CHARACTERIZATION OF NON-ISOLATED FORTS AND STABILITY OF AN ITERATIVE FUNCTIONAL EQUATION

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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Dedicated to

My Family and Teachers

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **CHARACTERIZATION OF NON-ISOLATED FORTS AND STABILITY OF AN ITERATIVE FUNCTIONAL EQUATION** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfilment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **CHARACTERIZATION OF NON-ISOLATED FORTS AND STABILITY OF AN ITERATIVE FUNCTIONAL EQUATION** submitted by **Mr. R. PALANIVEL.** (Register Number: 165067MA16F05) as the record of the research work carried out by him is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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ABSTRACT

The problem of finding a solution $f : X \to X$ of the iterative functional equation $f^n = F$ for a given positive integer $n \geq 2$ and a function $F : X \to X$ on a non-empty set X is known as the iterative root problem. The non-strictly monotone points (or forts) of F play an essential role in finding a continuous solution f of $f^n = F$ whenever X is an interval in the real line.

In this thesis, we define the forts for any continuous function $f : I \to J$, where I and J are arbitrary intervals in the real line $\mathbb{R}$. We study the non-monotone behavior of forts under composition and characterize the sets of isolated and non-isolated forts of iterates of any continuous self-map on an arbitrary interval I to study the continuous solutions of $f^n = F$. Consequently, we obtain an example of an uncountable measure zero dense set of non-isolated forts in the real line.

We define the notions of iteratively closed set in the space of continuous self-maps and the non-monotonicity height of any continuous self-map. We prove that continuous self-maps of non-monotonicity height 1 need not be strictly monotone on its range, unlike continuous piecewise monotone functions. Also, we obtain sufficient conditions for the existence of continuous solutions of $f^n = F$ for a class of continuous functions of non-monotonicity height 1. Further, we discuss the Hyers-Ulam stability of the iterative functional equation $f^n = F$ for continuous self-maps of non-monotonicity height 0 and 1.

Keywords: Functional equations, Iterative roots, Non-isolated forts, Cantor set, Measure zero dense set, Iteratively closed set, Non-monotonicity height, Characteristic interval, Non-PM functions, Hyers-Ulam stability.

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CHAPTER 1

INTRODUCTION

1.1 FUNCTIONAL EQUATIONS

A functional equation is an equation that involves known functions, unknown functions, and constants (Aczél (1966)). The theory of functional equations contributes to the development of strong tools in mathematics. Mathematicians such as Abel, Babbage, d'Alembert, Cauchy, Euler, Gauss, Jensen, Legendre, Schröder and many others have contributed to the growth of the theory of functional equations. Some of the classical functional equations are

Cauchy's equation: $f(x+y) = f(x) + f(y)$,

Jensen's equation: $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$,

D'Alembert's equation: $f(x+y) + f(x-y) = 2f(x)f(y)$.

A solution of a functional equation is a function which satisfies the given functional equation. For example, $f(x) = kx$, $f(x) = x^k$, and $f(x) = e^x$, where $k \in \mathbb{R}$ is a constant, are the solutions of the additive functional equation ($f(x+y) = f(x) + f(y)$), multiplicative functional equation ($f(xy) = f(x)f(y)$), and exponential functional equation ($f(x+y) = f(x)f(y)$), respectively.

Functional equations arise in many mathematics fields, such as geometry, statistics, and measure theory. One of the simple motivating applications of the additive functional equation $f(x+y) = f(x) + f(y)$ in geometry is finding the formula for the area of a rectangle. Functional equations find applications not only in mathematics, but also in the study of economics, neural networks, digital image processing, and many other fields (see Aczél (1966); Castillo et al. (2005); Iannella and Kindermann (2005)).

1.1.1 Iterative functional equations

Functional equations involving iterates or compositions of unknown functions are called *iterative functional equations*. The first ever study of the iterative functional equation was due to Charles Babbage (Babbage (1815)) and is of the form

$$f^n(x) = x, \tag{1.1.1}$$

where n is any natural number, $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$. The equation (1.1.1) is named after him as *Babbage functional equation*. Some other classical examples of iterative functional equations are

Abel's equation: $f(h(x)) = h(x+1)$,

Schröder's equation: $h(f(x)) = g(h(x))$,

Euler's equation: $f(x + f(x)) = f(x)$.

A detailed study on the existence of solutions of the above functional equations can be found in (Aczél (1966), Kuczma (1968), and Kuczma et al. (1990)). There are many other iterative functional equations, however, we concentrate on the existence of continuous solutions and stability of the iterative functional equation $f^n = F$, *the iterative root problem*.

1.2 ITERATIVE ROOT PROBLEM

Let F be a self-map on a non-empty set X and $n \geq 2$ be an integer. Finding a solution $f : X \to X$ of the iterative functional equation

$$f^n(x) = F(x),\ \forall x \in X \tag{1.2.1}$$

is known as the *iterative root problem*. We call a self-map f on X which satisfies the functional equation (1.2.1) as an *iterative root* of F of order n.

Example 1.2.1. *1.* $f(x) = x + \frac{1}{4}$ *is an iterative root of* $F(x) = x + 1$ *of order 4 on* $\mathbb{R}$.

2. $f(x) = x^{\sqrt{2}}$ *is a continuous solution of* $f^2(x) = F(x)$ *for* $F(x) = x^2$ *on* $[0, \infty)$.

Iterative functional equations find applications in embedding flow problem (Fort (1955)), invariant curves (Kuczma (1968); Kuczma et al. (1990)), neural networks (Iannella and Kindermann (2005) and Martin (2002)) and many engineering problems

(Castillo et al. (2005)). We present how the iterative root problem can be used in the embedding flow problem and finding invariant curves.

Let X be a topological space. For a function F on $X \times \mathbb{R}$, $x \in X$, $t \in \mathbb{R}$, denote $F(x,t) := F_t(x)$. A *(topological) flow* on X is a continuous function $F : X \times \mathbb{R} \to X$ such that

(i) $F_t(x)$ is a homeomorphism from X onto X for each $t \in \mathbb{R}$, and

(ii) $F(x,t+s) = F(F(x,s),t)$ for all $x \in X$ and $t,s \in \mathbb{R}$.

Embedding flow problem: *For a given topological space X and a given homeomorphism f from X onto itself, does there exist a flow F on X for which $F_1 = f$?*

If such a flow F exists, we say that f is embedded in F. Suppose that there exists a flow F on X for a given homeomorphism $f : X \to X$. Then for each $n \in \mathbb{N}$, the condition (ii) reduces into the following iterative functional equation

$$f^n(x) = F_n(x), \ \forall x \in X.$$

Therefore f is an iterative root of F_n of order n becomes a necessary condition for solving the embedding flow problem.

In the case of X is an interval, Fort (1955) proved that every order preserving homeomorphism of an interval onto itself can be embedded into a flow F.

Invariant curves problem: A set $E \subseteq \mathbb{R}^n$ is said to be invariant under a given function $f : \mathbb{R}^n \to \mathbb{R}^n$, if $f(E) \subseteq E$. The problem of finding the condition that the given curve is invariant under a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is known as the invariant curves problem. We discuss the invariant curves problem for $n = 2$.

Let $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ be the coordinate functions of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Let $E = \{(x, \phi(x)) \in \mathbb{R}^2 : x \in [0,1]\}$ be a curve (the graph of a function $\phi : [0,1] \to \mathbb{R}$). Now, the condition that $f(E) \subseteq E$ reduces to the following iterative functional equation:

$$\phi(f_1(x,\phi(x))) = f_2(x,\phi(x)), \ \forall x \in [0,1]. \tag{1.2.2}$$

If $f_1(x,y) = x+y,\ f_2(x,y) = \alpha y$ for all $(x,y) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$ is a constant, then (1.2.2) reduces to the iterative functional equation

$$\phi(x+\phi(x)) = \alpha\phi(x). \tag{1.2.3}$$

If $\alpha = 1$, then (1.2.3) is known as *Euler's functional equation*. Taking $\psi(x) = x+\phi(x)$, we get an another iterative functional equation

$$\psi^2(x) = (\alpha+1)\psi(x) - \alpha x. \tag{1.2.4}$$

If $\alpha = 0$ in (1.2.4), we get the equation of idempotent

$$\psi^2(x) = \psi(x).$$

Also, if $\alpha = -1$, the iterative functional equation (1.2.4) reduces to the Babbage functional equation

$$\psi^2(x) = x.$$

A detailed results on the embedding flow problem and invariant curves problem can be found in (Fort (1955), Nitecki (1971), Kuczma (1968) and Kuczma et al. (1990)).

1.2.1 Strictly monotone functions

In the early of the eighteenth century, Charles Babbage (Babbage (1815)) initiated the study of the existence of solutions of $f^n = F$ when $F(x) = x$. Bödewadt (1944); Łojasiewicz (1951); Haidukov (1958); Kuczma (1968) and Kuczma et al. (1990) and many others studied the continuous solutions of $f^n = F$ for strictly monotone functions on the interval. We present some of the basic results on the continuous solutions of $f^n = F$ for continuous strictly monotone functions.

Throughout the thesis, we use the following notations. Let I,J denote arbitrary intervals in $\mathbb{R}$ and $K := [a,b]$ with $a < b$. Let $C(I,J)$ denote the set of all continuous functions from I into J and $C(I) := C(I,I)$.

Theorem 1.2.2. *(Bödewadt (1944)) Let $F \in C(K)$ be strictly increasing. Then for any integer $n \geq 2$ and $A,B \in (a,b)$ with $A < B$, the equation $f^n = F$ has a strictly increasing solution $f \in C(K)$ such that*

$$F(a) \leq f(A) < f(B) \leq F(b).$$

In McShane (1961), it was proved that every monotone solution f of (1.1.1) either $f(x) = x$ for all $x \in I$ or n has to be even and f is a strictly decreasing and $f^2(x) = x$ for all $x \in I$. Further, Kuczma (1968) investigated the existence of solutions of $f^n = F$ for continuous strictly monotone functions.

Lemma 1.2.3. *(Kuczma (1968)) Let $F : I \to I$ be strictly monotone. Assume $f : I \to I$ to be a monotone solution of $f^n = F$ and fix $x_0 \in I$.*

(a) If f is increasing, then the following conditions are equivalent:

(i) $F(x_0) = x_0$,

(ii) $f(x_0) = x_0$,

(iii) $f(x_0) = F(x_0)$.

(b) If f and F are decreasing, then $f(x_0) = F(x_0)$ if and only if $F^2(x_0) = x_0$.

Theorem 1.2.4. *(Kuczma (1968)) Let $F : I \to I$ be strictly monotone and onto. Suppose that $f : I \to I$ is a solution of $f^n = F$. Then $f \in C(I)$ if and only if f is strictly monotone.*

The following theorem is an important result on the existence of continuous solutions of $f^n = F$ for strictly increasing functions which paved the way to develop the theory further.

Theorem 1.2.5. *(Kuczma (1968)) Let $F \in C(I)$. If F is strictly increasing, then $f^n = F$ has a strictly increasing solution $f \in C(I)$ for any $n \geq 2$.*

Proof. Let $\mathbb{G} = \{x \in I \ : \ F(x) = x\}$, the set of all fixed points of F. Clearly

$$I = \mathbb{G} \bigcup \left(\bigcup_{c,d \in \mathbb{G}} (c,d) \right),$$

where (c,d) is a pairwise disjoint interval with $c,d \in \mathbb{G}$ or endpoints of I. Clearly $F|_{(c,d)} : (c,d) \to (c,d)$ is strictly increasing and continuous on (c,d), and for each $x \in (c,d)$, either $c < F(x) < x < d$ or $c < x < F(x) < d$. If f is a strictly increasing solution of $f^n = F$ on I, then by Lemma 1.2.3, $f(x) = x$ for all $x \in \mathbb{G}$. Conversely, if there is a strictly increasing $f_{c,d} \in C([c,d])$ such that

$$f^n_{c,d}(x) = F(x), \ \forall x \in [c,d],$$

then $f : I \to I$ defined by

$$f(x) := \begin{cases} f_{c,d}(x), & \text{if} \quad x \in (c,d), \\ x, & \text{if} \quad x \in \mathbb{G}, \end{cases}$$

is a strictly increasing continuous function on I and satisfies

$$f^n(x) = F(x), \ \forall x \in I.$$

So the problem is reduced to finding a continuous solution of $f^n = F$ on each $[c,d]$. We proceed with the case $c < F(x) < x < d$ for all $x \in (c,d)$; other case ($c > F(x) > x > d$ for all $x \in (c,d)$) follows similarly.

Fix $x_0 \in [c,d]$, choose any $x_1, x_2, \ldots, x_{n-1} \in (F(x_0), x_0)$ with $x_{n-1} < \cdots < x_2 < x_1$ and let $x_{k+n} := F(x_k)$ for all $k \geq 0$ and $x_k := F^{-1}(x_{k+n})$ for all $k \leq -1$. Observe that $x_{k+1} < x_k$ for all $k \in \mathbb{Z}$.

Let $J_k := [x_{k+1}, x_k]$, $k \in \mathbb{Z}$. Now, for each $k \in \{0, 1, \ldots, n-2\}$, let f_k be an arbitrary strictly increasing continuous function from J_k onto J_{k+1}. Define

$$f_k(x) := F \circ f_{k-n+1}^{-1} \circ \cdots \circ f_{k-1}^{-1}(x),\ \forall x \in J_k,\ k \geq n-1,$$

and

$$f_k(x) := f_{k+1}^{-1} \circ \cdots \circ f_{k+n-1}^{-1} \circ F(x),\ \forall x \in J_k,\ k \in (-\infty, -1].$$

Thus for each k, f_k is a strictly increasing and continuous solution of $f^n = F$ on J_k. Therefore the function $f : [c,d] \to [c,d]$ defined by

$$f(x) := f_k(x),\ \forall x \in J_k,\ k \in \mathbb{Z}$$

is a strictly increasing continuous solution of $f^n = F$ on $[c,d]$. □

Remark 1.2.6. *The solution constructed in Theorem 1.2.5 depends on arbitrary strictly increasing continuous functions, and there are infinitely many such functions. Therefore solutions of $f^n = F$ for strictly increasing continuous functions are not necessarily unique.*

Since the composition of two strictly decreasing functions is a strictly increasing function, a strictly decreasing function cannot have strictly decreasing iterative roots of even order $n \geq 2$.

