(G_1)+K^2(G_2)$ is an induced subgraph of $K^2(G)$.*

Proof. Let $G = G_1+G_2$ be a graph such that $K(G_1)$ and $K(G_2)$ are not complete. Let $X_1, X_2, \ldots, X_n$ be the cliques of $K(G_1)$, and $Y_1, Y_2, \ldots Y_m$ be the cliques of $K(G_2)$. By Lemma 3.2.4, it follows that for every clique X_i of $K(G_1)$ there is a clique X_i' in $K(G)$, $1 \leq i \leq n$ and for every clique Y_j of $K(G_2)$ there is a clique Y_j' in $K(G)$, $1 \leq j \leq m$.

Claim 1: $X_i \cap X_j \neq \emptyset$ in $K(G_1)$ if and only if $X_i' \cap X_j' \neq \emptyset$ in $K(G)$ for $i \neq j$.

Let X_i, X_j be two cliques in $K(G_1)$ and $X_i \cap X_j \neq \emptyset$. Let v be a vertex in $X_i \cap X_j$. By Lemma 3.2.4, it follows that if v is a vertex in the clique X_i in $K(G_1)$, then for any vertex u in $K(G_2)$, $v+u$ is a vertex in the clique X_i' in $K(G)$ corresponding to the clique X_i in $K(G_1)$. Therefore $v+u$ is a vertex in $X_i' \cap X_j'$.

Conversely, suppose that X_i', X_j' be two cliques in $K(G)$ and $X_i' \cap X_j' \neq \emptyset$. Let w be a vertex in $X'_i \cap X'_j$. By Theorem 3.2.1, it follows that $w = v + u$, where v is a vertex of $K(G_1)$ and u is a vertex of $K(G_2)$. Since $w = v + u$ is a vertex of the clique X'_i in $K(G)$, it follows that v is a vertex of the clique X_i in $K(G_1)$. Similarly v is a vertex of the clique X_j in $K(G_1)$. Therefore v is in $X_i \cap X_j$.

Similarly we can prove that, $Y_i \cap Y_j \neq \emptyset$ in $K(G_2)$ if and only if $Y'_i \cap Y'_j \neq \emptyset$ in $K(G)$ for $i \neq j$.

Claim 2: $X'_i \cap Y'_j \neq \emptyset$ in $K(G)$ for $1 \leq i \leq n$, $1 \leq j \leq m$.

Let X'_i, Y'_j be two cliques in $K(G)$, $1 \leq i \leq n$, $1 \leq j \leq m$ and X_i, Y_j are the cliques in $K(G_1), K(G_2)$ corresponding to the maximal cliques X'_i, Y'_j in $K(G)$ respectively. Let v be a vertex in X_i and u be a vertex in Y_j, then by Lemma 3.2.4 $v + u$ be the vertex in X'_i as well as in Y'_j. Therefore $X'_i \cap Y'_j \neq \emptyset$.

By claims 1 and 2 it follows that $K^2(G_1) + K^2(G_2)$ is an induced subgraph of $K^2(G)$.

□

Note: Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If G is K-divergent, then G_1, G_2 don't need to be K-divergent

Example 3.2.7. *If H is a graph consisting of just two nonadjacent vertices and we define for every $n > 1$ the graph $J_n = \underbrace{(((H+H)+H)+\ldots)+H}_{n\ times}$, it turns out that $K(J_n) = J_{2^{n-1}}$. Suppose $G_1 = J_2 = C_4$, $G_2 = H$ then $G_1 + G_2 = J_3$ and $K(G_1 + G_2) = J_4$. Therefore $K^2(G_1 + G_2) = J_8$. Which implies that $G_1 + G_2$ is K-divergent. But G_1 and G_2 are not K-divergent.*

3.2.1 Observations

Let $G = G_1 + G_2$ be a graph and $\mathscr{K}_{G_1} = \{X_1, X_2, \ldots, X_n\}$ be the set of all cliques of G_1 and $\mathscr{K}_{G_2} = \{Y_1, Y_2, \ldots, Y_m\}$ be the set of all cliques of G_2. By Theorem 3.2.1, it follows that $\mathscr{K}_G = \{Q_{ij} = X_i + Y_j : 1 \leq i \leq n; 1 \leq j \leq m\}$ is the set of all cliques of G. Let v_{ij} be the vertex of $K(G)$ corresponding to the clique Q_{ij} of G. Arrange the vertices of $K(G)$ as a matrix $M = [m_{ij}]$, where $m_{ij} = v_{ij}$, i.e.,

$$M=\begin{pmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1m} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nm} \end{pmatrix}.$$

From the above matrix one can observe that the i^{th} row corresponds to the clique X_i of G_1 and j^{th} column corresponds to the clique Y_j of G_2, $1 \leq i \leq n$, $1 \leq j \leq m$.

Claim 1: Any two elements in the same row or same column in M are adjacent in $K(G)$.

Let Q_{ij}, Q_{ik} be any two elements in the i^{th} row. Since $Q_{ij} = X_i + Y_j$, $Q_{ik} = X_i + Y_k$, $Q_{ij} \cap Q_{ik} = X_i \neq \emptyset$. Therefore Q_{ij}, Q_{ik} are adjacent in $K(G)$. Similarly, any two elements in the same column are adjacent.

Claim 2: If $X_i \cap X_j \neq \emptyset$, then every vertex of i^{th} row is adjacent to every vertex of j^{th} row, $1 \leq i \neq j \leq n$.

Let $X_i \cap X_j \neq \emptyset$ and v_{ik}, v_{jl} be any two elements of i^{th} and j^{th} rows respectively in M. Since $Q_{ik} = X_i + Y_k$, $Q_{jl} = X_j + Y_l$ are the cliques of G corresponding to the vertices v_{ik}, v_{jl} of $K(G)$ and $X_i \cap X_j \neq \emptyset$, we have $Q_{ik} \cap Q_{jl} \neq \emptyset$. Therefore v_{ik}, v_{jl} are adjacent in $K(G)$.

Similarly if $Y_i \cap Y_j \neq \emptyset$, then every vertex of i^{th} column is adjacent to every vertex of j^{th} column, $1 \leq i \neq j \leq m$.

One can see that the following observations will follow from Claim 1 and Claim 2.

1. If $G = G_1 + G_2$, then $K(G)$ is Hamiltonian.

2. If $G = G_1 + G_2$, then $K(G)$ is planar if it satisfies one of the following:

i). The number of cliques in G_1 and G_2 is less than 3.

ii). If the number of cliques in G_1 is 3, then either G_2 is a complete graph or G_2 has exactly two cliques and $K(G_1) = \overline{K_3}$, $K(G_2) = \overline{K_2}$.

iii). If the number of cliques in G_1 is 4, then G_2 is a complete graph.

3. If $G = G_1 + G_2$ and n, m are the number of cliques in G_1, G_2 respectively, then the degree of any vertex in $K(G)$ is $(n+m-2) + k(n-1) + l(m-1) - kl$, $0 \leq k < m$ and $0 \leq l < n$.

4. Let G_1, G_2 be two graphs and $G = G_1 + G_2$,

i) If both G_1 and G_2 have odd number of cliques, then $K(G)$ is Eulerian if one of $K(G_1)$ or $K(G_2)$ is Eulerian.

ii) If both G_1 and G_2 have even number of cliques, then $K(G)$ is Eulerian if $K(G_1)$, $K(G_2)$ are Eulerian.

iii) If G_1 has even number of cliques and G_2 has odd number of cliques, then $K(G)$ is Eulerian if degree of each vertex in $K(G_2)$ is odd and $K(G_1)$ is Eulerian.

3.3 Cartesian product of graphs

In this section we are considering G_1, G_2 be connected graphs only.