Theorem 1.2.7. *(Kuczma (1968)) Let $F \in C(I)$ be strictly decreasing and onto. Then for each odd $n \geq 3$, there exists a strictly decreasing solution $f \in C(I)$ of $f^n = F$.*

Consider the complete metric space $C(K)$ with the uniform metric

$$\rho(f,g) := \sup\{|f(x) - g(x)| : x \in K\}.$$

For each $n \in \mathbb{N}$, let

$$W(n) = \{f^n : f \in C(K)\} \text{ and } W = \bigcup_{n=2}^{\infty} W(n).$$

It is known from Simon (1989) that W is of first category and $\mathrm{cl}(W) \neq C(K)$, where $\mathrm{cl}(W)$ is the closure of W. Moreover, Blokh (1992) proved that the set W is nowhere dense in $C(K)$. Even though the set of all continuous functions possessing continuous iterative roots on a compact interval are topologically small with respect to the topology induced by ρ, finding an iterative root becomes complicated yet interesting for continuous non-monotone functions.

1.2.2 Piecewise monotone functions

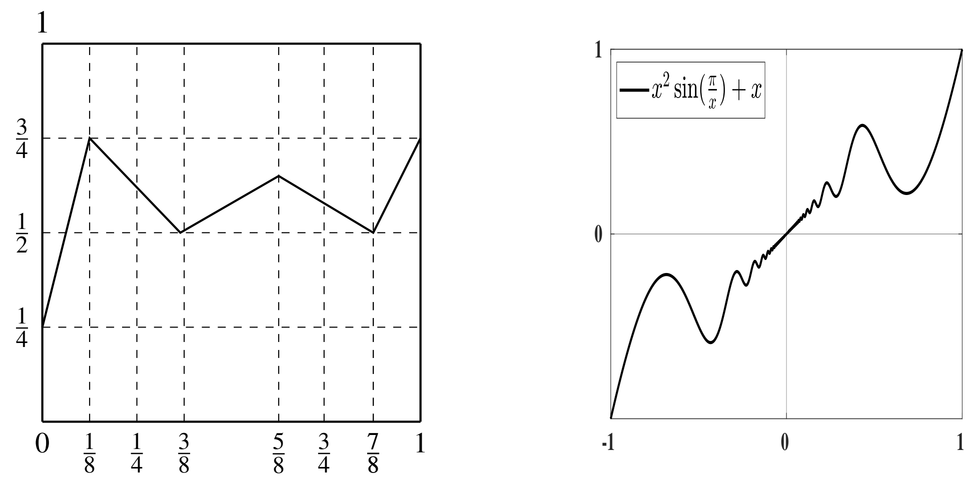
For finding continuous solutions of $f^n = F$, the difficulty lies in the behavior of non-monotone points of F. Zhang and Yang (1983) (in Chinese) and Zhang (1997) defined a non-monotone point of a function $f : K \to K$ in (a,b) and proved the fundamental results on the existence of continuous solutions of $f^n = F$.

Definition 1.2.8. *(Zhang (1997)) A point $x \in (a,b)$ is called a non-monotone point (or fort) of a function $f : K \to K$ if f is not strictly monotone in any neighborhood of x.*

A continuous self-map defined on a compact interval with finitely many forts is called a *piecewise monotone function* (*PM function*). For example, the graph of the function given in Figure 1.1(a) is a PM function.

Let $PM(K)$ denote the set of all PM functions in $C(K)$. For $f \in PM(K)$, let $N(f)$ be the number of forts of f and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $f \in PM(K)$, from (2.4) in Zhang (1997), we have

$$0 = N(f^0) \leq N(f) \leq \cdots \leq N(f^k) \leq N(f^{k+1}) \leq \cdots .$$



(a) A function with finite forts

(b) A function with infinitely many forts

Figure 1.1 : Non-monotone functions

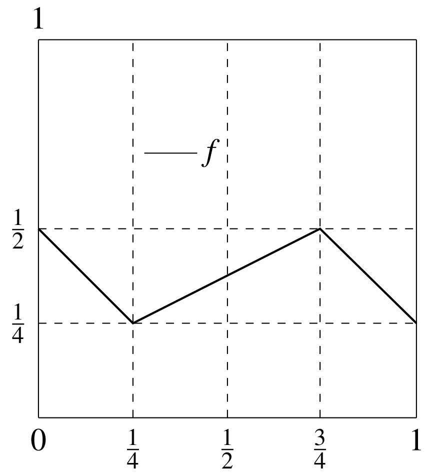
Zhang and Yang (1983) and Zhang (1997) introduced the concept of *non-monotonicity height* and *characteristic interval* for PM functions, and studied the existence and non-existence of continuous solutions of $f^n = F$ for $F \in PM(K)$.

Definition 1.2.9. *(Zhang (1997)) Let $f \in PM(K)$. The non-monotonicity height $H(f)$ denote the least non-negative integer k such that $N(f^k) = N(f^{k+1})$, if it exists and $H(f) = \infty$, if the sequence $\{N(f^k)\}_{k \in N_0}$ is strictly increasing.*

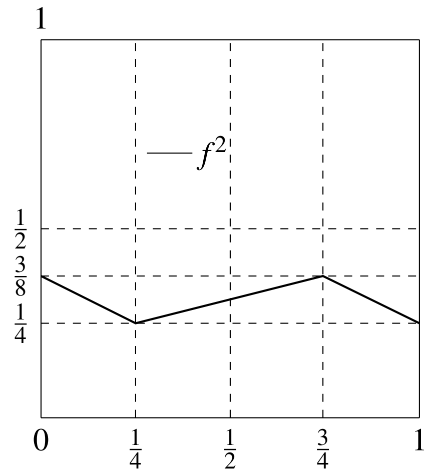
Example 1.2.10. *Consider the continuous function $f : [0,1] \to [0,1]$ defined by*

$$f(x) := \begin{cases} \frac{1}{2} - x, & \text{if } x \in [0, \frac{1}{4}], \\ \frac{x}{2} + \frac{1}{8}, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ \frac{5}{4} - x, & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

We can see that $f \in PM([0,1])$, $\{\frac{1}{4}, \frac{3}{4}\}$ is the set of forts of f, $R(f) = [\frac{1}{4}, \frac{1}{2}]$, and f is strictly increasing on $[\frac{1}{4}, \frac{1}{2}]$ (see Figure 1.2(a)). Also, $\frac{1}{4}$ and $\frac{3}{4}$ are the only forts of f^2 (see Figure 1.2(b)). This implies $H(f) = 1$.



(a) $\frac{1}{4}, \frac{3}{4}$ are forts of f and $R(f) = [\frac{1}{4}, \frac{1}{2}]$



(b) $\frac{1}{4}, \frac{3}{4}$ are forts of f^2 and $R(f^2) = [\frac{1}{4}, \frac{3}{8}]$

Figure 1.2 : A PM function f with $H(f) = 1$

The following results are obtained by a quick observation from the non-monotonicity height of PM functions.

Lemma 1.2.11. *(Zhang (1997)) Let $f \in PM(K)$. $H(f) \leq 1$ if and only if f is strictly monotone on its range.*

Theorem 1.2.12. *(Zhang (1997)) For any $F \in PM(K)$ with $H(F) > 1$, $f^n = F$ has no continuous solution for $n > N(F)$.*

Characteristic interval for PM functions: Let $f \in PM(K)$ and $H(f) \leq 1$. From the non-monotonicity of f, it follows that $N(f) = N(f^2)$. Thus f is strictly monotone on $[m,M]$, where m and M are the minimum and maximum value of f on K respectively. Extending appropriately the interval on which f is monotone, there exist two points $a', b' \in K$ such that

(i) a', b' are either forts of f or $a', b' \in \{a, b\}$,

(ii) f is strictly monotone on (a', b'),

(iii) $[m,M] \subseteq [a', b']$.

Definition 1.2.13. *(Zhang (1997)) Let $f \in PM(K)$ and $H(f) \leq 1$. The unique interval $[a', b']$ obtained above is referred to as the characteristic interval of f.*

Example 1.2.14. *For the continuous function f defined as in Example 1.2.10, the characteristic is equals to $[\frac{1}{4}, \frac{3}{4}]$.*

In the same paper, they introduced the extension method (extending a solution of $f^n = F$ from characteristic interval to the whole domain) for obtaining a continuous solution f of $f^n = F$ and proved the following result.

Theorem 1.2.15. *(Zhang (1997)) Let $F \in PM(K)$ and $H(F) \leq 1$.*

1. *Suppose F is strictly increasing on $[a', b']$, and $F(x)$ cannot reach a' and b' on K unless $F(a') = a'$ or $F(b') = b'$. Then $f^n = F$ has a solution $f \in PM(K)$ for all $n \geq 2$. Moreover, the conditions are necessary for $n > N(F) + 1$.*

2. *Suppose F is strictly decreasing on $[a', b']$. If either $F(a') = b'$ and $F(b') = a'$, or $a' < F(x) < b'$ for all $x \in K$, then $f^n = F$ has a solution $f \in PM(K)$ for only odd $n \geq 3$.*

The following open problems were raised in Zhang (1997).

Problem 1.2.16. *Let $F \in PM(K)$ with $H(F) \geq 2$. Does $f^n = F$ have a solution $f \in C(K)$ for all $n \leq N(F)$?*

Problem 1.2.17. *Let $F \in PM(K)$ and $H(F) \leq 1$. Does $f^n = F$ have a solution $f \in C(K)$ for $n \leq N(F) + 1$ when $F(x') = a'$ or $F(x') = b'$ for some $x' \in K$ but $x \notin [a', b']$?*

Problem 1.2.16 is solved in (Sun and Xi (1996); Sun (2000)) in the case $n = 2$. In Li et al. (2008), the Problem 1.2.17 solved partly in the case of F is strictly increasing on the characteristic interval of F. Later, Liu et al. (2012) provides the necessary and sufficient conditions for the existence of solutions of Problem 1.2.16 for the case

$n = N(F) \geq 3$ by characterizing the set of forts of the composition of continuous functions f and g as the union of forts of g and inverse image of forts of f under g in (a,b). The Problems 1.2.16 and 1.2.17 are further investigated in (Li and Zhang (2018), Liu et al. (2018) and Li and Liu (2019)).

1.2.3 Continuous non-PM functions

Finding continuous solutions of $f^n = F$ for continuous non-PM functions is more complicated than for PM functions. In Lin (2014), the existence and non-existence of continuous solutions of $f^n = F$ were studied for a class of continuous non-PM functions known as *sickle-like functions*. Also, the solutions of $f^n = F$ was described in (Lin et al. (2017)) for another class of continuous functions called *clenched single-plateau functions* on a compact interval $K = [a,b]$. Recently, Cho et al. (2018) defined the fort for functions in $C(K)$ and generalized the concept of characteristic interval from PM functions to continuous functions on K, and studied the solutions of $f^n = F$ for continuous non-PM functions, which are non-constant in any interval of its domain.

Definition 1.2.18. *(Cho et al. (2018)) A point $x_0 \in K$ is said to be a fort of $f \in C(K)$ if f is strictly monotone in no neighborhood of x_0.*

A fort $x_0 \in K$ of f is called a *non-isolated fort* if f has a fort in every neighborhood of x_0 other than x_0. Otherwise, x_0 is called an *isolated fort*.

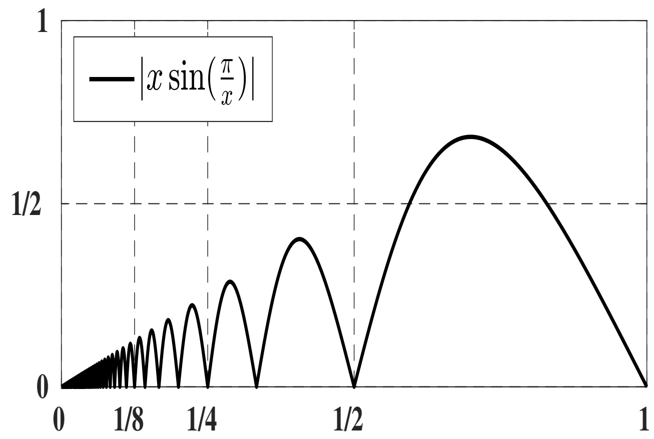


Figure 1.3 : A continuous function with endpoint as a fort

Let $T(f)$ be the set of all forts of $f \in C(K)$. Definition 1.2.18 is a generalization of Definition 1.2.8, which includes the endpoints of $[a,b]$. Note that continuous non-PM functions may exhibit non-monotonic behavior at the endpoints of their domain. For example, 0 is a non-isolated fort (see Figure 1.3) of the continuous non-PM function $f : [0,1] \to [0,1]$ defined by

$$f(x) := \begin{cases} 0, & \text{if } x = 0, \\ |x \sin(\frac{\pi}{x})|, & \text{if } x \neq 0. \end{cases}$$

Characteristic interval for continuous functions: For $f \in C(K)$, let $R(f) = [m, M]$ be the range of f, where m and M are the minimum and maximum value of f respectively, and $m \leq M$. For $f \in C(K)$, define

$$a' := \sup\{x \in T(f) \cup \{a\} : x \leq m\} \text{ and } b' := \inf\{y \in T(f) \cup \{b\} : y \geq M\}.$$

Clearly, a' and b' are well-defined, unique and $a', b' \in T(f) \cup \{a, b\}$.

Definition 1.2.19. *(Cho et al. (2018)) The unique interval $[a', b']$ for $f \in C(K)$ is called the characteristic interval of f and $Ch_f := [a', b']$.*

Note that $[m, M] \subseteq [a', b']$ and $(a', m) \cap T(f) = \emptyset = (M, b') \cap T(f)$. This implies that if $m \in T(f)$ (resp. $M \in T(f)$), then $a' = m$ (resp. $b' = M$). It follows from Proposition 3 (a) in (Cho et al. (2018)) that

$$Ch_f \supseteq Ch_{f^2} \supseteq \ldots \supseteq Ch_{f^k} \supseteq Ch_{f^{k+1}} \supseteq \ldots. \tag{1.2.5}$$

Note that for $f \in PM(K)$ with $H(f) \leq 1$, Definition 1.2.19 and Definition 1.2.13 are equivalent.

In the following theorem, Cho et al. (2018) obtained the continuous solutions of $f^n = F$ by extending solutions from the characteristic interval to the whole domain.

Theorem 1.2.20. *(Cho et al. (2018)) Let $F \in C(K)$ and $F_0 = F|_{Ch_F}$. Suppose there is a solution $f_0 \in C(Ch_F)$ of $f^n = F_0$ on Ch_F with the following properties:*

(i) *there exist $x', y' \in T(f_0)$ such that $T(f_0) \cap ((a', x') \cup (y', b')) = \emptyset$ with $a < a' < x' < y' < b' < b$,*

(ii) *$F([a, a']) \subseteq F_0([a', x']) \subseteq f_0([a', x']) \subseteq [a', x']$ and $F([b', b]) \subseteq F_0([y', b']) \subseteq f_0([y', b']) \subseteq [y', b']$.*

Then $f^n = F$ has a solution $f \in C(K)$ such that $f|_{Ch_F} = f_0$.

1.3 STABILITY OF FUNCTIONAL EQUATIONS

In 1940, during a talk in the Mathematical Colloquium at the University of Wisconsin, S. M. Ulam posted a problem regarding the stability of Cauchy's functional equation as follows (cf. Forti (1995)):

Given a group $(G_1, \cdot)$ and a metric group $(G_2, *)$ with metric d and a positive number ε, does there exists a $\delta > 0$ such that, if a function $g : G_1 \to G_2$ satisfies

$$d(g(x \cdot y), g(x) * g(y)) < \delta, \ \forall x, y \in G_1,$$

then there is a function $f : G_1 \to G_2$ such that

$$f(x \cdot y) = f(x) * f(y)$$

and

$$d(f(x), g(x)) < \varepsilon, \ \forall x \in G_1?$$

In case of a positive answer to the above problem, we say that the Cauchy functional equation $f(x \cdot y) = f(x) * f(y)$ is *stable*. D. H. Hyers (Hyers (1941)) solved Ulam's problem when G_1 and G_2 are Banach spaces. Due to Ulam's question and Hyers' answer, this type of stability is called the *Hyers-Ulam stability* of functional equations.

1.3.1 Direct method

The following theorem gives the partial answer to Ulam's question.

Theorem 1.3.1. *(Hyers (1941)) Let E_1 and E_2 be Banach spaces and suppose that a mapping $g : E_1 \to E_2$ satisfies*

$$\|g(x+y) - g(x) - g(y)\| \leq \varepsilon, \ \forall x, y \in E_1. \tag{1.3.1}$$

Then

$$f(x) = \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$$

exists for each x in E_1 such that

$$f(x+y) = f(x) + f(y) \tag{1.3.2}$$

and

$$\|f(x) - g(x)\| \leq \varepsilon, \forall x \in E_1. \tag{1.3.3}$$

Moreover, $f(x)$ is unique.

Proof. Take $x = y$ in (1.3.1) and dividing by 2, we get

$$\left\| \frac{g(2x)}{2} - g(x) \right\| \leq \frac{\varepsilon}{2}. \tag{1.3.4}$$

In (1.3.4), replace x by $2x$ and divide by 2, will get

$$\left\| \frac{g(2^2x)}{2^2} - \frac{g(2x)}{2} \right\| \leq \frac{\varepsilon}{2^2}, \tag{1.3.5}$$

and

$$\left\| \frac{g(2^2x)}{2^2} - g(x) \right\| \leq \varepsilon\left(\frac{1}{2} + \frac{1}{2^2}\right) = \varepsilon\left(1 - \frac{1}{2^2}\right).$$

Then by induction, repeating the same procedure will get

$$\left\| \frac{g(2^nx)}{2^n} - g(x) \right\| \leq \varepsilon\left(1 - \frac{1}{2^n}\right),\ n \in \mathbb{N}. \tag{1.3.6}$$

Let $f_n(x) = \frac{g(2^nx)}{2^n}$. Replace x by $2^m x$ in (1.3.6) and divide by 2^m, $m \in \mathbb{N}$, we get

$$\left\| \frac{g(2^{m+n}x)}{2^{(m+n)}} - \frac{g(2^mx)}{2^m} \right\| \leq \frac{\varepsilon}{2^m}.$$

Hence by the Cauchy's criterion, the limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each x in E_1 also $f(x)$ satisfies (1.3.3). To prove (1.3.2) replace x by $2^n x$ and y by $2^n y$ in (1.3.1) and divide by 2^n, to get

$$\left\| \frac{g(2^n(x+y))}{2^n} - \frac{g(2^nx)}{2^n} - \frac{g(2^ny)}{2^n} \right\| \leq \frac{\varepsilon}{2^n}.$$

Taking limit as $n \to \infty$, we get $f(x+y) = f(x) + f(y)$. The uniqueness follows from the additive property of f. □

Note that both functions f and g could be discontinuous everywhere on E_1. However, the continuity of f follows from the continuity of g. In particular, if g is continuous at some point x_0, then f is continuous everywhere on E_1. The method used by Hyers is called the *direct method.* Th. M. Rassias (Rassias (1978)) generalized Hyers's result of Theorem 1.3.1, and this type of stability is called the *Hyers-Ulam-Rassias Stability*.

Theorem 1.3.2. *(Rassias (1978)) Let E_1 be a normed linear space and E_2 be a Banach space and a mapping $g : E_1 \to E_2$ such that*

$$\|g(x+y) - g(x) - g(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p),\ \forall x, y \in E_1,$$

where $\varepsilon > 0$ and $p < 1$ are constants. Then

$$f(x) = \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$$

exists for all x in E_1 and f is unique and

$$\|f(x) - g(x)\| \leq k\varepsilon\|x\|^p, \ \forall x \in E_1,$$

where $k = \frac{2}{2-2^p}$.

In 1990, Th. M. Rassias (see Hyers et al. (1998)) asked the question whether Theorem 1.3.2 can also be proved for $p \geq 1$. Gajda (1991) gave a solution to Rassias's question for $p > 1$ using the same approach as in Theorem 1.3.2. In 1993, G. Isac and Th. M. Rassias proved the more generalization of Theorem 1.3.2 by introducing a function ψ mapping the real interval $\mathbb{R}_+ = (0, \infty)$ into itself instead of a function t^p with the following conditions:

$$\lim_{t\to\infty} \frac{\psi(t)}{t} = 0, \ \ \psi(ts) \leq \psi(t)\psi(s), \ \forall t > 0, \ s > 0 \text{ and } \psi(t) < t, \forall t > 1.$$

The detailed study on the Hyers-Ulam stability and its generalizations for different kind of functional equations in several variables can be found in (Forti (1995); Hyers et al. (1998); Rassias (2000) and the references therein).

Consider the functional equation

$$E_1(f) = E_2(f), \tag{1.3.7}$$

where f is unknown. As in (Xu and Zhang (2002)), we say the functional equation (1.3.7) has the Hyers-Ulam stability, if for every function g satisfies

$$\|E_1(g) - E_2(g)\| \leq \ \delta$$

for some constant δ, there exists a solution f of (1.3.7) such that

$$\|f - g\| \leq \varepsilon$$

for some $\varepsilon > 0$ depends only on δ.

In 2002, Bing Xu and Weinian Zhang (Xu and Zhang (2002)) studied the Hyers-

Ulam stability of a non-linear functional equation

$$G(f^{n_1},\ldots,f^{n_k}) = F$$

on K, where $n_i, k \in \mathbb{N}, i = 1,\ldots,k$ for a Lipschitz mapping $F : K \to K$ such that $F(a) = a$ and $F(b) = b$. Further, Agarwal et al. (2003) generalized the results of Xu and Zhang (2002) and investigated the Hyers-Ulam stability of linear and non-linear functional equations in single variable.

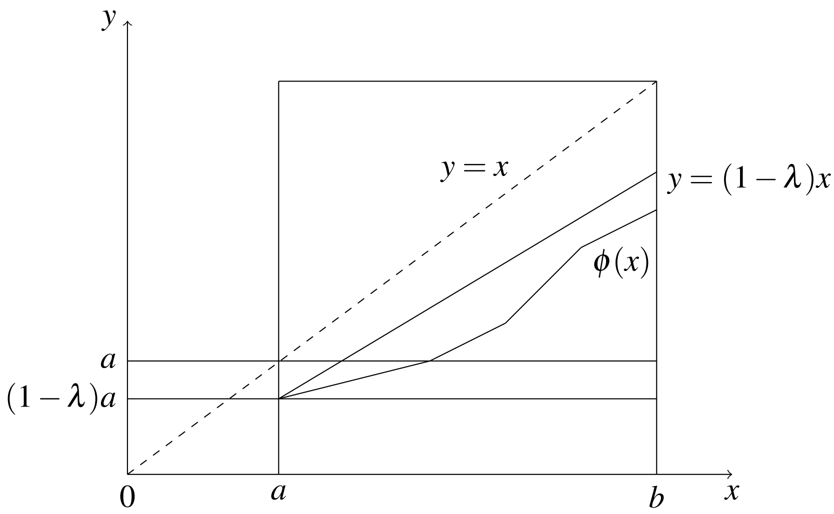


Figure 1.4 : $\phi \in R_{a,\lambda}(|a,b|)$

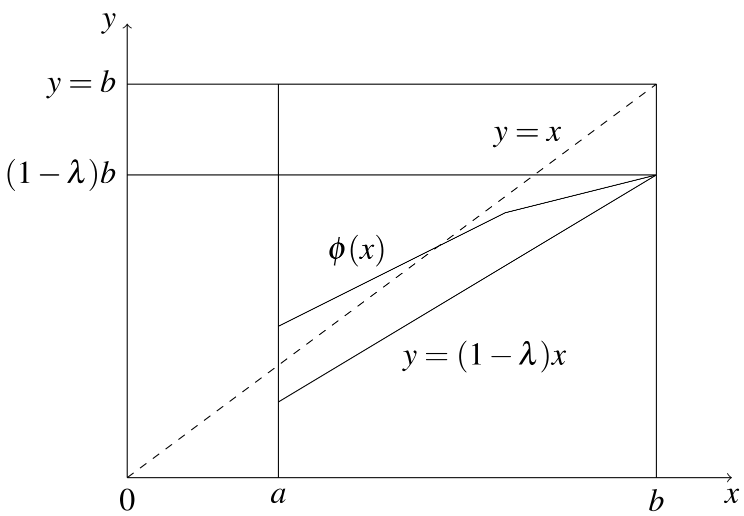


Figure 1.5 : $\phi \in R_{b,\lambda}(|a,b|)$

Figures 1.4 and 1.5 refer to the functions in the classes $R_{a,\lambda}(|a,b|)$ and $R_{b,\lambda}(|a,b|)$.

Later, Xu and Zhang (2007) constructed continuous solutions and discussed the Hyers-Ulam stability of the polynomial-like equation

$$f^n(x) = \sum_{i=1}^{n-1} \lambda_i f^i(x) + F(x),\ x \in |a,b|,\ n \in \mathbb{N} \text{ and } n \geq 2 \tag{1.3.8}$$

on the interval $(a,x_0|$ or $|x_0,b)$, $x_0 \in |a,b|$ ($|a,b|$ means either an open interval (a,b), a semi-closed interval $[a,b)$ or $(a,b]$, or a closed interval $[a,b]$) in $\mathbb{R}$, and one or both endpoints of $|a,b|$ my be infinite with constants $\lambda_i \in [0,\infty)$ and

$$\lambda := \sum_{i=1}^{n-1} \lambda_i < 1 \tag{1.3.9}$$

for F in the class $R_{\xi,\lambda}(|a,b|)$ of strictly increasing functions on $|a,b|$. Let $\mathrm{cl}(|a,b|)$ be the closure of $|a,b|$. For $\xi \in \mathrm{cl}(|a,b|)$ and the constant λ defined as in (1.3.9), let $R_{\xi,\lambda}(|a,b|)$ denote the set all functions ϕ, which are continuous and strictly increasing on $|a,b|$ and satisfies that

$$(\phi(x) - (1-\lambda)x)(\xi - x) > 0,\ \forall x \in |a,b|,\ x \neq \xi, \text{ and}$$

$$(\phi(x) - (1-\lambda)\xi)(\xi - x) < 0,\ \forall x \in |a,b|,\ x \neq \xi.$$

If $\lambda_i = 0,\ i = 1,\ldots,n-1$ in (1.3.8), then (1.3.8) becomes

$$f^n(x) = F(x).$$

If $\lambda = 0$ in $R_{a,\lambda}(|a,b|)$ and $R_{b,\lambda}(|a,b|)$, then

$$R_{a,0}(|a,b|) = \{\phi \in C(|a,b|) : \phi \text{ is strictly increasing and } \phi(x) < x,\ \forall x \in |a,b|,\ x \neq a\},$$

$$R_{b,0}(|a,b|) = \{\phi \in C(|a,b|) : \phi \text{ is strictly increasing and } \phi(x) > x,\ \forall x \in |a,b|,\ x \neq b\}.$$

Let $F : I \to I$ and $n \geq 2$. Taking $E_1(f) = f^n$ and $E_2(f) = F$ for all f in (1.3.7), we say the iterative functional equation $f^n = F$ has the Hyers-Ulam stability if for every $g : I \to I$ such that

$$|g^n(x) - F(x)| \leq \delta,\ \forall x \in I \tag{1.3.10}$$

for some fixed constant $\delta > 0$, there exists a solution $f : I \to I$ of $f^n = F$ and satisfies

$$|f(x) - g(x)| \leq \varepsilon(\delta),\ \forall x \in I, \tag{1.3.11}$$

where the constant $\varepsilon(\delta) > 0$ which depends only on δ.