Theorem 3.3.1. *If G_1, G_2 are Clique-Helly graphs different from K_1 and $G = G_1 \Box G_2$, then $K^2(G) = G$.*

Proof. Let G_1, G_2 be Clique-Helly graphs different from K_1 and $G = G_1 \Box G_2$. Let $V(G_1) = \{v_1, v_2, \ldots v_{n_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots u_{n_2}\}$, then by the definition of $G_1 \Box G_2$, it follows that $V(G) = \{V_{ij} : V_{ij} = (v_i, u_j)$ where $1 \leq i \leq n_1, 1 \leq j \leq n_2\}$, $|V(G)| = n_1 n_2$. Also, G has n_2 copies of G_1 (say, $G_1^1, G_1^2, \ldots, G_1^{n_2}$) are vertex disjoint induced subgraphs and n_1 copies of G_2 (say, $G_2^1, G_2^2, \ldots, G_2^{n_1}$) are vertex disjoint induced subgraphs. Clearly one can observe that $V(G_2^i) \cap V(G_1^j) = V_{ij}$, V_{ij} is not in $V(G_2^n)$ and $V(G_1^m)$ for $n \neq i$, $m \neq j$ for all $1 \leq i \leq n_1$, $1 \leq j \leq n_2$. As $G = G_1 \Box G_2$, we can see that every clique in G_1 and G_2 are cliques in G. Let $\mathscr{K}_{G_1} = \{Q_1, Q_2, \ldots, Q_{l_1}\}$ and $\mathscr{K}_{G_2} = \{P_1, P_2, \ldots, P_{l_2}\}$, then

$\mathscr{K}_G = \{Q_1^1, Q_2^1, \ldots, Q_{l_1}^1, Q_1^2, Q_2^2, \ldots, Q_{l_1}^2, \ldots Q_1^{n_2}, Q_2^{n_2}, \ldots, Q_{l_1}^{n_2},$
$P_1^1, P_2^1, \ldots, P_{l_2}^1, P_1^2, P_2^2, \ldots, P_{l_2}^2, \ldots, P_1^{n_1}, P_2^{n_1}, \ldots, P_{l_2}^{n_1}\}$.

Claim 1: For every vertex V_{ij} in G there is a clique in $K(G)$.

Let V_{ij} be a vertex in G for some i, j, $1 \leq i \leq n_1, 1 \leq j \leq n_2$. Define $A_{ij} = \{Q : V_{ij} \in Q\} \subseteq \mathscr{K}_G$. Clearly intersection of any two cliques in A_{ij} is non-empty. Therefore the vertices corresponding to these cliques in $K(G)$ form a complete subgraph in $K(G)$. Suppose it is not a maximal complete subgraph in $K(G)$, then there exists a vertex V in $K(G)$ such that V is adjacent to all the vertices of A_{ij}. Let Q_V be the clique in G

corresponding to the vertex V in $K(G)$. Clearly V_{ij} is not in Q_V. Since every clique in G is either a clique in G_1 or a clique in G_2, assume that Q_V is a clique in G_1^j. Let Q be a clique in G_2^i having the vertex V_{ij}, then Q is in A_{ij}. Since $V(G_2^i) \cap V(G_1^j) = V_{ij}$, Q is a clique in G_2^i and $V_{ij} \in V(Q)$ and $V(Q) \cap V(G_1^j) = V_{ij}$. Which implies that $V(Q) \cap (V(G_1^j) \setminus \{V_{ij}\}) = \emptyset$. Since V_{ij} is not in Q_V and Q_V is a clique in G_1^j, $V(Q_V) \subseteq (V(G_1^j) \setminus V_{ij})$. Therefore $V(Q) \cap V(Q_V) = \emptyset$, a contradiction to the fact that Q_V is adjacent to all the vertices of A_{ij} in $K(G)$. Hence the elements of A_{ij} form a clique in $K(G)$.

Claim 2: For any clique Q in $K(G)$, intersection of all the cliques of G corresponding to the vertices of Q is non-empty and a singleton.

Let Q be a clique in $K(G)$ and $V(Q) = \{x_1, x_2, \ldots x_n\}$. Suppose all x_k's are cliques in G_1^j for some j, $1 \le j \le n_2$, then the intersection of all x_k's is non-empty in G, where $x_k \in V(Q)$, as G_1^j satisfies Clique-Helly property. Let $V \in \cap_{x_k \in Q} x_k$, then V is in G_2^i for some i, $1 \le i \le n_1$. Let P be any clique in G_2^i having a vertex V, then P intersects with every element of $V(Q)$. Therefore $V(Q) \cup \{P\}$ forms a complete graph in $K(G)$, a contradiction to the assumption that Q is maximal complete subgraph. Thus the elements of Q are the cliques of G_1 and cliques of G_2. Since G_1^j's are vertex disjoint and G_2^i's are vertex disjoint, any element of Q is either a clique of G_1^j or a clique of G_2^i for fixed i, j, $1 \le i \le n_1$, $1 \le j \le n_2$. Let $x_1, x_2, \ldots, x_l$ be the cliques of G_1^j and $x_{l+1}, x_{l+2}, \ldots, x_n$ be the cliques of G_2^i. Since $V(G_1^j) \cap V(G_2^i) = V_{ij}$, x_{l_1} is a clique of G_1^j, x_{l_2} is a clique of G_2^i and $V(x_{l_1}) \cap V(x_{l_2}) \neq \emptyset$, $1 \le l_1 \le l$, $l+1 \le l_2 \le n$, $V(x_{l_1}) \cap V(x_{l_2}) = V_{ij}$. Which implies that V_{ij} belongs to every x_k in Q. Therefore $\cap_{x_k \in Q} x_k = V_{ij}$.

As the cliques of $K(G)$ are the vertices of $K^2(G)$, by Claims 1 and 2 one can see that there is a one to one correspondence between the vertices of G and $K^2(G)$.

Claim 3: Let U, V be any two adjacent vertices in G. Then the intersection of the cliques in $K(G)$ corresponding to these vertices is non-empty.

Let U, V be any two adjacent vertices in G and Q_U, Q_V be the cliques in $K(G)$ corresponding to the vertices U, V in G respectively. Since there is an edge between U,

V in G, there exists a clique Q in G such that the vertices U, V are in Q. By Claims 1 and 2 it follows that, the vertices of Q_U in $K(G)$ are the cliques of G having the vertex U in G is in common. Therefore Q is in $V(Q_U)$. Similarly Q is in $V(Q_V)$. Which implies that $Q_U \cap Q_V \neq \emptyset$. Since cliques of $K(G)$ are the vertices of $K^2(G)$, the vertices corresponding to the cliques Q_U and Q_V of $K(G)$ are adjacent in $K^2(G)$.

Claim 4: Let P, Q be any two cliques in $K(G)$. If the intersection of P and Q is non-empty, then the vertices in G corresponding to these two cliques are adjacent.

Let P, Q be any two cliques in $K(G)$, $P \cap Q \neq \emptyset$ and U, V be the vertices in G corresponding to the cliques P, Q of $K(G)$ respectively. Since $P \cap Q \neq \emptyset$, there exists a vertex Q_1 belonging to $V(P) \cap V(Q)$. By Claims 1 and 2, one can observe that Q_1 is a clique in G and $\cap_{P_i \in V(P)} P_i = U$, $\cap_{Q_i \in V(Q)} Q_i = V$. Thus U, V belongs to $V(Q_1)$ in G. Therefore U, V are adjacent in G.

By Claims 3 and 4 it follows that, two vertices are adjacent in G if and only if the corresponding vertices are adjacent $K^2(G)$.

Therefore $K^2(G)$ is the same as G, if $G = G_1 \Box G_2$ and G_1, G_2 are Clique-Helly graphs such that G_1, G_2 are different from K_1. $\square$

Corollary 3.3.2. *Let G_1, G_2 be two graphs and $G = G_1 \Box G_2$. If G_1, G_2 are Clique-Helly graphs different from K_1, then*

i) *G is a Clique-Helly graph.*

ii) *G is K-periodic.*

iii) *G is K-convergent.*

Chapter 4

FOREST GRAPH

In 1966, Cummins introduced the "tree graph": the tree graph $\mathbf{T}(G)$ of a graph G (possibly infinite) has all its spanning trees as vertices, and distinct such trees correspond to adjacent vertices if they differ in just one edge. i.e., two spanning trees T_1 and T_2 are adjacent if $T_2 = T_1 - e + f$ for some edges $e \in T_1$ and $f \notin T_1$. The tree graph of a connected graph need not be connected. To obviate this difficulty, we define the "forest graph": let G be a labeled graph of order α, finite or infinite, and let $\mathfrak{N}(G)$ be the set of all labeled maximal forests of G. The forest graph of G, denoted by $\mathbf{F}(G)$, is the graph with vertex set $\mathfrak{N}(G)$ in which two maximal forests F_1, F_2 of G form an edge if and only if they differ exactly by one edge, i.e., $F_2 = F_1 - e + f$ for some edges $e \in F_1$ and $f \notin F_1$.

We write $\mathbf{F}^2(G)$ to denote $\mathbf{F}(\mathbf{F}(G))$, and in general $\mathbf{F}^n(G) = \mathbf{F}(\mathbf{F}^{n-1}(G))$ for $n \geq 1$, with $\mathbf{F}^0(G) = G$.