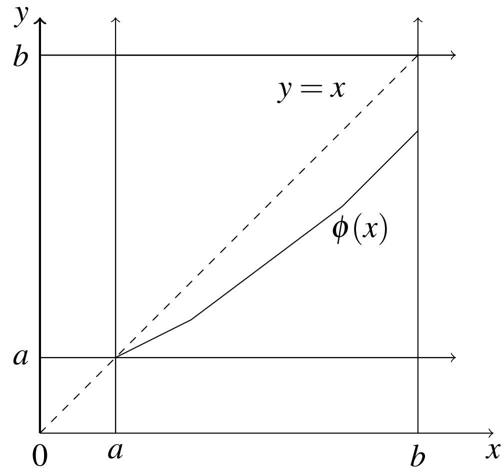


Figure 1.6 : $\phi \in R_{a,0}(I)$

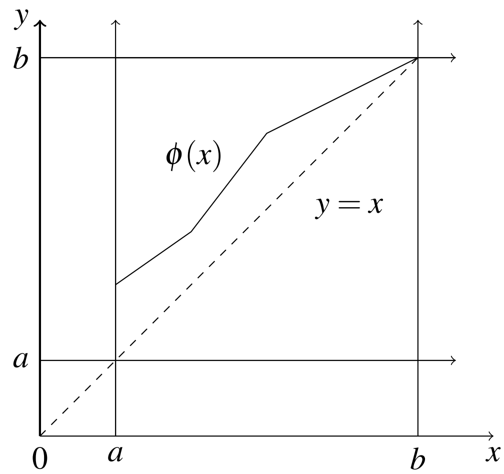


Figure 1.7 : $\phi \in R_{b,0}(I)$

Figures 1.6 and 1.7 refer to the functions in the classes $R_{a,0}(|a,b|)$ and $R_{b,0}(|a,b|)$ respectively.

The following results discussed the Hyers-Ulam stability of $f^n = F$ for F in the classes $R_{a,0}(|a,b|)$ and $R_{b,0}(|a,b|)$.

Theorem 1.3.3. *(Xu and Zhang (2007)) Let $F \in R_{a,0}(|a,b|)$ and $\lambda < 1$. If g is a self-map on $|a,b|$ such that*

$$|g(x) - g(y)| \leq l|x-y|, \ \forall x,y \in |a,b| \text{ and } r := \sum_{j=1}^{n-1} l^j < 1 \tag{1.3.12}$$

and satisfies that

(i) *there exists* $x_0 \in |a,b|$ *such that* $a < g^n(x_0) < g^{n-1}(x_0) < \cdots < g(x_0) < x_0$ *and* $g^n(x_0) = F(x_0)$,

(ii) g *is strictly increasing on* $[g^{n-1}(x_0), x_0]$, *and*

(iii) $|g^n(x) - F(x)| \leq \delta$ *for all* $x \in (a, x_0]$, *for a constant* $\delta > 0$,

then $f^n = F$ *has a solution* $f \in R_{a,0}(|a,x_0])$ *such that*

$$|g(x) - f(x)| \leq \delta(1-r)^{-1}, \ \forall x \in (a, x_0].$$

Theorem 1.3.4. *(Xu and Zhang (2007)) Let* $F \in R_{b,0}(|a,b|)$ *and* $\lambda < 1$. *If* g *is a self-map on* $|a,b|$ *such that* (1.3.12) *holds and satisfies that*

(i) *there exists* $x_0 \in |a,b|$ *such that* $x_0 < g(x_0) < \cdots < g^{n-1}(x_0) < g^n(x_0) < b$ *and* $g^n(x_0) = F(x_0)$,

(ii) g *is strictly increasing on* $[x_0, g^{n-1}(x_0)]$, *and*

(iii) $|g^n(x) - F(x)| \leq \delta$ *for all* $x \in [x_0, b)$, *for a constant* $\delta > 0$,

then $f^n = F$ *has a solution* $f \in R_{b,0}([x_0, b|)$ *such that*

$$|g(x) - f(x)| \leq \delta(1-r)^{-1}, \ \forall x \in [x_0, b).$$

Remark 1.3.5. *The strictly increasing homeomorphism* F *(strictly increasing, continuous and onto) does not belong to* $R_{a,0}(|a,b|) \cup R_{b,0}(|a,b|)$.

Remark 1.3.6. *In Theorem 1.3.3, the Hyers-Ulam stability of* $f^n = F$ *is discussed only on* $(a,b]$ *for* F *in the class* $R_{a,0}(|a,b|)$ *in the case* $x_0 = b$.

Remark 1.3.7. *In Theorem 1.3.4, the Hyers-Ulam stability of* $f^n = F$ *is discussed only on* $[a,b)$ *for* $F \in R_{b,0}(|a,b|)$ *in the case* $x_0 = a$.

In 2015, Li et al. (2015) discussed the Hyers-Ulam stability of $f^n = F$ for a class of PM functions F with $H(F) = 1$ as follows:

Theorem 1.3.8. *(Li et al. (2015)) Let* $F \in PM(K)$ *with* $H(F) = 1$. *If* $g \in PM(K)$ *and* $l, L > 0$ *such that*

$$l|x-y| \leq |g(x) - g(y)| \leq L|x-y|, \ \forall x, y \in Ch_F,$$

and satisfies that

(i) $H(g) = 1$ *and* $Ch_g = Ch_F$,

(ii) g *is a solution of* $f^n = F$ *on* Ch_F *and* $g : Ch_F \to Ch_F$ *is a homeomorphism,*

(iii) $|g^n(x) - F(x)| \leq \delta$ *for all* $x \in K$, *and for a constant* $\delta > 0$,

then $f^n = F$ *has a solution* $f \in PM(K)$ *for any* $n \geq 2$ *such that*

$$|g(x) - f(x)| \leq (1+L)\delta l^{-n}, \ \forall x \in K.$$

1.3.2 Fixed point method

The fixed point method is another most used method to prove the stability of functional equations, which was used for the first time by J. A. Baker (Baker (1991)). In Baker (1991), a variant of Banach's fixed point theorem is used to obtain the stability of a functional equation

$$f(t) = F(t, f(\varphi(t))) \tag{1.3.13}$$

using the stability of the equation $T(x_0) = x_0$.

Theorem 1.3.9. *(Baker (1991)) Let* $T : X \to X$ *be a contraction map on a complete metric space* (X,d). *If* $u \in X, \delta > 0$ *and* $d(u, T(u)) \leq \delta$, *then* T *has a unique fixed point* $p \in X$ *and* $d(u, p) \leq \frac{\delta}{1-\lambda}$.

Theorem 1.3.10. *(Baker (1991)) Let* S *be a non-empty set,* (X,d) *be a complete metric space,* $\varphi : S \to S$, $F : S \times X \to X$, $\lambda \in [0,1)$ *and* $d(F(t,u), F(t,v)) \leq \lambda d(u,v)$ *for all* $t \in S, u, v \in X$. *If* $g : S \to X$, $\delta > 0$, *and*

$$d(g(t), F(t, g(\varphi(t)))) \leq \delta, \ \forall t \in S,$$

then there is a unique function $f : S \to X$ *such that* $f(t) = F(t, f(\varphi(t)))$ *for all* $t \in S$. *and*

$$d(f(t), g(t)) \leq \delta(1-\lambda)^{-1}, \ \forall t \in S.$$

In 2003, Radu (2003) proved the Hyers-Ulam stability of (1.3.2) using the fixed point method. Găvruta (2008) used Matkowski's fixed point theorem to prove the Hyers-Ulam stability of (1.3.13). In 2009, Cădariu et al. (2009) used a variant of Banach's fixed point theorem to prove the Hyers-Ulam stability of the iterative functional equation

$$f(t) = F(f(t), f(\varphi(t))),$$

where f is unknown. Also, Akkouchi (2011) proved the Hyers-Ulam stability of (1.3.13) using the variant of Ćirić's fixed point theorem. The detailed study on the fixed point method can be found in (Găvruta and Găvruta (2010); Jung (2011); Cădariu and Radu (2012); Ciepliński (2012); Brzdęk et al. (2014); Xu et al. (2015) and the references therein).

1.4 OUTLINE OF THE THESIS

The proposed thesis consists of six chapters, the first of which provides a brief introduction to the iterative root problem and Hyers-Ulam stability of functional equations. From the literature survey, we observe that no characterization is obtained for the set of non-isolated forts of iterates of a continuous function on an arbitrary interval I, and no study is done on the Hyers-Ulam stability of $f^n = F$ for strictly increasing homeomorphisms and continuous non-PM functions.

In Chapter 2, we generalize the notion of a fort for functions in $C(I,J)$ and study the properties of isolated and non-isolated forts of continuous functions to study the existence of continuous solutions of $f^n = F$. We show how large and complicated can be the set of non-isolated forts of nowhere constant (non-constant in any interval) continuous functions (Example 2.1.6). We also prove that continuous nowhere differentiable functions have the whole domain as the set of non-isolated forts.

In Chapter 3, we observe that the non-monotone behavior of isolated and non-isolated forts under composition. Our main results are to characterize the sets of isolated and non-isolated forts of iterates of continuous functions on an arbitrary interval I (Theorem 3.1.8 and Corollary 3.1.9). Consequently, an example of an uncountable measure zero dense set of non-isolated forts whose complement is also dense in the real line is obtained (Theorem 3.2.2).

In Chapter 4, we introduce the concept of iteratively closed set in $C(K)$ and generalize the notion of non-monotonicity height for maps in $C(K)$. We prove that continuous non-PM functions of non-monotonicity height 1 is not necessarily strictly monotone on its range, unlike PM functions. Further, we discuss the existence of continuous solutions of $f^n = F$ for a class of non-constant continuous functions of non-monotonicity height 1 (Theorem 4.2.1).

In Chapter 5, we study the Hyers-Ulam stability of $f^n = F$ for strictly increasing homeomorphisms (Theorem 5.1.1) and for continuous functions of non-monotonicity height 1 (Theorem 5.2.1).

Chapter 6 concludes the thesis by describing the scope for future research in the area.

CHAPTER 2

THE SET OF FORTS OF CONTINUOUS FUNCTIONS

It is fundamental and essential to study the set of forts of continuous functions and its iterates to study the existence of continuous solutions of $f^n = F$. In this chapter, we generalize the notion of the forts for $f \in C(I,J)$ and study the properties of isolated and non-isolated forts of f. Also, we give an example of a continuous function on $[0,1]$ having the Cantor ternary set as the set of non- isolated forts. Moreover, we discuss the difference between the forts and non-differentiable points of a continuous function.

Definition 2.0.1. *A point $x \in I$ is called a non-strictly monotone point (or fort) of $f \in C(I,J)$ if for each $\varepsilon > 0$, f is not strictly monotone in the neighborhood $N_\varepsilon(x)$ of x, where $N_\varepsilon(x) := \{y \in I \, : \, |y-x| < \varepsilon\}$.*

For $f \in C(I,J)$, let $S(f)$, $\Lambda(f)$ and $\Lambda^*(f)$ denote the set of all forts, isolated forts and non-isolated forts of f respectively. The following fact can be easily observed for $f \in C(I,J)$.

Fact 2.0.2. (i) $\Lambda(f) \cap \Lambda^*(f) = \emptyset$.

(ii) $S(f) = \Lambda(f) \cup \Lambda^*(f)$.

(iii) $x \in \Lambda^*(f)$ *if and only if x is a limit point of* $S(f)$.

(iv) $S(f) = \emptyset$ *if and only if f strictly monotone.*

Local extremum points of a continuous function f are forts of f, and isolated forts of f are points of local extremum of f. The following example shows that a non-isolated fort of a continuous f need not be a point of local extremum of f.

Example 2.0.3. *Define* $f : [-1,1] \to (-1,1)$ *by*

$$f(x) := \begin{cases} \frac{-x}{36}, & \text{if } x \in [-1,0], \\ f_n(x), & \text{if } x \in I_n, \text{ even } n \in \mathbb{N},\ n \geq 4, \\ -f_n(x), & \text{if } x \in I_n, \text{ odd } n \in \mathbb{N},\ n \geq 4, \end{cases}$$

where $I_n = \left[\frac{4}{n+1}, \frac{4}{n}\right]$ *and* $f_n : I_n \to (-1,1)$ *defined by*

$$f_n(x) = -\left(x - \frac{4}{n+1}\right)\left(x - \frac{4}{n}\right) = -x^2 + \frac{4x(2n+1)}{n(n+1)} - \frac{4^2}{n(n+1)}. \qquad (2.0.1)$$

It is easy to see that each f_n *is continuous on* I_n*,* $f_n(x) \geq 0$ *for all* $x \in I_n$ *and*

$$f_{n+1}\left(\frac{4}{n+1}\right) = 0 = f_n\left(\frac{4}{n+1}\right).$$

Thus f *is continuous on* $[-1,0) \cup (0,1]$*. To prove the continuity of* f *at* 0*, first we observe that*

$$f_n'(x) = -2x + \frac{4(2n+1)}{n(n+1)} > 0,\ \forall x \in \left[\frac{4}{n+1}, y_n\right) \text{ and } f_n'(x) < 0,\ \forall x \in \left(y_n, \frac{4}{n}\right],$$

where $y_n = \frac{2(2n+1)}{n(n+1)}$ *is the midpoint of* I_n*. Thus* f_n *is strictly increasing on* $[\frac{4}{n+1}, y_n)$ *and strictly decreasing on* $\left(y_n, \frac{4}{n}\right]$*. This implies that* f_n *attains a local maximum at* y_n *on* I_n*. Hence* $-f_n$ *attains a local minimum at* y_n *on* I_n *(see Figure 2.1).*

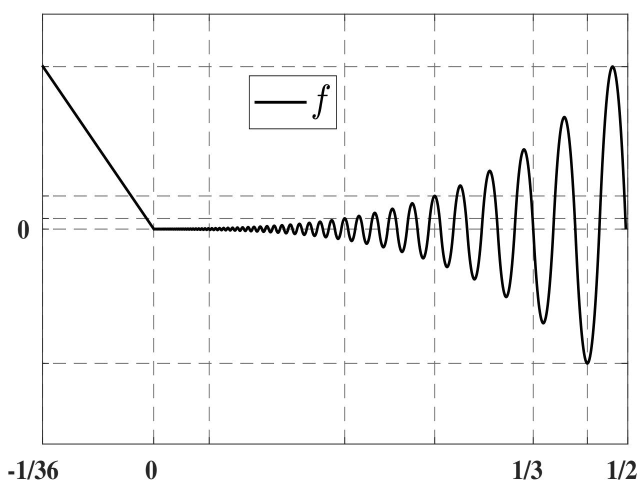


Figure 2.1 : A continuous functions with a non-isolated fort

It follows from (2.0.1) *that*

$$f_{n+1}\left(y_{n+1}\right)=\left(\frac{2}{(n+1)(n+2)}\right)^{2}<\left(\frac{2}{n(n+1)}\right)^{2}=f_{n}(y_{n}), \tag{2.0.2}$$

$y_{n+1}=\frac{2(2n+3)}{(n+1)(n+2)}$ *is the midpoint of* I_{n+1}. *Now, for each element* $x\in\left(0,\frac{4}{k}\right)$, $k\in\mathbb{N}, k\geq 4$, *we have* $x\in I_m$ *for some* $m\geq k$. *By the fact that* f_m *attains a local maximum at* y_m *on* I_m*, from (2.0.2), we get*

$$|f(x)|=|f_m(x)|\leq|f_m(y_m)|\leq|f_k(y_k)|=\left(\frac{2}{k(k+1)}\right)^{2}.$$

Hence f *is continuous at* 0. *By the non-monotonicity of* f_n *on* I_n *(even* $n\geq 4$*), we have* f *is strictly decreasing on* $[\frac{4}{n+2},y_{n+1})$, *strictly increasing on* $[y_{n+1},y_n]$ *and strictly decreasing on* $(y_n,\frac{4}{n}]$ *(see Figure 2.1). Thus*

$$\Lambda(f)=\{y_n : n\in\mathbb{N},\ n\geq 4\},$$

and by $\lim_{n\to\infty}y_n=0$, $\Lambda^*(f)=\{0\}$. *Moreover, the point* $0\in\Lambda^*(f)$ *but* f *does not attains a local extremum at* 0 *(see Figure 2.1).*

2.1 THE SET OF NON-ISOLATED FORTS

In this section, we will discuss about some basic properties of isolated and non-isolated forts of $f\in C(I,J)$.

2.1.1 Properties of isolated and non-isolated forts

Proposition 2.1.1. *Let* $f\in C(I,J)$. *A point* $x\in S(f)$ *if and only if for any* $\varepsilon>0$ *there exist two distinct points* $x_1,x_2\in N_\varepsilon(x)$ *such that* $f(x_1)=f(x_2)$.

Proof. Let $x\in S(f)$ and $\varepsilon>0$. It follows from the non-monotonicity of f that f is not one-to-one on $N_\varepsilon(x)$. Thus there exist distinct $x_1,x_2\in N_\varepsilon(x)$ such that $f(x_1)=f(x_2)$. Conversely, if there exist two distinct points $x_1,x_2\in N_\varepsilon(x)$ for each $\varepsilon>0$ such that $x_1<x_2$ and $f(x_1)=f(x_2)$, then f attains a local extremum at some point $x_3\in(x_1,x_2)$. Therefore $x\in S(f)$. □

Proposition 2.1.2. *Let* $f\in C(I,J)$ *and* $I'\subseteq I$ *be a non-empty open interval. Suppose that there are no distinct* $x_1,x_2,x_3\in I'$ *with* $f(x_1)=f(x_2)=f(x_3)$, *and* $f(p)=f(q)$ *for some* $p,q\in I'$ *with* $p<q$. *Then there exists* $r\in(p,q)$ *with the following properties.*

(i) $S(f) \cap (p,q) = \{r\}$.

(ii) *If* $f(r) > f(p)$ *(resp.* $f(r) < f(p)$*), then* $f(x) < f(p)$ *(resp.* $f(x) > f(p)$*) for all* $x \in I'$ *and* $x \notin [p,q]$.

Proof. (i) By the continuity of f, there exists $r \in (p,q)$ such that either

$$f(r) \leq f(x) < f(p) \text{ or } f(r) \geq f(x) > f(p),\ \forall x \in (p,q). \tag{2.1.1}$$

This implies $r \in S(f)$. Suppose $s \in S(f) \cap (p,r)$, choose $\delta > 0$ such that $N_\delta(s) \subseteq (p,r)$. By Proposition 2.1.1, there are $p_0, q_0 \in N_\delta(s)$ such that $p_0 < q_0$ and $f(p_0) = f(q_0)$. Note that by (2.1.1),

$$f(r) < f(p_0) < f(p) = f(q) \text{ or } f(r) > f(p_0) > f(p) = f(q).$$

By applying the Intermediate Value Theorem (IMVT) on (r,q), we get a point $r_0 \in (r,q)$ such that $f(p_0) = f(r_0)$. This implies

$$f(p_0) = f(q_0) = f(r_0)$$

for distinct $p_0, q_0, r_0 \in (p,q)$, a contradiction. Thus $S(f) \cap (p,r) = \emptyset$. Similarly, we can prove that $S(f) \cap (r,q) = \emptyset$. Hence $S(f) \cap (p,q) = \{r\}$.

(ii) It follows from the hypotheses that $f(x) \neq f(p)$ for all $x \in I'$ and $x \notin [p,q]$. Suppose $f(t) > f(p)$ (resp. $f(t) < f(p)$) for some $t \in I'$ with $t \notin [p,q]$, choose $t_1 \in I'$ with $t_1 \notin [p,q]$ such that

$$f(r) > f(t_1) > f(p) = f(q) \text{ (resp.} f(r) < f(t_1) < f(p) = f(q)).$$

By IMVT, there exist $t_2 \in (p,r)$ and $t_3 \in (r,q)$ such that $f(t_2) = f(t_1) = f(t_3)$, a contradiction. Hence the proof. □

The following lemma gives the equivalent condition for a non-isolated fort of a continuous function $f \in C(I,J)$.

Lemma 2.1.3. *Let* $f \in C(I,J)$. *An element* $x_0 \in I$ *is a non-isolated fort of* f *if and only if for each* $\varepsilon > 0$, *there exist three distinct points* $x_1, x_2, x_3 \in N_\varepsilon(x_0)$ *such that* $f(x_1) = f(x_2) = f(x_3)$.

Proof. Let $x_0 \in \Lambda^*(f)$ and $\varepsilon > 0$. The result is trivial when f is constant in some non-empty open interval of $N_\varepsilon(x_0)$. Assume that f is nowhere constant in $N_\varepsilon(x_0)$. To the contrary, we assume that

$$\text{there are no distinct } x_1, x_2, x_3 \in N_\varepsilon(x_0) \text{ such that } f(x_1) = f(x_2) = f(x_3). \qquad (2.1.2)$$

By Proposition 2.1.1 and (2.1.2), there exist two points $p_1, q_1 \in N_\varepsilon(x_0)$ such that $p_1 < q_1$, $f(p_1) = f(q_1)$, and $x_0 \notin \{p_1, q_1\}$. Then by Proposition 2.1.2 (i), there exists $r_1 \in (p_1, q_1)$ such that

$$S(f) \cap (p_1, q_1) = \{r_1\}. \qquad (2.1.3)$$

Without loss of generality, we assume that $f(r_1) > f(p_1)$. By Proposition 2.1.2 (ii),

$$f(x) < f(p_1), \ \forall x \in N_\varepsilon(x_0) \text{ and } x \notin [p_1, q_1]. \qquad (2.1.4)$$

If $x_0 \in (p_1, q_1)$, then $x_0 = r_1$ and $x_0 \in \Lambda(f)$ by (2.1.3), a contradiction to $x_0 \in \Lambda^*(f)$. Otherwise (i.e., $x_0 \notin (p_1, q_1)$), there exists $\delta > 0$ such that

$$N_\delta(x_0) \cap (p_1, q_1) = \emptyset.$$

Then we have $f(p_2) = f(q_2)$ for some $p_2, q_2 \in N_\delta(x_0)$ with $p_2 < q_2$ and $x_0 \notin \{p_2, q_2\}$ by Proposition 2.1.1 and (2.1.2). Then either $p_2 < q_2 < p_1$ or $q_1 < p_2 < q_2$. We discuss the case $p_2 < q_2 < p_1$, the other case is similar.

By Proposition 2.1.2 (i), there exists $r_2 \in (p_2, q_2)$ such that

$$S(f) \cap (p_2, q_2) = \{r_2\} \text{ and } f(r_2) < f(p_2) \text{ or } f(r_2) > f(p_2).$$

Suppose that $f(r_2) > f(p_2)$, since $f(q_2) < f(p_1)$ (by (2.1.4)), choose $y_1 \in (q_2, p_1)$ with $f(p_2) < f(y_1) < f(r_2)$. By IMVT, there exist two points $y_2 \in (p_2, r_2)$ and $y_3 \in (r_2, q_2)$ such that $f(y_2) = f(y_1) = f(y_3)$, a contradiction to (2.1.2). Therefore $f(r_2) < f(p_2)$. By Proposition 2.1.2 (ii), we have

$$f(p_2) < f(x), \ \forall x \in N_\varepsilon(x_0) \text{ and } x \notin [p_2, q_2]. \qquad (2.1.5)$$

Now, we claim that

$$S(f) \cap N_\varepsilon(x_0) = \{r_1, r_2\}. \qquad (2.1.6)$$

Suppose that $t \in S(f) \cap N_\varepsilon(x_0)$ and $t \notin (p_1, q_1) \cup (p_2, q_2)$. Then choose $\eta > 0$ such that

$$N_\eta(t) \subseteq N_\varepsilon(x_0) \text{ and } N_\eta(t) \cap \{r_1, r_2\} = \emptyset.$$

From Proposition 2.1.1, there exist $p_3, q_3 \in N_\eta(t)$ such that $p_3 < q_3$ and $f(p_3) = f(q_3)$.

Since

$$S(f) \cap (p_1, q_1) = \{r_1\} \text{ and } S(f) \cap (p_2, q_2) = \{r_2\}$$

with

$$f(p_1) = f(q_1) < f(x) < f(r_1),\ \forall x \in (p_1, q_1)$$

and

$$f(r_2) < f(x) < f(p_2) = f(q_2), \forall x \in (p_2, q_2),$$

we have $p_3, q_3 \notin [p_1, q_1] \cup [p_2, q_2]$ by Proposition 2.1.2 (i) and (2.1.2). It follows from (2.1.4) and (2.1.5) that

$$f(q_2) < f(x) < f(p_1),\ \forall x \in (p_3, q_3).$$

From Proposition 2.1.2 (i), there exists $r_3 \in (p_3, q_3)$ such that $S(f) \cap (p_3, q_3) = \{r_3\}$. Choose $z_1 \in (q_2, p_1)$ with $z_1 \notin [p_3, q_3]$ such that either

$$f(p_3) < f(z_1) < f(r_3) \text{ or } f(p_3) > f(z_1) > f(r_3).$$

By IMVT and the equality $f(p_3) = f(q_3)$, there exist $z_2 \in (p_3, r_3)$ and $z_3 \in (r_3, q_3)$ such that $f(z_1) = f(z_2) = f(z_3)$, a contradiction to (2.1.2). Hence (2.1.6) is proved. This contradicts $x_0 \in \Lambda^*(f)$. Therefore there exist distinct $x_1, x_2, x_3 \in N_\varepsilon(x_0)$ such that $f(x_1) = f(x_2) = f(x_3)$.

Conversely, assume that for each $\varepsilon > 0$, there exist distinct points $x_1, x_2, x_3 \in N_\varepsilon(x_0)$ such that

$$x_1 < x_2 < x_3 \text{ and } f(x_1) = f(x_2) = f(x_3).$$

Without loss of generality, we assume that $x_2, x_3 \in [x_0, x_0 + \varepsilon)$. By the continuity of f and the fact $f(x_2) = f(x_3)$, f attains a local extremum at some $x_\varepsilon \in (x_2, x_3)$. This implies $x_\varepsilon \in S(f)$ and $x_\varepsilon \neq x_0$. Hence $x_0 \in \Lambda^*(f)$. □

The above lemma is a generalization of Lemma 2 in Cho et al. (2018) from a continuous self-map on a compact interval K into any continuous function on an arbitrary interval I.

It is worth to mention here that if $x_0 \in \Lambda^*(f)$, then it is not necessarily true that there exist four distinct points $x_1, x_2, x_3, x_4 \in N_\varepsilon(x_0)$ such that $f(x_1) = f(x_2) = f(x_3) = f(x_4)$ (see Remark 3 in Cho et al. (2018)).