Definition 4.0.3. *A graph G is said to be* $\mathbf{F}$-convergent *if $\{\mathbf{F}^n(G) : n \in \mathbb{N}\}$ is finite; otherwise it is* $\mathbf{F}$-divergent.

A graph H is said to be an $\mathbf{F}$-root *of G if $\mathbf{F}(H)$ is isomorphic to G, $\mathbf{F}(H) \cong G$. The* $\mathbf{F}$-depth *of G is*

$$\sup\{n \in \mathbb{N} : G \cong \mathbf{F}^n(H) \textit{ for some graph } H\}.$$

The $\mathbf{F}$-depth of a graph G that has no $\mathbf{F}$-root is said to be zero.

The graph G is said to be $\mathbf{F}$-periodic *if there exists a positive integer n such that $\mathbf{F}^n(G) = G$. The least such integer is called the* $\mathbf{F}$-periodicity *of G. If $n = 1$, G is called*

F-stable.

This chapter is organized as follows. In Section 4.1 we give some basic results. In later sections, using Zorn's lemma, transfinite induction, the well ordering principle and the theory of cardinal numbers, we study the number of **F**-roots and determine the **F**-convergence, **F**-divergence, **F**-depth and **F**-stability of any graph G. In particular, we show that:

(i) A graph G is **F**-convergent if and only if G has at most one cycle of length 3.

(ii) The **F**-depth of any graph G different from K_3 and K_1 is finite.

(iii) The **F**-stable graphs are precisely K_3 and K_1.

(iv) A graph that has one **F**-root has innumerably many, but only some **F**-roots are important.

4.1 Preliminaries

For standard notation and terminology in graph theory we follow Diestel (Diestel, 2005) and Prisner (Prisner, 1995).

Some elementary properties of infinite cardinal numbers that we use are (see, e.g., Kamke (Kamke, 1950)):

1. $\alpha + \beta = \alpha.\beta = \max(\alpha, \beta)$ if α, β are cardinal numbers and β is infinite. In particular, $2.\beta = \aleph_0.\beta = \beta$.

2. $\beta^n = \beta$ if β is an infinite cardinal and n is a positive integer.

3. $\beta < 2^\beta$ for every cardinal number.

4. The number of finite subsets of an infinite set of cardinality β is equal to β.

We consider finite and infinite labeled graphs *without multiple edges or loops*. An *isthmus* of a graph G is an edge e such that deleting e divides one component of G into two of $G-e$. Equivalently, an isthmus is an edge that belongs to no cycle. Each isthmus is in every maximal forest, but no non-isthmus is.

Let $\mathfrak{C}(G)$ and $\mathfrak{N}(G)$ denote the set of all possible cycles and the set of all maximal forests of a graph G, respectively. Note that a maximal forest of G consists of a spanning tree in each component of G. A fundamental fact, whose proof is similar to that of the existence of a maximal forest, is the following forest extension lemma:

Lemma 4.1.1. *In any graph G, every forest is contained in a maximal forest.*

Proof. Let G be a graph and F be a forest of G. If F is maximal forest of G we are done. Suppose F is not maximal forest of G. If G is connected, maximal forest is same as spanning tree. Since F is not maximal forest, then F must be acyclic and disconnected. Add the edges from $E(G) \setminus E(F)$ to F such that it remains acyclic and connected, call it as F'. Clearly F' is maximal forest of G. By the above construction it follows that F is contained in F'. If G is disconnected, repeat the above process to each connected component in G, we will get a maximal forest F' which contains F. □

Lemma 4.1.2. *If G is a complete graph of infinite order α, then $|\mathfrak{N}(G)| = 2^{\alpha}$.*

Proof. Let $G = (V,E)$ be a complete graph of order α (α infinite), i.e., $G = K_{\alpha}$. Let v_1, v_2 be two vertices of G and $V' = V \setminus \{v_1, v_2\}$. Then for every $A \subseteq V'$ there is a spanning tree T_A such that every vertex of A is adjacent only to v_1 and every vertex of $V' \setminus A$ is adjacent only to v_2. It is easy to see that $T_A \neq T_B$ whenever $A \neq B$. As the cardinality of the power set of V' is 2^{α}, there are at least 2^{α} spanning trees of G. Since G is connected, the maximal forests are the spanning trees; therefore $|\mathfrak{N}(G)| \geq 2^{\alpha}$. Since the degree of each vertex is α and G contains α vertices, the total number of edges in G is $\alpha.\alpha = \alpha$. The edge set of a maximal forest of G is a subset of E and the number of all possible subsets of E is 2^{α}. Therefore, G has at most 2^{α} maximal forests, i.e., $|\mathfrak{N}(G)| \leq 2^{\alpha}$. Hence $|\mathfrak{N}(G)| = 2^{\alpha}$. □

For two maximal forests of G, F_1 and F_2, let $d(F_1, F_2)$ denote the distance between them in $\mathbf{F}(G)$. We connect this distance to the number of edges by which F_1, F_2 differ; the result is elementary but we could not find it anywhere in the literature. We say F_1, F_2 *differ by l edges* if $|E(F_1) \setminus E(F_2)| = |E(F_2) \setminus E(F_1)| = l$.

Lemma 4.1.3. *Let l be a natural number. For two maximal forests F_1, F_2 of a graph G, if $|E(F_1) \setminus E(F_2)| = l$, then $|E(F_2) \setminus E(F_1)| = l$. Furthermore, F_1 and F_2 differ by exactly l edges if and only if $d(F_1, F_2) = l$.*

Proof. We prove the first part by induction on l. Let F_1, F_2 be maximal forests of G and let $E(F_1) \setminus E(F_2) = \{e'_1, e'_2, \ldots, e'_k\}$, $E(F_2) \setminus E(F_1) = \{e_1, e_2, \ldots, e_l\}$. If $l = 0$ then $k = 0 = l$ because $F_2 = F_1$. Suppose $l > 0$; then $k > 0$ also. Deleting e_l from F_2 divides a tree of F_2 into two trees. Since these trees are in the same component of G, there is an edge of F_1 that connects them; this edge is not e_1 so it is not in F_2; therefore, it is an e'_i, say e'_k. Let $F'_2 = F_2 - e_l + e'_k$. Then $E(F_1) \setminus E(F'_2) = \{e'_1, e'_2, \ldots, e'_{k-1}\}$, $E(F_2) \setminus E(F_1) = \{e_1, e_2, \ldots, e_{l-1}\}$. By induction, $k - 1 = l - 1$.

We also prove the second part by induction on l. Assume F_1, F_2 differ by exactly l edges and define F'_2 as above. If $l = 0, 1$, clearly $d(F_1, F_2) = l$. Suppose $l > 1$. In a shortest path from F_1 to F_2, whose length is $d(F_1, F_2)$, each successive edge of the path can increase the number of edges not in F_1 by at most 1. Therefore, F_1 and F_2 differ by at most $d(F_1, F_2)$ edges. That is, $l \leq d(F_1, F_2)$. Conversely, $d(F_1, F'_2) = l - 1$ by induction and there is a path in $\mathbf{F}(G)$ from F_1 to F'_2 of length $l - 1$, then continuing to F_2 and having total length l. Thus, $d(F_1, F_2) \leq l$. □

Lemma 4.1.4. *For any graph G, $\mathbf{F}(G)$ is connected if and only if any two maximal forests of G differ by at most a finite number of edges.*

Proof. Proof of this Lemma follows by the Lemma 4.1.3 □

Lemma 4.1.5. *If $G = K_\alpha$, α infinite, then $\mathbf{F}(G)$ is disconnected.*

Proof. Proof of this Lemma follows by the Lemma 4.1.3 □

Lemma 4.1.6. *Let G be a graph with α vertices and β edges and with no isolated vertices. If either α or β is infinite, then $\alpha = \beta$.*

Proof. We know that $|E(G)| \leq |V(G)|^2$, i.e., $\beta \leq \alpha^2$ so if β is infinite, α must also be infinite. We also know, since each edge has two endpoints, that $|V(G)| \leq 2|E(G)|$, i.e.,

$\alpha \leq 2.\beta$ so if α is infinite, then β must be infinite. Now assuming both are infinite, $\alpha^2 = \alpha$ and $2.\beta = \beta$, hence $\alpha = \beta$. $\square$

The following lemmas are used to prove **F**-convergence and **F**-divergence in Section 4.4 and **F**-depth in Section 4.5.

Lemma 4.1.7. *Let G be a graph. If K_n (for finite $n \geq 2$) is a subgraph of G, then $K_{\lfloor n^2/4 \rfloor}$ is a subgraph of* $\mathbf{F}(G)$.

Proof. Let G be a graph such that K_n ($n \geq 2$, finite) is a subgraph of G with vertex labels $v_1, v_2, \ldots, v_n$. Then there is a path $L = v_1, v_2, \ldots, v_n$ of order n in G. Let F be a maximal forest of G such that F contains the path L. In F if we replace the edge $v_{\lfloor n/2 \rfloor} v_{\lfloor n/2 \rfloor + 1}$ by any other edge $v_i v_j$ where $i = 1, \ldots, \lfloor n/2 \rfloor$ and $j = \lfloor n/2 \rfloor + 1, \ldots, n$, we get a maximal forest F_{ij}. Since there are $\lfloor n^2/4 \rfloor$ such edges $v_i v_j$, there are $\lfloor n^2/4 \rfloor$ maximal forests F_{ij} (of which one is F). Any two forests F_{ij} differ by one edge. It follows that they form a complete subgraph in $\mathbf{F}(G)$. Therefore $K_{\lfloor n^2/4 \rfloor}$ is a subgraph of $\mathbf{F}(G)$. $\square$

Lemma 4.1.8. *If G has a cycle of (finite) length n with $n \geq 3$, then $\mathbf{F}(G)$ contains K_n.*

Proof. Suppose that G has a cycle C_n of length n with edge set $\{e_1, e_2, \ldots, e_n\}$. Let $P_i = C_n - e_i$ for $i = 1, 2, \ldots, n$ and let F_1 be a maximal forest of G containing the path P_1. Define $F_i = F_1 \setminus P_1 \cup P_i$ for $i = 2, 3, \ldots, n$. These F_i's are maximal forests of G and any two of them differ by exactly one edge, so they form a complete graph K_n in $\mathbf{F}(G)$. $\square$

In particular, $\mathbf{F}(C_n) = K_n$.

Lemma 4.1.9. *Suppose that G contains K_n, where $n \geq 3$. Then $\mathbf{F}^2(G)$ contains $K_{n^{n-2}}$.*

Proof. Cayley's formula states that K_n has n^{n-2} spanning trees. Cummins (Cummins, 1966) proved that the tree graph of a finite connected graph is Hamiltonian. Therefore, $\mathbf{F}(K_n)$ contains $C_{n^{n-2}}$. Let F_{T_0} be a spanning tree of G that extends one of the spanning trees T_0 of the K_n subgraph. Replacing the edges of T_0 in F_{T_0} by the edges of any other spanning tree T of K_n, we have a spanning tree F_T that contains T. The F_T's for all spanning trees T of K_n are n^{n-2} spanning trees of G that differ only within K_n; thus, the graph of the F_T's is the same as the graph of the T's, which is Hamiltonian. That is, $\mathbf{F}(G)$ contains $C_{n^{n-2}}$. By Lemma 4.1.8, $\mathbf{F}^2(G)$ contains $K_{n^{n-2}}$. □

Lemma 4.1.10. *If G has two edge disjoint triangles, then* $\mathbf{F}^2(G)$ *contains* K_9.

Proof. Suppose that G has two edge disjoint triangles whose edges are e_1, e_2, e_3 and f_1, f_2, f_3, respectively. The union of the triangles has exactly 9 maximal forests F'_{ij}, obtained by deleting one e_i and one f_j from the triangles. Extend F'_{11} to a maximal forest F_{11} and let F_{ij} be the maximal forest $F_{11} \setminus E(F'_{11}) \cup F_{ij}$, for each $i, j = 1,2,3$. The nine maximal forests F'_{ij}, and consequently the maximal forests F_{ij} in $\mathbf{F}(G)$, form a Cartesian product graph $C_3 \times C_3$, which contains a cycle of length 9. By Lemma 4.1.8, $\mathbf{F}^2(G)$ contains K_9. □

We now show that repeated application of the forest graph operator to many graphs creates larger and larger complete subgraphs.

Lemma 4.1.11. *If G has a cycle of (finite) length n with* $n \geq 4$ *or it has two edge disjoint triangles, then for any finite* $m \geq 1$, $\mathbf{F}^m(G)$ *contains* K_{m^2}.

Proof. We prove this lemma by induction on m.

Case 1: Suppose that G has a cycle C_n of length n ($n \geq 4$, n finite). By Lemma 4.1.8, $\mathbf{F}(G)$ contains K_n as a subgraph, which implies that $\mathbf{F}(G)$ contains K_4. By Lemma 4.1.9, $\mathbf{F}^3(G)$ contains K_{16} and in particular it contains K_{3^2}.

Case 2: Suppose that G has two edge disjoint triangles. By Lemma 4.1.10 $\mathbf{F}^2(G)$ contains K_9 as a subgraph. It follows by Lemma 4.1.7 that $\mathbf{F}^3(G)$ contains $K_{\lfloor 9^2/4 \rfloor} = K_{20}$ as a subgraph. This implies that $\mathbf{F}^3(G)$ contains K_{3^2} as a subgraph.

By Cases 1 and 2 it follows that the result is true for $m = 1,2,3$. Let us assume that the result is true for $m = l \geq 3$, i.e., that $\mathbf{F}^l(G)$ contains K_{l^2} as a subgraph. By Lemma 4.1.7 it follows that $\mathbf{F}(\mathbf{F}^l(G))$ has a subgraph $K_{\lfloor l^4/4 \rfloor}$. Since $\lfloor l^4/4 \rfloor > (l+1)^2$, it follows that $\mathbf{F}^{l+1}(G)$ contains $K_{(l+1)^2}$. By the induction hypothesis $\mathbf{F}^m(G)$ contains K_{m^2} for any finite $m \geq 1$. $\square$

With Lemma 4.1.9 it is clearly possible to prove a much stronger lower bound on complete subgraphs of iterated forest graphs, but Lemma 4.1.11 is good enough for our purposes.

Lemma 4.1.12. *A forest graph that is not K_1 has no isolated vertices and no isthmi.*

Proof. Let $G = \mathbf{F}(H)$ for some graph H. Consider a vertex F of G, that is, a maximal forest in H. Let e be an edge of F that belongs to a cycle C in H. Then there is an edge f in C that is not in F and $F' = F - e + f$ is a second maximal forest that is adjacent to F in G. Since C has length at least 3, it has a third edge g. If g is not in F, let $F'' = F - e + g$. If g is in F, let $F'' = F - g + f$. In both cases F'' is a maximal forest that is adjacent to F and F'. Thus, F is not isolated and the edge FF' in G is not an isthmus.

Suppose $F, F' \in \mathfrak{N}(H)$ are adjacent in G. That means there are edges $e \in E(F)$ and $e' \in E(F')$ such that $F' = F - e + e'$. Thus, e belongs to the unique cycle in $F + e'$. As shown above, there is an $F'' \in \mathfrak{N}(H)$ that forms a cycle with F and F'. Therefore the edge FF' of G is not an isthmus.

Let $F \in \mathfrak{N}(H)$ be an isolated vertex in G. If H has an edge e not in F, then $F + e$ contains a cycle so F has a neighboring vertex in G, as shown above. Therefore, no such e can exist; in other words, $H = F$ and G is K_1. $\square$

4.2 Basic Properties of an Infinite Forest Graph

We now present a crucial foundation for the proof of the main theorem in Section 4.4. The *cyclomatic number* $\beta_1(G)$ of a graph G can be defined as the cardinality $|E(G) \setminus E(F)|$ where F is a maximal forest of G.

Proposition 4.2.1. *Let G be a graph such that $|\mathfrak{C}(G)| = \beta$, an infinite cardinal number. Then:*

(i) $\beta_1(G) = \beta$ *and* $\beta_1(\mathbf{F}(G)) = 2^\beta$.

(ii) *Both the order of* $\mathbf{F}(G)$ *and its number of edges equal* 2^β. *Both the order and the number of edges of G equal* β, *provided that G has no isolated vertices and no isthmi.*

(iii) $\mathbf{F}(G)$ *is* β*-regular.*

(iv) *The order of any connected component of* $\mathbf{F}(G)$ *is* β, *and it has exactly* β *edges.*

(v) $\mathbf{F}(G)$ *has exactly* 2^β *components.*

(vi) *Every component of* $\mathbf{F}(G)$ *has exactly* β *cycles.*

(vii) $|\mathfrak{C}(\mathbf{F}(G))| = 2^\beta$.