Proposition 2.1.4. *Let $f \in C(I, J)$. Then $\Lambda^*(f)$ and $S(f)$ are closed subsets of I.*

Proof. Let $\{x_n\}$ be a sequence in $\Lambda^*(f)$ and $\lim\limits_{n\to\infty} x_n = x$. Then for each $\varepsilon > 0$, there is

$x_\varepsilon \in \{x_n\}$ such that $x_\varepsilon \in N_\varepsilon(x)$. Choose $\eta > 0$ such that

$$N_\eta(x_\varepsilon) \subseteq N_\varepsilon(x).$$

Since f has a fort $x_\eta \in N_\eta(x_\varepsilon)$ such that $x_\eta \neq x_\varepsilon$, f has a fort x_η or x_ε in $N_\varepsilon(x)$ different from x. Thus $x \in \Lambda^*(f)$ and hence $\Lambda^*(f)$ is closed. The proof for $S(f)$ is similar. □

We remark here that $\Lambda(f)$ is not closed for the continuous function f defined in Example 2.0.3. In fact,

$$\Lambda(f) = \left\{ \frac{2(2n+1)}{n(n+1)} : n \in \mathbb{N},\ n \geq 4 \right\},\ \lim_{n\to\infty} \frac{2(2n+1)}{n(n+1)} = 0, \text{ and } 0 \in \Lambda^*(f).$$

Every interval $I \subseteq \mathbb{R}$ is second countable. If $\Lambda(f)$ is uncountable, then there is a sequence $\{x_n\}$ of distinct elements in $\Lambda(f)$ such that $\lim_{n\to\infty} x_n = x$ and $x \in \Lambda(f)$. Therefore $x \in \Lambda(f) \cap \Lambda^*(f)$, a contradiction to Fact 2.0.2 (i). Thus for each continuous function $f \in C(I,J)$, the set $\Lambda(f)$ is countable and nowhere dense by Fact 2.0.2 (i) and (iii). The periodicity helps us to come up with a continuous function on an unbounded interval with countably infinite non-isolated forts, and it is not easy to visualize the same on a bounded interval.

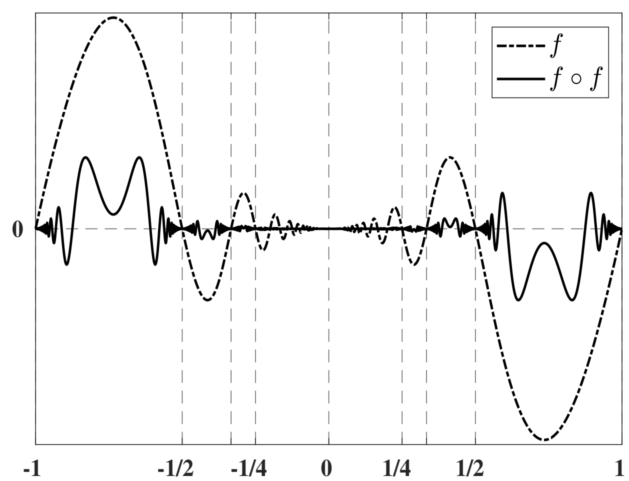


Figure 2.2 : Countably infinite non-isolated forts on a bounded interval

Example 2.1.5. *Consider the function* $f : [-1,1] \to (-1,1)$ *defined by*

$$f(x) := \begin{cases} 0, & \text{if } x = 0, \\ x^2 \sin(\frac{\pi}{x}), & \text{if } x \neq 0. \end{cases}$$

It is easy to see that $\Lambda^*(f) = \{0\}$ *and*

$$\Lambda^*(f \circ f) = \{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{Z} \setminus \{0\} \right\}$$

is countably infinite (see Figure 2.2).

2.1.2 Nowhere dense set of non-isolated forts

It is to be noted that if $f \in C(I,J)$ is constant on a non-empty open interval $I' \subseteq I$ and strictly monotone elsewhere, then $I' = S(f) = \Lambda^*(f)$. Hence $\Lambda^*(f)$ is uncountable. So, the following questions arise naturally:

(Q1) Is there a continuous function f, which is nowhere constant such that $S(f) = \Lambda^*(f)$?

(Q2) For each nowhere constant function $f \in C(I,J)$, is $\Lambda^*(f)$ countable?

The answer is “yes” for (Q1) and “no” for (Q2). Here we mention that there is no continuous function f such that $S(f) = \Lambda^*(f)$ and $\Lambda^*(f)$ is countable ($S(f) = \Lambda^*(f)$ implies $\Lambda^*(f)$ is perfect, and it is known that perfect sets are uncountable).

The following example answers (Q1) and (Q2) simultaneously.

Example 2.1.6. *Consider the Cantor ternary set* $\mathscr{C}$ *in* $[0,1]$. *We construct a nowhere constant continuous function* f *on* $[0,1]$ *such that*

$$S(f) = \Lambda^*(f) = \mathscr{C}.$$

From the construction of $\mathscr{C}$, *for* $n \in \mathbb{N}$, *we denote the set of all deleted open intervals of* $[0,1]$ *in the* n^{th} *stage by* D_n *and* $C_n = [0,1] \setminus D_n$. *Then*

$$D_n = \bigcup_{k=1}^{2^n-1} D_{n,k} \text{ and } C_n = \bigcup_{k=1}^{2^n} C_{n,k},$$

where

$$D_{n,k} := \begin{cases} D_{(n-1),l}, & \text{if } k = 2l, \text{ for some } l \in \{1,\dots,2^{n-1}-1\}, \\ \left(\frac{x_{n,k}}{3^n}, \frac{x_{n,k}+1}{3^n}\right), & \text{otherwise}, \end{cases} \tag{2.1.7}$$

and

$$x_{n,k} := \begin{cases} x_{(n-1),k}, & \text{if } k < 2^{n-1}, \\ 3^n - (1 + x_{n,l}), & \text{if } k \geq 2^{n-1} \text{ and } l = 2^n - k. \end{cases} \tag{2.1.8}$$

Also, for $k \in \{1,3,\ldots,2^n-1\}$ *and* $k' \in \{2,4,\ldots,2^n\}$,

$$C_{n,k} = \left[\frac{x_{n,k}-1}{3^n}, \frac{x_{n,k}}{3^n}\right] \text{ and } C_{n,k'} = \left[\frac{x_{n,(k'-1)}+1}{3^n}, \frac{x_{n,(k'-1)}+2}{3^n}\right].$$

Clearly, $C_{n+1} \subsetneq C_n$, $D_n \subsetneq D_{n+1}$ *and* $\mathscr{C} = \bigcap_{n=1}^{\infty} C_n$. *For* $n=2$, *we get*

$$D_{2,1} = \left(\frac{1}{3^2}, \frac{2}{3^2}\right),\ D_{2,2} = \left(\frac{3}{3^2}, \frac{6}{3^2}\right) = D_{1,1} \text{ and } D_{2,3} = \left(\frac{21}{3^3}, \frac{24}{3^3}\right) = D_{3,6},$$

and

$$C_{2,1} = \left[0, \frac{1}{3^2}\right],\ C_{2,2} = \left[\frac{2}{3^2}, \frac{3}{3^2}\right],\ C_{2,3} = \left[\frac{6}{3^2}, \frac{7}{3^2}\right] \text{ and } C_{2,4} = \left[\frac{8}{3^2}, 1\right].$$

Let $f_0(x) = x$ *on* $C_0 = [0,1]$. *For* $n \in \mathbb{N} \cup \{0\}$, *define*

$$f_{n+1}(x) := \begin{cases} \frac{2}{3} f_n(3x), & \text{if } x \in [0, \frac{1}{3}], \\ 1-x, & \text{if } x \in (\frac{1}{3}, \frac{2}{3}), \\ \frac{1}{3} + \frac{2}{3} f_n(3x-2), & \text{if } x \in [\frac{2}{3}, 1]. \end{cases} \tag{2.1.9}$$

It is easy to see that each f_n *is well-defined and continuous on* $[0,1]$. *Let* $f = \lim_{n\to\infty} f_n$. *To claim* f *is continuous on* $[0,1]$, *first we claim that*

$$\max_{x\in[0,1]} |f_{i+1}(x) - f_i(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^i, \quad i \in \mathbb{N} \cup \{0\}. \tag{2.1.10}$$

The inequality (2.1.10) is trivial for $i=0$. *Assume that (2.1.10) is holds for* $i=m$. *Now, for* $x \in [0, \frac{1}{3}]$, *from (2.1.9), we have*

$$|f_{m+2}(x) - f_{m+1}(x)| = \left|\frac{2}{3} f_{m+1}(3x) - \frac{2}{3} f_m(3x)\right| \le \frac{2}{3}\left(\frac{1}{3}\left(\frac{2}{3}\right)^m\right) = \frac{1}{3}\left(\frac{2}{3}\right)^{m+1},$$

and for $x \in [\frac{2}{3}, 1]$,

$$|f_{m+2}(x) - f_{m+1}(x)| = \left|\frac{2}{3} f_{m+1}(3x-2) - \frac{2}{3} f_m(3x-2)\right| \le \frac{1}{3}\left(\frac{2}{3}\right)^{m+1}.$$

Also, we have $f_{m+2}(x) = f_{m+1}(x)$ *for all* $x \in (\frac{1}{3}, \frac{2}{3})$. *This implies*

$$\max_{x\in[0,1]} |f_{m+2}(x) - f_{m+1}(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{m+1}.$$

Hence (2.1.10) is proved by induction on i. Now, for $n > m$, it follows from (2.1.10) that

$$\max_{x\in[0,1]} |f_n(x) - f_m(x)| \le \sum_{i=m}^{n-1} \max_{x\in[0,1]} |f_{i+1}(x) - f_i(x)| \le \frac{1}{3}\sum_{i=m}^{n-1}\left(\frac{2}{3}\right)^i.$$

By the Cauchy's criterion, f_n converges uniformly to f. Hence f is continuous on $[0,1]$.

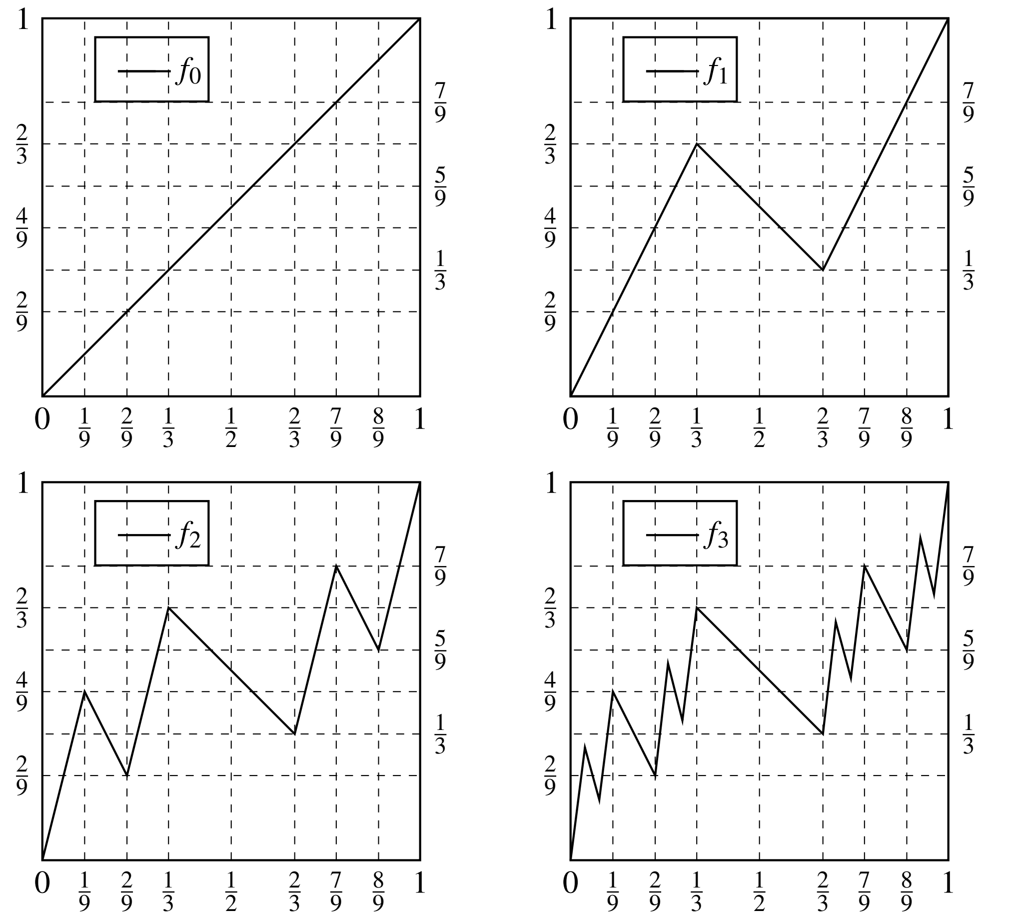


Figure 2.3 : First four steps of the construction of f with $S(f) = \Lambda^*(f) = \mathscr{C}$

Observe from (2.1.7) and (2.1.8) that for each $x \in D_{n,k}$, $n \ge 2$, $k < 2^{n-1}$ (resp. $k > 2^{n-1}$) with $x \notin D_{n-1}$, we get

$$3x \in D_{n-1,k} \;(resp.\; 3x - 2 \in D_{(n-1),(k-2^{n-1})}). \tag{2.1.11}$$

Similarly, for each $x \in C_{n,k}$, $k \le 2^{n-1}$ (resp. $k > 2^{n-1}$), we get

$$3x \in C_{n-1,k} \;(resp.\; 3x - 2 \in C_{(n-1),(k-2^{n-1})}).$$

Since $f_2 = f_1$ on $D_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$, by induction, we can prove that $f_{n+1} = f_n$ on $D_{n,k}$. This *implies* $f = f_n$ *on* D_n. *Since* f_1 *is strictly decreasing in* $D_{1,1} = (\frac{1}{3}, \frac{2}{3})$, *by induction on* n, *assume that* f_{n-1} *is strictly decreasing on each* $D_{(n-1),l}$. *Then from* (2.1.9), (2.1.11), *and the assumption, we get* f_n *is strictly decreasing on each* $D_{n,k}$. *Similarly, we can prove that* f_n *is strictly increasing on each* $C_{n,k}$ *(see Figure 2.3). Thus* f *is strictly decreasing on each* $D_{n,k}$. *This implies that* f *is nowhere constant continuous on* $[0,1]$.