Proof. Let G be a graph with $|\mathfrak{C}(G)| = \beta$ (β infinite).

(i) Let F be a maximal forest of G. The number of cycles in G is not more than the number of finite subsets of $E(G) \setminus E(F)$. This number is finite if $E(G) \setminus E(F)$ is finite, but it cannot be finite because $|\mathfrak{C}(G)|$ is infinite. Therefore $E(G) \setminus E(F)$ is infinite and the number of its finite subsets equals $|E(G) \setminus E(F)| = \beta_1(G)$. Thus, $\beta_1(G) \geq |\mathfrak{C}(G)|$. The number of cycles is at least as large as the number of edges not in F, because every such edge makes a different cycle with F. Thus, $|\mathfrak{C}(G)| \geq \beta_1(G)$. It follows that $\beta_1(G) = |\mathfrak{C}(G)| = \beta$. Note that this proves $\beta_1(G)$ does not depend on the choice of F.

The value of $\beta_1(\mathbf{F}(G))$ follows from this and part (vii).

(ii) For the first part, let F be a maximal forest of G and let F_0 be a maximal forest of $G \setminus E(F)$. As $G \setminus E(F)$ has $\beta_1(G) = \beta$ edges by part (i), it has β non-isolated vertices by Lemma 4.1.6. F_0 has the same non-isolated vertices, so it too has β edges.

Any edge set $A \subseteq F_0$ extends to a maximal forest F_A in $F \cup A$. Since $F_A \setminus F = A$, the F_A's are distinct. Therefore, there are at least 2^β maximal forests in $F_0 \cup F$. The maximal forest F consists of a spanning tree in each component of G; therefore, the vertex sets of components of F are the same as those of G, and so are those of $F_0 \cup F$. Therefore, a maximal forest in $F_0 \cup F$, which consists of a spanning tree in each component of $F_0 \cup F$, contains a spanning tree of each component of G.

We conclude that a maximal forest in $F_0 \cup F$ is a maximal forest of G and hence that there are at least 2^β maximal forests in G, i.e., $|\mathfrak{N}(G)| \geq 2^\beta$. Since G is a subgraph of K_β, and since $|\mathfrak{N}(K_\beta)| = 2^\beta$ by Lemma 4.1.2, we have $|\mathfrak{N}(G)| \leq 2^\beta$. Therefore $|\mathfrak{N}(G)| = 2^\beta$. That is, the order of $\mathbf{F}(G)$ is 2^β. By Lemmas 4.1.12 and 4.1.6, that is also the number of edges of $\mathbf{F}(G)$.

For the second part, note that G has infinite order or else $\beta_1(G)$ would be finite. If G has no isolated vertices and no isthmi, then $|V(G)| = |E(G)|$ by Lemma 4.1.6. By part (i) there are β edges of G outside a maximal forest; hence $\beta \leq |E(G)|$.

Since every edge of G is in a cycle, by the axiom of choice we can choose a cycle $C(e)$ containing e for each edge e of G. Let $\mathfrak{C} = \{C(e) : e \in E(G)\}$. The total number of pairs (f,C) such that $f \in C \in \mathfrak{C}$ is no more than $\aleph_0 \cdot |\mathfrak{C}| \leq \aleph_0 \cdot |\mathfrak{C}(G)| = \aleph_0 \cdot \beta = \beta$. This number of pairs is not less than the number of edges, so $|E(G)| \leq \beta$. It follows that G has exactly β edges.

(iii) Let F be a maximal forest of G. By part (i), $|E(G) \setminus E(F)| = \beta$. By adding any edge e from $E(G) \setminus E(F)$ to F we get a cycle C. Removing any edge other than e from the cycle C gives a new maximal forest which differs by exactly one edge with F. The number of maximal forests we get in this way is $\beta_1(G)$ because there are $\beta_1(G)$ ways to choose e and a finite number of edges of C to choose to remove, and $\beta_1(G)$ is infinite. Thus we get β maximal forests of G, each of which differs by exactly one edge with F. Every such maximal forest is generated by this construction. Therefore, the degree of any vertex in $\mathbf{F}(G)$ is β.

(iv) Let A be a connected component of $\mathbf{F}(G)$. As $\mathbf{F}(G)$ is β-regular by part (iii), it follows that $|V(A)| \geq \beta$. Fix a vertex v in A and define the n^{th} neighborhood $D_n = \{v' : d(v,v') = n\}$ for each n in $\mathbb{N}$. Since every vertex has degree β, $|D_0| = 1$, $|D_1| = \beta$ and $|D_k| \leq \beta |D_{k-1}|$. Thus, by induction on n, $|D_n| \leq \beta$ for $n > 0$.

Since A is connected, it follows that $V(A) = \bigcup_{i \in \mathbb{N} \cup \{0\}} D_i$, i.e., $V(A)$ is the countable union of sets of order β. Therefore $|A| = \beta$, as $|\mathbb{N}|.\beta' = \beta'$. Hence any connected component of $\mathbf{F}(G)$ has β vertices. By Lemma 4.1.6 it has β edges.

(v) By parts (ii, iv) the order of $\mathbf{F}(G)$ is 2^β and the order of each component of $\mathbf{F}(G)$ is β. Since $|\mathbf{F}(G)| = 2^\beta$, $\mathbf{F}(G)$ has at most 2^β components. Suppose that $\mathbf{F}(G)$ has β' components where $\beta' < 2^\beta$. As each component has β vertices, it follows that $\mathbf{F}(G)$ has order at most $\beta'.\beta = \max\{\beta', \beta\}$. This is a contradiction to part (ii). Therefore $\mathbf{F}(G)$ has exactly 2^β components.

(vi) Let A be a component of $\mathbf{F}(G)$. Since it is infinite, by part (iv) it has exactly β edges. Suppose that $|\mathfrak{C}(A)| = \beta'$. Then β' is at most the number of finite subsets of $E(A)$, which is β since $|E(A)| = \beta$ is infinite; that is, $\beta' \leq \beta$. By the argument in part (iii) every edge of $\mathbf{F}(G)$ lies on a cycle. The length of each cycle is finite. Thus A has at most $\aleph_0.\beta' = \max\{\beta', \aleph_0\} = \beta'$ edges if β' is infinite and it has a finite number of edges if β' is finite. Since $|E(A)| = \beta$, which is infinite, $\beta' \geq \beta$. We conclude that $\beta' = \beta$.

(vii) By parts (v, vi) $\mathbf{F}(G)$ has 2^β components and each component has β cycles. Since every cycle is contained in a component, $|\mathfrak{C}(\mathbf{F}(G))| = \beta.2^\beta = 2^\beta$. □

From the above proposition it follows that an infinite graph cannot be a forest graph unless every component has the same infinite order β and there are 2^β components. A consequence is that the infinite graph itself must have order 2^β. Hence,

Lemma 4.2.2. *Any infinite graph whose order is not a power of* 2*, including* $\aleph_0$ *and all other limit cardinals, is not a forest graph.*

Lemma 4.2.3. *For a graph G the following statements are equivalent.*

i) $\mathbf{F}(G)$ *is connected.*

ii) $\mathbf{F}(G)$ *is finite.*

iii) *The union of all cycles in G is a finite graph.*

Proof. (i) $\Longrightarrow$ (iii). Suppose that $\mathbf{F}(G)$ is connected. If G has infinitely many cycles then by Proposition 4.2.1(v) $\mathbf{F}(G)$ is disconnected. Therefore G has finitely many cycles. Let $A = \{e \in E(G) : \text{edge } e \text{ lies on a cycle in } G\}$. Then $|A|$ is finite because the length of each cycle is finite. That proves (iii).

(iii) $\Longrightarrow$ (ii). As every maximal forest of G consists of a maximal forest of A and all the edges of G which are not in A, G has at most 2^n maximal forests where $n = |A|$. Hence $\mathbf{F}(G)$ has a finite number of vertices and consequently is finite.

(ii) $\Longrightarrow$ (i). By identifying vertices in different components (Whitney vertex identification; see Section 4.3) we can assume G is connected so $\mathbf{F}(G) = \mathbf{T}(G)$. Cummins (Cummins, 1966) proved that the tree graph of a finite graph is Hamiltonian; therefore it is connected. □

4.3 F-Roots

In this section we establish properties of **F**-roots of graphs. We begin with the question of what an **F**-root should be.

Since any graph H' that is isomorphic to an **F**-root H of G is immediately also an **F**-root, the number of non-isomorphic **F**-roots is a better question than the number of labeled **F**-roots. We now show in some detail that a still better question is the number of non-isomorphic **F**-roots without isthmi.