Let y_0 *be the left endpoint of* $D_{n,k}$. *From the monotonicity of* f_n *on* $D_{n,k}$ *and* $C_{n,k}$, *for each* $\varepsilon \leq \frac{1}{3^n}$, *we have*

$$f_n(y) \leq f_n(y_0), \; \forall y \in N_\varepsilon(y_0). \tag{2.1.12}$$

By taking limit on (2.1.12), we get

$$f(y) \leq f(y_0), \; \forall y \in N_\varepsilon(y_0).$$

Thus f *attains a local maximum at* y_0 *and hence* $y_0 \in S(f)$. *Similarly, we can prove* f *attains a local minimum at the right endpoint of* $D_{n,k}$. *Thus by the monotonicity of* f *on each* $D_{n,k}$, *and the property that every point of* $\mathscr{C}$ *is the limit point of the endpoints of* $D_{n,k}$ *and Fact 2.0.2 (iii), we have*

$$S(f) = \Lambda^*(f) = \mathscr{C}.$$

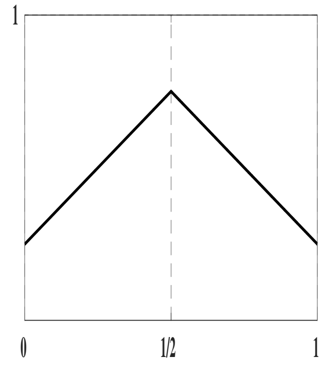
Remark 2.1.7. *(i) In Example 2.1.6, the set of points of local extrema of* f, $Extr(f)$, *is the set of endpoints of* $D_{n,k}$, $n \in \mathbb{N}$. *Hence* $\Lambda^*(f) \setminus Extr(f)$ *is uncountable.*

(ii) It is known from Theorem 2 of (Behrends et al., 2008) that the interior of $Extr(f)$ *is empty for any non-constant continuous real-valued function* $f : I \to \mathbb{R}$. *Therefore if* f *is nowhere constant continuous on* I *and* $\Lambda^*(f) = I$, *then the set of non-isolated forts which are not in* $Extr(f)$ *forms a dense subset of* I.

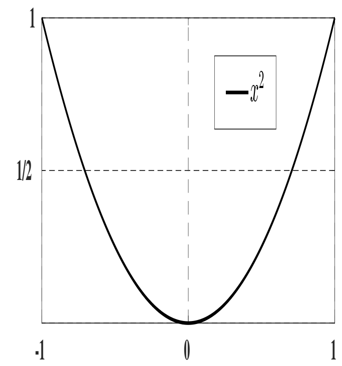
(iii) There is no continuous function $f : \mathbb{R} \to \mathbb{R}$ *such that* $Extr(f)$ *is a nowhere dense, non-empty perfect set (see Proposition 2 (ii) in (Balcerzak et al., 2017)). Define* $\varphi : \mathbb{R} \to \mathbb{R}$ *by* $\varphi(x) = f(x)$ *and* $\varphi(x) = x$, *otherwise, where* f *is the function as defined in Example 2.1.6. Then* $\Lambda^*(\varphi)$ *is a nowhere dense, non-empty perfect set in* $\mathbb{R}$.

2.2 CONTINUOUS FUNCTIONS WITH $\Lambda^*(f) = I$

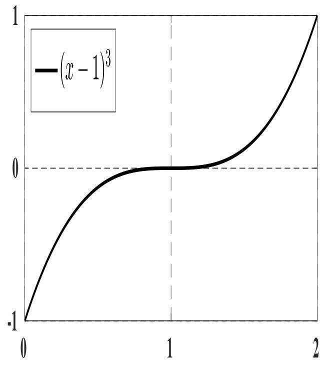
In this section, we discuss the difference between forts and non-differentiable points of a continuous function $f \in C(I,J)$.



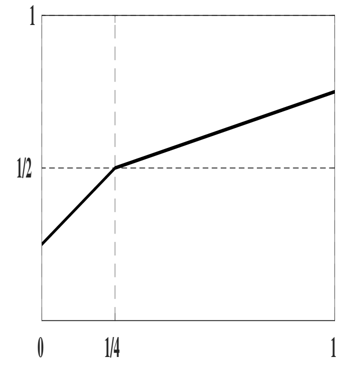
(a) $\frac{1}{2}$ is an isolated fort and a non Diff. point



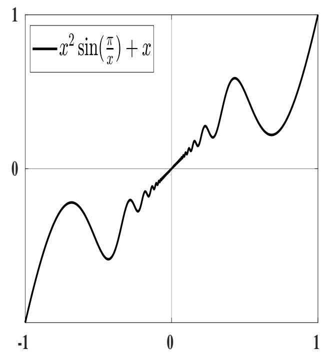
(b) $0 \in \Lambda(\phi_1)$ and $\phi_1'(0) = 0$



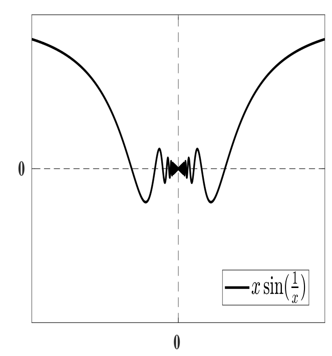
(c) $\phi_2'(1) = 0$ and $1 \notin \Lambda(\phi_2)$



(d) ϕ_3 is not Diff. at $\frac{1}{4}$ and $\frac{1}{4} \notin \Lambda(\phi_3)$



(e) $\phi_4'(0) = 1$ and $0 \in \Lambda^*(\phi_4)$



(f) ϕ_5 is not Diff. at 0 and $0 \in \Lambda^*(\phi_5)$

Figure 2.4 : Forts and non-differentiable points

Let $\phi_1 \in C(I,J)$. If $x \in \Lambda(\phi_1)$, then either $\phi_1'(x) = 0$ (if exists) or ϕ_1 is not differentiable at x. But the converse is not necessarily true. For example, $\phi_2'(1) = 0$ and $1 \notin \Lambda(\phi_2)$ for the function $\phi_2(x) := (x-1)^3$ on $[0,2]$ (see Figure 2.4(c)). Consider a function $\phi_3 : [0,1] \to [0,1]$ defined by

$$\phi_3(x) := \begin{cases} x + \frac{1}{4}, & \text{if } x \in [0, \frac{1}{4}], \\ \frac{5}{12} + \frac{x}{3}, & \text{if } x \in [\frac{1}{4}, 1]. \end{cases}$$

Here ϕ_3 is not differentiable at $\frac{1}{4}$ and strictly increasing on $[0,1]$ (see Figure 2.4(d)).

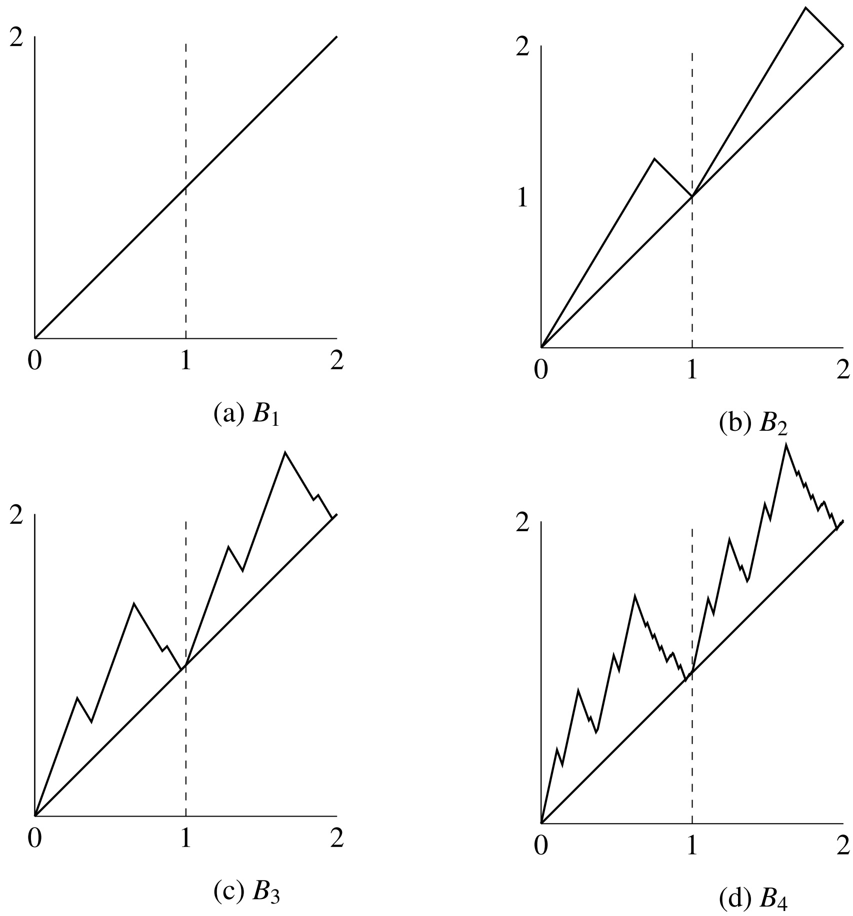
Let $\phi_4 \in C(I,J)$. If $x^* \in \Lambda^*(\phi_4)$, then we cannot conclude that either ϕ_4 is not differentiable at x^* or $\phi_4'(x^*) = 0$ (if exists). Consider the function ϕ_4 defined on $[-1,1]$ by

$$\phi_4(x) := \begin{cases} 0, & \text{if } x = 0, \\ x^2 \sin(\frac{\pi}{x}) + x, & \text{if } x \neq 0. \end{cases}$$

Here 0 is a non-isolated fort of ϕ_4 but $\phi_4'(0) = 1$ (see Figure 2.4(e)), and for the function

$$\phi_5(x) := \begin{cases} 0, & \text{if } x = 0, \\ x\sin(\frac{1}{x}), & \text{if } x \neq 0, \end{cases}$$

0 is a non-isolated fort of ϕ_5 and ϕ_5 is not differentiable at 0 (see Figure 2.4(f)).



(a) B_1 (b) B_2 (c) B_3 (d) B_4

Figure 2.5 : Construction of the Bolzano function B on $[0,2]$

From the construction of the Bolzano function $B = \lim_{n\to\infty} B_n$ (see (Jarnicki and Pflug, 2015, pp. 65-67)), we observe that every point of I is a non-isolated fort of B (see Figure 2.5). However, B is not only the function having the set of non-isolated forts as the whole domain. The known behavior of B is continuous everywhere and nowhere differentiable on I. In fact, $\Lambda^*(f) = I$ for each continuous everywhere and nowhere differentiable function f defined on I. Otherwise, there exists $x \in I$ and $\varepsilon > 0$ such that f is strictly monotone on $N_\varepsilon(x)$. By Monotone differentiation theorem (Theorem 1.6.25 in Tao (2011)), f is differentiable almost everywhere in $N_\varepsilon(x)$, a contradiction to f is nowhere differentiable on I.

We conclude from Hunt (1994) that the set of all functions $f \in C(I,\mathbb{R})$ with the property that $\Lambda^*(f) = I$ is dense in $C(I,\mathbb{R})$. In Lynch (2013), Lynch gave an example of a continuous function on $[0,1]$, which is differentiable only at rationals, and hence it has the whole domain as the set of non-isolated forts by Monotone differentiation theorem. Also, Katznelson and Stromberg (1974) constructed a function H, which is everywhere differentiable on $\mathbb{R}$ and $\Lambda^*(H) = \mathbb{R}$. Recently, Ciesielski (2018) provided a simple construction of an everywhere differential function f on $\mathbb{R}$ with $\Lambda^*(f) = \mathbb{R}$. Thus the continuous function f on I is not necessarily nowhere differentiable when $\Lambda^*(f) = I$.

CHAPTER 3

CHARACTERIZATION OF NON-ISOLATED FORTS

To study the existence of continuous solutions of $f^n = F$ for a given continuous function F on an interval I, we need to characterize the set of isolated and non-isolated forts of iterates of f. Liu et al. (2012) defined a fort for a continuous function in (a,b) and characterized the set of forts of the composition of continuous functions f and g ($f \circ g$) as the union of forts of g and inverse image of forts of f under g in (a,b). This characterization is not necessarily true for the set of forts of $f \circ g$ in an arbitrary interval I (i.e., the set $S(f \circ g)$ is not necessarily equal to $S(g) \cup g^{-1}(S(f))$). Moreover, there is no study on the characterization of $\Lambda^*(f^k)$, $f \in C(I)$, $k \in \mathbb{N}$. In this chapter, we characterize the sets of isolated and non-isolated forts of the composition of continuous functions. Applying the characterization result, we obtain an uncountable measure zero dense set of non-isolated forts in $\mathbb{R}$.

3.1 FORTS OF COMPOSITION OF CONTINUOUS FUNCTIONS

3.1.1 Non-monotone behavior of forts under composition

Let I_1, I_2 and I_3 be any intervals in $\mathbb{R}$ with non-empty interior and $\mathrm{int}(I_1)$ denotes the interior of I_1.

Theorem 3.1.1. *Let $f_1 \in C(I_1, I_2)$ and $f_2 \in C(I_2, I_3)$. Then the following hold:*

(i) $S(f_1) \subseteq S(f_2 \circ f_1)$.

(ii) $S(f_2 \circ f_1) \subseteq S(f_1) \cup f_1^{-1}(S(f_2))$.

(iii) *If $f_1^{-1}(S(f_2)) \subseteq \mathrm{int}(I_1)$, then $S(f_2 \circ f_1) = S(f_1) \cup f_1^{-1}(S(f_2))$.*

(iv) *Let $x \in \Lambda(f_2)$ and $y \in f_1^{-1}(\{x\})$. If $y \in \mathrm{int}(I_1)$ with $y \notin \Lambda^*(f_1)$, then $y \in \Lambda(f_2 \circ f_1)$.*

(v) $\Lambda^*(f_1) \subseteq \Lambda^*(f_2 \circ f_1)$.

(vi) *Let* $x \in \Lambda^*(f_2)$ *and* $y \in f_1^{-1}(\{x\})$. *If* $y \in \mathrm{int}(I_1)$ *with* $y \notin S(f_1)$, *then* $y \in \Lambda^*(f_2 \circ f_1)$.

Proof. (i) The result follows from Proposition 2.1.1.