Let t_β be the number of non-isomorphic rooted trees of order β. We note that $t_{\aleph_0} \geq 2^{\aleph_0}$, by a construction of Reinhard Diestel (personal communication, July 10, 2015). (We do not know a corresponding lower bound on t_β for $\beta > \aleph_0$.) Let P be a one-way infinite path whose vertices are labelled by natural numbers, with root 1; choose any subset S of $\mathbb{N}$ and attach two edges at every vertex in S, forming a rooted tree T_S (rooted at 1). Then S is determined by T_S because the vertices in S are those of degree at least 3 in T_S. (If $2 \in S$ but $1 \notin S$, then vertex 1 is determined only up to isomorphism by T_S, but S itself is determined uniquely.) The number of sets S is $2^{\aleph_0}$, hence $t_{\aleph_0} \geq 2^{\aleph_0}$.

Proposition 4.3.1. *Let G be a graph with an* **F***-root of order α. If α is finite, then G*

has infinitely many non-isomorphic finite $\mathbf{F}$*-roots. If* α *is finite or infinite, then G has at least* t_β *non-isomorphic* $\mathbf{F}$*-roots of order* β *for every infinite* $\beta \geq \alpha$*.*

Proof. Let G be a graph which has an $\mathbf{F}$-root H, i.e., $\mathbf{F}(H) \cong G$, and let α be the order of H. We may assume H has no isthmi and no isolated vertices unless it is K_1.

Suppose α is finite; then let T be a tree, disjoint from H, of any finite order n. Identify any vertex v of H with any vertex w of T. The resulting graph H_T also has G as its forest graph since T is contained in every maximal forest of H_T. As the order of H_T is $\alpha + n - 1$ and n can be any natural number, the graphs H_T are an infinite number of non-isomorphic finite graphs with the same forest graph up to isomorphism.

Suppose α is finite or infinite and $\beta \geq \alpha$ is infinite. Let T be a rooted tree of order β with root vertex w; for instance, T can be a star rooted at the star center. Attach T to a vertex v of H by identifying v with the root vertex w. Denote the resulting graph by H_T; it is an $\mathbf{F}$-root of G and it has order β because it has order $\alpha + \beta$, which equals β because β is infinite and $\beta \geq \alpha$. As H has no isthmi, T and w are determined by H_T; therefore, if we have a non-isomorphic rooted tree T' with root w' (that means there is no isomorphism of T with T' in which w corresponds to w'), $H_{T'}$ is not isomorphic to H_T. (The one exception is when $H = K_1$, which is easy to treat separately.) The number of non-isomorphic $\mathbf{F}$-roots of G of order β is therefore at least the number of non-isomorphic rooted trees of order β, i.e., t_β. □

Proposition 4.3.1 still does not capture the essence of the number of $\mathbf{F}$-roots. Whitney's *2-operations* on a graph G are the following (Whitney, 1933):

1. *Whitney vertex identification.* Identify a vertex in one component of G with a vertex in a another component of G, thereby reducing the number of components by 1. For an infinite graph we modify this by allowing an infinite number of vertex identifications; specifically, let W be a set of vertices with at most one from each component of G, and let $\{W_i : i \in I\}$ be a partition of W into $|I|$ sets (where I is any index set); then for each $i \in I$ we identify all the vertices in W_i with each other.

2. *Whitney vertex splitting.* The reverse of vertex identification.

3. *Whitney twist.* If u, v are two vertices that separate G—that is, $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \{u, v\}$ and $|V(G_1)|, |V(G_2)| > 2$, then reverse the names u and v in G_2 and then take the union $G_1 \cup G_2$ (so vertex u in G_1 is identified with the former vertex v in G_2 and v with the former vertex u). Call the new graph G'. For an infinite graph we allow an infinite number of Whitney twists.

It is easy to see that the edge sets of maximal forests in G and G' are identical, hence $\mathbf{F}(G)$ and $\mathbf{F}(G')$ are naturally isomorphic. It follows by Whitney vertex identification that every graph with an **F**-root has a connected **F**-root, and it follows from Whitney vertex splitting that every graph with an F-root has an **F**-root without cut vertices.

We may conclude from Proposition 4.3.1 that the most interesting question about the number of **F**-roots of a graph G that has an **F**-root is not the total number of non-isomorphic **F**-roots (which by Proposition 4.3.1 cannot be assigned any cardinality); it is not the number of a given order; it is not even the number that have no isthmi; it is the number of non-2-isomorphic, connected **F**-roots with no isthmi and (except when $G = K_1$) no isolated vertices.

We do not know which graphs have **F**-roots, but we do know two large classes that cannot have **F**-roots.

Theorem 4.3.2. *No infinite connected graph has an* **F**-*root.*

Proof. This follows by Lemma 4.2.3. □

Theorem 4.3.3. *No bipartite graph G has an* **F**-*root.*

Proof. Let G be a bipartite graph of order p $(p \geq 2)$ and let H be a root of G, i.e., $\mathbf{F}(H) \cong G$. Suppose H has no cycle; then $\mathbf{F}(H)$ is K_1, which is a contradiction. Therefore H has a cycle of length ≥ 3. It follows by Lemma 4.1.8 that $\mathbf{F}(H)$ contains K_3, a contradiction. Hence no bipartite graph G has a root. □

4.4 F-Convergence and F-Divergence

In this section we establish the necessary and sufficient conditions for **F**-convergence of a graph.

Lemma 4.4.1. *Let G be a finite graph that contains a C_n (for $n \geq 4$) or at least two edge disjoint triangles; then G is* **F***-divergent.*

Proof. Let G be a finite graph. By Lemma 4.1.11, $\mathbf{F}^m(G)$ contains K_{m^2} as a subgraph. Therefore, as m increases the clique size of $\mathbf{F}^m(G)$ increases. Hence G is **F**-divergent. □

Lemma 4.4.2. *If $|\mathfrak{C}(G)| = \beta$ where β is infinite, then G is* **F***-divergent.*

Proof. Assume $|\mathfrak{C}(G)| = \beta$ (β infinite). By Proposition 4.2.1(vii), as $2^\beta < 2^{2^\beta} < 2^{2^{2^\beta}} < \cdots$, it follows that $|\mathfrak{C}(\mathbf{F}(G))| < |\mathfrak{C}(\mathbf{F}^2(G))| < |\mathfrak{C}(\mathbf{F}^3(G))| < \cdots$. Therefore, as n increases $|\mathfrak{C}(\mathbf{F}^n(G))|$ increases. Hence G is **F**-divergent. □

Theorem 4.4.3. *Let G be a graph. Then,*

i) *G is* **F***-convergent if and only if either G is acyclic or G has only one cycle, which is of length* 3.

ii) *If G is* **F***-convergent, then it converges in at most two steps.*

Proof. i) If G has no cycle, then it is a forest and $\mathbf{F}(G)$ is K_1. If G has only one cycle and that cycle has length 3, then $\mathbf{F}(G)$ is K_3. Therefore in each case G is **F**-convergent.

Conversely, suppose that G has a cycle of length greater than 3 or has at least two triangles. If G has infinitely many cycles, then it follows by Lemma 4.4.2 that G is **F**-divergent. Therefore we may assume that G has a finite number of cycles. If G has a finite number of vertices, then it is finite and by Lemma 4.4.1 it is **F**-divergent. Therefore G has an infinite number of vertices. However, it can have only a finite number of edges that are not isthmi, because each cycle is finite. Thus G consists of a finite graph G_0 and any number of isthmi and isolated vertices. Since $\mathbf{F}(G)$ depends only on the edges that are not isthmi and the vertices that are not isolated, $\mathbf{F}(G) = \mathbf{F}(G_0)$ (under the natural identification of maximal forests in G_0 with their extensions in G by adding all isthmi of G). Therefore, G is $\mathbf{F}$-divergent.

ii) If G has no cycle, then G is a forest and $\mathbf{F}(G) \cong \mathbf{F}^2(G) \cong K_1$. If G has only one cycle, which is of length 3, then $\mathbf{F}(G) \cong \mathbf{F}^2(G) \cong K_3$. Therefore G converges in at most 2 steps. □

Corollary 4.4.4. *A graph G is* $\mathbf{F}$*-stable if and only if $G = K_1$ or K_3.*

4.5 F-Depth

In this section we establish results about the $\mathbf{F}$-depth of a graph.

Theorem 4.5.1. *Let G be a finite graph. The* $\mathbf{F}$*-depth of G is infinite if and only if G is K_1 or K_3.*

Proof. Let G be a finite graph. Suppose that G is K_1 or K_3. Then by Corollary 4.4.4, it follows that G is $\mathbf{F}$-stable. Therefore, the $\mathbf{F}$-depth of G is infinite.

Conversely, suppose that G is different from K_1 and K_3.

Case 1: Let $|V| < 4$. Then G has no $\mathbf{F}$-root so its $\mathbf{F}$-depth is zero.

Case 2: Let $|V| = 4$. Suppose G has an $\mathbf{F}$-root H (i.e., $\mathbf{F}(H) \cong G$). Then H should have exactly 4 maximal forests. That is possible only when H has only one cycle, which is of length 4. By Lemma 4.1.8 it follows that $\mathbf{F}(H)$ contains K_4, hence it is K_4. Therefore G has an $\mathbf{F}$-root if and only if it is K_4. Hence the $\mathbf{F}$-depth of G is zero, except that the depth of K_4 is 1.

Case 3: Let $|V| = n$ where $n > 4$. Suppose that G has infinite $\mathbf{F}$-depth. Then for every m there is a graph H_m such that $\mathbf{F}^m(H_m) = G$. If H_m does not have two triangles or a cycle of length greater than 3, then H_m has only one cycle which is of length 3, or no cycle and H_m converges to K_1 or K_3 in at most two steps, a contradiction. Therefore H_m has two triangles or a cycle of length greater than 3. By Lemma 4.1.11 it follows that $\mathbf{F}^m(H_m)$ contains K_{m^2} for each $m \geq 2$, so that in particular $\mathbf{F}^n(H_n)$ contains K_{n^2}.

That is, G contains K_{n^2}. This is impossible as G has order n. Hence the **F**-depth of G is finite. □

Theorem 4.5.2. *The* **F***-depth of any infinite graph is finite.*

Proof. Let G be a graph of infinite order α. If G has an **F**-root, then G is without isthmi or isolated vertices.

If G is connected, Theorem 4.3.2 implies that G has no root. Therefore its **F**-depth is zero.

If G is disconnected, assume it has infinite depth. Then for each natural number n there exists a graph H_n such that $G \cong \mathbf{F}^n(H_n)$. Let β_n denote the order of H_n. Since $\mathbf{F}(H_1) \cong G$, by Proposition 4.2.1(ii) $\alpha = 2^{\beta_1}$, from which we infer that $\beta_1 < \alpha$. This is independent of which root H_1 is, so in particular we can take $H_1 = \mathbf{F}(H_2)$ and conclude that $\beta_1 = 2^{\beta_2}$, hence that $\beta_2 < \beta_1$. Continuing in like manner we get an infinite decreasing sequence of cardinal numbers starting with α. The cardinal numbers are well ordered (Kamke, 1950), so they cannot contain such an infinite sequence. It follows that the **F**-depth of G must be finite. □

Chapter 5

CONCLUSION

This thesis provides a method for assigning colors to the graphs satisfying the hypothesis of the Erdös - Faber - Lovász conjecture. Also, it contains the results on iterated forest graphs and clique graphs.

We gave a method to construct H_n, then assign colors to the graph H_n using the symmetric Latin Squares and also gave two different approaches for assigning colors to the graphs satisfying the hypothesis of the Erdös - Faber - Lovász conjecture. One is using Symmetric Latin Squares and second one is using intersection matrix. Also, we gave theoritical proof of the conjecture for some class of graphs.

We provided a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1 + G_2$, gave a partial characterization for clique divergence of the join of graphs and proved that if G_1, G_2 are Clique-Helly graphs different from K_1 and $G = G_1 \Box G_2$, then $K^2(G) = G$. Further, one can extend these results to obtain when $G_1 + G_2$ is K-convergent.

We defined the "forest graph" $\mathbf{F}(G)$ of a graph G. Using the theory of cardinal numbers, Zorn's lemma, transfinite induction, the axiom of choice and the well-ordering principle, we established the results on the number of **F**-roots and determined the **F**-convergence, **F**-divergence, **F**-depth and **F**-stability of any graph G. In particular it is shown that a graph G (finite or infinite) is **F**-convergent if and only if G has at most one cycle of length 3. The **F**-stable graphs are precisely K_3 and K_1. The **F**-depth of any graph G different from K_3 and K_1 is finite. In future work one can characterise graphs using **F**-root.

Bibliography

"Journal of Combinatorial Designs website". http://www.emba.uvm.edu/~jcd/reports/426/index.html.

Acharya, B. D. (1979). "Some queries on the periodicity and convergence of a graph". In *Professor P. L. Bhatnagar commemoration volume*, 185–205. Nat. Acad. Sci., Allahabad.

Alcón, L., Faria, L., de Figueiredo, C. M. H., and Gutierrez, M. (2009). "The complexity of clique graph recognition". *Theoret. Comput. Sci.*, 410(21-23), 2072–2083.

Berge, C. (1990). "On two conjectures to generalize Vizing's theorem". *Matematiche (Catania)*, 45(1), 15–23 (1991). Graphs, designs and combinatorial geometries (Catania, 1989).

Broersma, H. and Xueliang, L. (1996). "The connectivity of the leaf-exchange spanning tree graph of a graph". *Ars Combinatoria*, 43, 225–231.

Bryant, D. and Rodger, C. (2004). "On the completion of Latin rectangles to symmetric Latin squares". *Journal of the Australian Mathematical Society*, 76(1), 109–124.

Chang, W. I. and Lawler, E. L. (1988). "Edge coloring of hypergraphs and a conjecture of ErdÖs, Faber, Lovász". *Combinatorica*, 8(3), 293–295.

Chen, W.-K. (1967). "Hamilton circuits in directed-tree graphs". *Circuit Theory, IEEE Transactions on*, 14(2), 231–233.

Cummins, R. L. (1966). "Hamilton circuits in tree graphs". *Circuit Theory, IEEE Transactions on*, CT-13, 82–90.

Das, K. C. (2007). "A sharp upper bound for the number of spanning trees of a graph". *Graphs and Combinatorics*, 23(6), 625–632.

Das, K. C., Cevik, A. S., and Cangul, I. N. (2013). "The number of spanning trees of a graph". *Journal of Inequalities and Applications*, 2013(1), 1–13.

Deza, M., Erdös, P., and Frankl, P. (1978). "Intersection properties of systems of finite sets". *Proceedings of the London Mathematical Society*, 3(2), 369–384.

Diestel, R. (2005). *"Graph theory (Graduate texts in mathematics)"*. Springer Heidelberg.

Erdős, P. (1975). "Problems and results in graph theory and combinatorial analysis". *Proc. British Combinatorial Conj., 5th*, 169–192.

Erdős, P. (1981). "On the combinatorial problems which I would most like to see solved". *Combinatorica*, 1(1), 25–42.

Faber, V. (2010). "The Erdős-Faber-Lovász conjecture—the uniform regular case". *J. Comb.*, 1(2), 113–120.

Feng, L., Guihai, Y., Jiang, Z., and Ren, L. (2008). "Sharp upper bounds for the number of spanning trees of a graph". *Applicable Analysis and Discrete Mathematics*, 2(2), 255–259.

Feng, L., Xu, K., Das, K. C., Ilić, A., and Yu, G. (2014). "The number of spanning trees of a graph with given matching number". *International Journal of Computer Mathematics*, (ahead-of-print), 1–7.

Frías-Armenta, M. E., Larrión, F., Neumann-Lara, V., and Pizaña, M. A. (2005). "Local cutpoints and iterated clique graphs". In *Proceedings of GRACO2005*, 19 of *Electron. Notes Discrete Math.*, 345–349 (electronic), Amsterdam. Elsevier.

Frías-Armenta, M. E., Larrión, F., Neumann-Lara, V., and Pizaña, M. A. (2013). "Edge contraction and edge removal on iterated clique graphs". *Discrete Appl. Math.*, 161(10-11), 1427–1439.

Grannell, M., Griggs, T., and Whitehead, C. (2000). "The resolution of the anti-Pasch conjecture". *Journal of Combinatorial Designs*, 8(4), 300–309.

Gravier, S., Protti, F., and Sales, C. L. (2004). "On Clique-inverse graphs of K_p-free graphs". *Electronic Notes in Discrete Mathematics*, 18, 139–143.

Grimmett, G. (1976). "An upper bound for the number of spanning trees of a graph". *Discrete Mathematics*, 16(4), 323–324.

Haddad, L. and Tardif, C. (2004). "A clone-theoretic formulation of the Erdös-Faber-Lovász conjecture". *Discussiones Mathematicae Graph Theory*, 24(3), 545–549.

Hamelink, R. C. (1968). "A partial characterization of clique graphs". *J. Combinatorial Theory*, 5, 192–197.

Harary, F. (1969). *"Graph Theory"*. Addison-Wesley Publishing Company, Inc., California, London.

Harary, F. and Norman, R. Z. (1960). "Some properties of line digraphs". *Rendiconti del Circolo Matematico di Palermo*, 9(2), 161–168.

Hedetniemi, S. and Slater, P. (1972). "Line graphs of triangleless graphs and iterated clique graphs". In *Graph theory and Applications*, 139–147. Springer.

Hedman, B. (1984). "Clique graphs of time graphs". *J. Combin. Theory Ser. B*, 37(3), 270–278.

Hedman, B. (1986). "A polynomial algorithm for constructing the clique graph of a line graph". *Discrete Appl. Math.*, 15(1), 61–66.

Imrich, W., Klavzar, S., and Rall, D. F. (2008). *"Topics in graph theory: Graphs and their Cartesian product"*. AK Peters Ltd.

Jackson, B., Sethuraman, G., and Whitehead, C. (2007). "A note on the Erdős-Farber-Lovász conjecture". *Discrete Math.*, 307(7-8), 911–915.

Jensen, T. R. and Toft, B. (2011). *"Graph coloring problems"*, 39. John Wiley & Sons.

Kahn, J. (1992). "Coloring nearly-disjoint hypergraphs with $n+o(n)$ colors". *Journal of Combinatorial Theory, Series A*, 59(1), 31–39.

Kamke, E. (1950). *"Theory of sets"*. Courier Corporation.

Kishi, G. and Kajitani, Y. (1968). "On Hamilton circuits in tree graphs". *Circuit Theory, IEEE Transactions on*, 15(1), 42–50.

Krausz, J. (1943). "Démonstration nouvelle dâĂŹune théorème de Whitney sur les réseaux". *Mat. Fiz. Lapok*, 50(75-85), 11.

Kundu, S., Sampathkumar, E., Shearer, J., and Sturtevant, D. (1980). "Reconstruction of a pair of graphs from their concatenations". *SIAM Journal on Algebraic Discrete Methods*, 1(2), 228–231.

Laywine, C. F. and Mullen, G. L. (1998). *"Discrete mathematics using Latin squares"*, 17. Wiley New York.

Li, J., Shiu, W. C., and Chang, A. (2010). "The number of spanning trees of a graph". *Applied Mathematics Letters*, 23(3), 286–290.

Lim, C. K. (1982). "A result on iterated clique graphs". *J. Austral. Math. Soc. Ser. A*, 32(3), 289–294.

Liu, G. (1988). "On connectivities of tree graphs". *Journal of graph theory*, 12(3), 453–459.

Mayeda, W. and Seshu, S. (1965). "Generation of trees without duplications". *Circuit Theory, IEEE Transactions on*, 12(2), 181–185.

Mitchem, J. and Schmidt, R. L. (2010). "On the Erdős-Faber-Lovász conjecture". *Ars Combin.*, 97, 497–505.

Neumann-Lara, V. (1978). "On clique-divergent graphs". *Problems Combinatoires et Théorie des Graphes, Colloques internationaux du CNRS, Paris*, 260, 313–315.

Ore, O. (1962). "Theory of Graphs". In *American Mathematical Society Colloquium Publications*, 38.

Paul, V. and Germina, K. A. (2012). "On edge coloring of hypergraphs and Erdös-Faber-Lovász conjecture". *Discrete Math. Algorithms Appl.*, 4(1), 1250003, 5.

Prisner, E. (1995). *"Graph dynamics"*, 338. CRC Press.

Protti, F. and Szwarcfiter, J. L. (2000). "Clique-inverse graphs of K_3-free and K_4-free graphs". *J. Graph Theory*, 35(4), 257–272.

Roberts, F. S. and Spencer, J. H. (1971). "A characterization of clique graphs". *J. Combinatorial Theory Ser. B*, 10, 102–108.

Rodriguez, J. and Petingi, L. (1997). "A sharp upper bound for the number of spanning trees of a graph". *Congressus Numerantium*, 209–217.

Romero, D. and Alonso-Pecina, F. (2014). "The Erdős–Faber–Lovász conjecture is true for nâL'd' 12". *Discrete Mathematics, Algorithms and Applications*, 6(03), 1450039.

Romero, D. and Sánchez-Arroyo, A. (2007). "Adding evidence to the Erdős-Faber-Lovász conjecture". *Ars Combin.*, 85, 71–84.

Sánchez-Arroyo, A. (2008). "The Erdős-Faber-Lovász conjecture for dense hypergraphs". *Discrete Math.*, 308(5-6), 991–992.

Shank, H. (1968). "A note on hamilton circuits in tree graphs". *Circuit Theory, IEEE Transactions on*, 15(1), 86–86.

Suresh, D. V. V. P. R. V. B., Venkateshwarlu, D., and Rao, S. B. (2010). "Dynamics of spanning tree graph operator". In *International Congress of Mathematicians ICM 2010 Short Communications Abstracts Book*, 472–473. Hindustan Book Agency.

Szwarcfiter, J. L. (2003). "A survey on clique graphs". In *Recent advances in algorithms and combinatorics*, 109–136. Springer.

Teranishi, Y. (2005). "The number of spanning forests of a graph". *Discrete mathematics*, 290(2), 259–267.

West, D. B. et al. (2001). *"Introduction to graph theory"*, 2. Prentice hall Upper Saddle River.

Whitney, H. (1932). "Congruent graphs and the connectivity of graphs". *American Journal of Mathematics*, 54(1), 150–168.

Whitney, H. (1933). "2-isomorphic graphs". *American Journal of Mathematics*, 245–254.

Ye, X. and Xu, Y. (2011). "On the number of symmetric Latin squares". In *Computer Science and Service System (CSSS), 2011 International Conference on*, 2366–2369. IEEE.

Zhang, F. and Chen, Z. (1986). "Connectivity of (adjacency) tree graphs". *J. Xinjiang Univ. Natur. Sci*, 3(4), 1–5.

LIST OF PUBLICATIONS /COMMUNICATIONS BASED ON THESIS:

- Suresh Dara, S. M. Hegde, Venkateshwarlu Deva, S. B. Rao and Thomas Zaslavsky, “The Dynamic of the Forest Graph Operator”, *Discussiones Mathematicae Graph Theory*, 36 (2016) 899 - 913 (SCI Journal, Web of Science).
- S. M. Hegde, Suresh Dara, “On Clique Convergence of Graphs” *AKCE International Journal of Graphs and Combinatorics* 13 (3) (2016) 261 – 266 (Scopus Indexed Journal, Web of Science).
- Suresh Dara, Vance Faber, Suresh M. Hegde and Noah Streib, “First-Fit EFL", *AMO - Advanced Modeling and Optimization* 18 (2) (2016) 279 – 283.
- S. M. Hegde, Suresh Dara, “On Erdös - Faber - Lovász Conjecture", (Communicated).
- S. M. Hegde, Suresh Dara, “Further Results on Erdös - Faber - Lovász Conjecture", (Communicated).

Conference papers / Presentation

- S. M. Hegde, V. V. P. R. V. B. Suresh Dara, *On Clique Graph*, Extended Abstract published in Proceedings of the national conference on “Emerging Trends in Graph Connections" University of Kerala, Thiruvananthapuram, 2014, pp. 351 - 353.
- S. M. Hegde, V. V. P. R. V. B. Suresh Dara, *A Partial Solution of the Erdös - Faber - Lovász Conjecture*, 23^{rd} International Conference of Forum for Interdisciplinary Mathematics (FIM) on “Interdisciplinary Mathematical, Statistical and Computational Techniques - 2014” NITK, Surathkal, INDIA, from 18/12/2014 to 20/12/2014 (Presented).
- S. M. Hegde, Suresh Dara, *A Solution of the Erdös - Faber - Lovász Conjecture*, “The Asian Mathematical Conference (AMC) 2016”, BNDCC Nusa Dua, Bali, Indonesia, from 25/07/2016 to 29/07/2016 (Presented).

- S. M. Hegde, Suresh Dara, *On Clique Convergence of Graphs*, International Conference on "Current Trends in Graph Theory and Computation - 2016", Department of Mathematics, South Asian University, New Delhi, INDIA, from 17/09/2016 to 19/09/2016 (Presented).

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