(ii) Let $x \in S(f_2 \circ f_1)$ and $x \notin S(f_1)$. Then for each $\varepsilon > 0$, we have

$$f_1(N_\eta(x)) \subseteq N_\varepsilon(f_1(x))$$

and f_1 is strictly monotone on $N_\eta(x)$ for some $\eta > 0$. Since $x \in S(f_2 \circ f_1)$ and f_1 is strictly monotone on $N_\eta(x)$, f_2 is not strictly monotone in $f_1(N_\eta(x)) \subseteq N_\varepsilon(f_1(x))$. This implies $f_1(x) \in S(f_2)$.

(iii) Let $y \in f_1^{-1}(S(f_2)) \cap \mathrm{int}(I_1)$ with $y \notin S(f_1)$. Then there exists $\varepsilon > 0$ such that f_1 is strictly monotone on $N_\varepsilon(y)$ and $f_1(N_\varepsilon(y))$ is a neighborhood of $f_1(y)$. Since $f_1(y) \in S(f_2)$, there exist $x_1, x_2 \in f_1(N_\varepsilon(y))$ with $x_1 \neq x_2$ such that $f_2(x_1) = f_2(x_2)$ by Proposition 2.1.1. By the fact that f_1 is strictly monotone on $N_\varepsilon(y)$, we get distinct

$$y_1, y_2 \in f_1^{-1}(\{x_1, x_2\}) \cap N_\varepsilon(y) \text{ such that } f_2(f_1(y_1)) = f_2(f_1(y_2)).$$

Again by Proposition 2.1.1, $y \in S(f_2 \circ f_1)$. Thus by results (i) and (ii),

$$S(f_2 \circ f_1) = S(f_1) \cup f_1^{-1}(S(f_2)).$$

(iv) Let $x \in \Lambda(f_2)$ and $y \in f_1^{-1}(\{x\}) \cap \mathrm{int}(I_1)$ with $y \notin \Lambda^*(f_1)$. It follows from result (iii) that $y \in S(f_2 \circ f_1)$. Since $x \in \Lambda(f_2)$, there exists $\varepsilon > 0$ such that

$$S(f_2) \cap N_\varepsilon(x) = \{x\}. \tag{3.1.1}$$

Since $y \notin \Lambda^*(f_1)$, there exists $\delta > 0$ such that

$$f_1(N_\delta(y)) \subseteq N_\varepsilon(x) \text{ and } S(f_1) \cap N_\delta(y) \subseteq \{y\}. \tag{3.1.2}$$

This implies $f_1(y') \neq f_1(y)$ for all $y' \in N_\delta(y)$ and $y' \neq y$. Suppose that $y \in \Lambda^*(f_2 \circ f_1)$. Then there is $y_\delta \in N_\delta(y) \cap S(f_2 \circ f_1)$ such that $y_\delta \neq y$. Thus by result (ii) and (3.1.2), we have

$$f_1(y_\delta) \in N_\varepsilon(x) \cap S(f_2) \text{ with } f_1(y_\delta) \neq x,$$

contrary to (3.1.1). Therefore $y \in \Lambda(f_2 \circ f_1)$.

(v) Let $x \in \Lambda^*(f_1)$ and $\varepsilon > 0$. By Lemma 2.1.3, there exist three distinct points

$x_1, x_2, x_3 \in N_\varepsilon(x)$ such that $f_1(x_1) = f_1(x_2) = f_1(x_3)$. This implies

$$f_2(f_1(x_1)) = f_2(f_1(x_2)) = f_2(f_1(x_3)).$$

Thus again by Lemma 2.1.3, $x \in \Lambda^*(f_2 \circ f_1)$.

(vi) Let $x \in \Lambda^*(f_2)$ and $y \in f_1^{-1}(\{x\}) \cap \operatorname{int}(I_1)$ with $y \notin S(f_1)$. Then for each $\varepsilon > 0$, f_1 is strictly monotone on $N_{\varepsilon'}(y)$ and $f_1(N_{\varepsilon'}(y))$ is a neighborhood of $f_1(y) = x$ for some $\varepsilon' \leq \varepsilon$. As $x \in \Lambda^*(f_2)$, we get $x_0 \in S(f_2) \cap f_1(N_{\varepsilon'}(y))$ with $x_0 \neq x$. Since f_1 is strictly monotone on $N_{\varepsilon'}(y)$, there exists a point

$$y_0 \in f_1^{-1}(\{x_0\}) \cap N_{\varepsilon'}(y) \text{ such that } y_0 \neq y.$$

Note that $y_0 \in \operatorname{int}(I_1)$. Thus by result (iii), $y_0 \in S(f_2 \circ f_1)$. Hence $y \in \Lambda^*(f_2 \circ f_1)$. □

In view of statements (iii), (iv) and (vi) of Theorem 3.1.1, the following example shows that the elements of $f_1^{-1}(\{x\})$ for some $x \in S(f_2)$ (resp. $x \in \Lambda^*(f_2)$) are not necessarily in $S(f_2 \circ f_1) \setminus S(f_1)$ (resp. $\Lambda^*(f_2 \circ f_1) \setminus \Lambda^*(f_1)$).

Example 3.1.2. *Consider the continuous functions $f_2, f_1 : [\frac{-\pi}{16}, \frac{\pi}{8}] \to [\frac{-\pi}{16}, \frac{\pi}{8}]$ defined by*

$$f_2(x) := \begin{cases} \frac{\pi}{16} + \left| x \sin\left(\frac{\pi}{x}\right) \right|, & \text{if } x \in [\frac{-\pi}{16}, 0), \\ \frac{\pi}{16} - \frac{x}{2}, & \text{if } x \in [0, \frac{\pi}{8}], \end{cases}$$

and

$$f_1(x) := \begin{cases} x + \frac{\pi}{16}, & \text{if } x \in [\frac{-\pi}{16}, \frac{-\pi}{32}], \\ -x, & \text{if } x \in [\frac{-\pi}{32}, 0], \\ x, & \text{if } x \in [0, \frac{\pi}{32}], \\ \frac{5\pi}{96} - \frac{2x}{3}, & \text{if } x \in [\frac{\pi}{32}, \frac{\pi}{8}]. \end{cases}$$

Then

$$(f_2 \circ f_1)(x) = \begin{cases} \frac{\pi}{32} - \frac{x}{2}, & \text{if } x \in [\frac{-\pi}{16}, \frac{-\pi}{32}), \\ \frac{x}{2} + \frac{\pi}{16}, & \text{if } x \in [\frac{-\pi}{32}, 0), \\ \frac{\pi}{16} - \frac{x}{2}, & \text{if } x \in [0, \frac{\pi}{32}), \\ \frac{x}{3} + \frac{7\pi}{192}, & \text{if } x \in [\frac{\pi}{32}, \frac{5\pi}{64}], \\ \frac{\pi}{16} + \left| (\frac{5\pi}{96} - \frac{2x}{3}) \sin\left(\frac{\pi}{\frac{5\pi}{96} - \frac{2x}{3}}\right) \right|, & \text{if } x \in (\frac{5\pi}{64}, \frac{\pi}{8}]. \end{cases}$$

Here $0 \in \Lambda^*(f_2)$, $\frac{-\pi}{16}$ is an endpoint of $[\frac{-\pi}{16}, \frac{\pi}{8}]$, and $\frac{-\pi}{16}, 0 \in f_1^{-1}(\{0\})$. However, $\frac{-\pi}{16} \notin S(f_2 \circ f_1)$ and $0 \notin \Lambda^*(f_2 \circ f_1) = \{\frac{5\pi}{64}\}$ (see Figures 3.1 and 3.2).

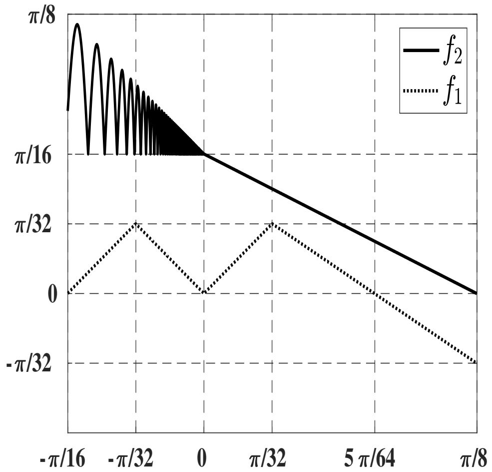


Figure 3.1 : $\frac{-\pi}{16}, 0 \in f_1^{-1}(\{0\})$

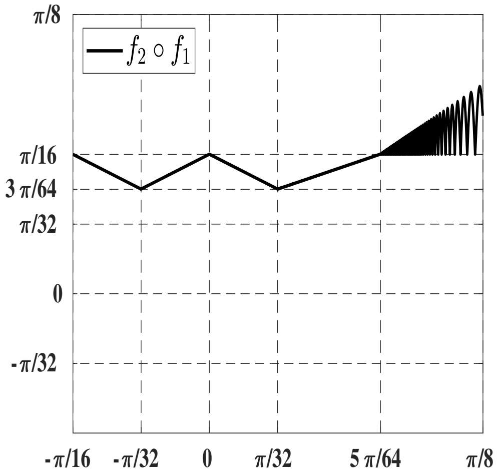


Figure 3.2 : $\frac{-\pi}{16} \notin S(f_2 \circ f_1)$ and $0 \notin \Lambda^*(f_2 \circ f_1)$

A sufficient condition on each $x \in \Lambda^*(f_2)$ such that $f_1^{-1}(\{x\}) \subseteq \Lambda^*(f_2 \circ f_1)$ requires the following definition. For $f \in C(I,J)$, define

$$\Lambda_L^*(f) := \{x \in \Lambda^*(f) : x = \lim_{n\to\infty} x_n, \text{ where } x_n \in S(f) \text{ and } x_n < x, \forall n \in \mathbb{N}\}$$

and

$$\Lambda_R^*(f) := \{x \in \Lambda^*(f) : x = \lim_{n\to\infty} x_n, \text{ where } x_n \in S(f) \text{ and } x_n > x, \forall n \in \mathbb{N}\}.$$

Example 3.1.3. *1. For the functions f_1 and f_2 as defined in Example 3.1.2, we have $\Lambda_R^*(f_2 \circ f_1) = \{\frac{5\pi}{64}\}$ and $\Lambda_L^*(f_2) = \{0\}$ (see Figures 3.1 and 3.2).*

2. *For a constant function* $f : [0,1] \to [1,2]$ *defined by* $f(x) := 1$, $\Lambda_R^*(f) = [0,1)$, $\Lambda_L^*(f) = (0,1]$ *and* $\Lambda_L^*(f) \cap \Lambda_R^*(f) = (0,1)$.

Lemma 3.1.4. *Let* $f \in C(I,J)$ *and* $x_0 \in \Lambda^*(f)$. *Then the following hold:*

(i) $x_0 \in \Lambda_L^*(f)$ *if and only if for each* $\varepsilon > 0$ *there exist distinct* $x_1, x_2, x_3 \in (x_0 - \varepsilon, x_0]$ *such that* $f(x_1) = f(x_2) = f(x_3)$.

(ii) $x_0 \in \Lambda_R^*(f)$ *if and only if for each* $\varepsilon > 0$ *there exist distinct* $x_1', x_2', x_3' \in [x_0, x_0 + \varepsilon)$ *such that* $f(x_1') = f(x_2') = f(x_3')$.

Proof. Let $x_0 \in \Lambda^*(f)$ and $\varepsilon > 0$. Then the proof of (i) and (ii) follow by applying Lemma 2.1.3 for the functions $g_1 \in C((x_0 - \varepsilon, x_0], J)$ and $g_2 \in C([x_0, x_0 + \varepsilon), J)$ respectively, where $g_1 := f|_{(x_0-\varepsilon,x_0]}$ and $g_2 := f|_{[x_0,x_0+\varepsilon)}$. □

The following lemma imposes the condition on a non-isolated fort x of f_2 such that $f_1^{-1}(\{x\}) \subseteq \Lambda^*(f_2 \circ f_1)$.

Lemma 3.1.5. *Let* $f_1 \in C(I_1, I_2), f_2 \in C(I_2, I_3)$ *and* $x \in \Lambda^*(f_2)$. *If* $x \in \Lambda_L^*(f_2) \cap \Lambda_R^*(f_2)$, *then* $f_1^{-1}(\{x\}) \subseteq \Lambda^*(f_2 \circ f_1)$.

Proof. Let $y \in f_1^{-1}(\{x\})$ and $\delta > 0$. If f_1 is a constant function on $N_\delta(y)$, then we have $y \in \Lambda^*(f_2 \circ f_1)$ by Lemma 2.1.3. Otherwise, by the continuity of f_1, there exists $\varepsilon > 0$ such that

$$\text{either } (x-\varepsilon, x] \subseteq f_1(N_\delta(y)) \text{ or } [x, x+\varepsilon) \subseteq f_1(N_\delta(y)).$$

Since $x \in \Lambda_L^*(f_2) \cap \Lambda_R^*(f_2)$, by Lemma 3.1.4 (i) and (ii), we have three distinct $x_1, x_2, x_3 \in (x-\varepsilon, x]$ and $x_1', x_2', x_3' \in [x, x+\varepsilon)$ such that

$$f_2(x_1) = f_2(x_2) = f_2(x_3) \text{ and } f_2(x_1') = f_2(x_2') = f_2(x_3'). \tag{3.1.3}$$

Now, choose $y_1, y_2, y_3 \in N_\delta(y)$ such that either

$$f_1(y_1) = x_1, f_1(y_2) = x_2 \text{ and } f_1(y_3) = x_3 \text{ (for } (x-\varepsilon, x] \subseteq f_1(N_\delta(y)))$$

or

$$f_1(y_1) = x_1', f_1(y_2) = x_2' \text{ and } f_1(y_3) = x_3' \text{ (for } [x, x+\varepsilon) \subseteq f_1(N_\delta(y))).$$

Clearly, y_1, y_2, y_3 are distinct in $N_\delta(y)$ and by (3.1.3),

$$f_2(f_1(y_1)) = f_2(f_1(y_2)) = f_2(f_1(y_3)).$$

Therefore $y \in \Lambda^*(f_2 \circ f_1)$ by Lemma 2.1.3. □

Here we remark that the converse of Lemma 3.1.5 is not necessarily true. For example, consider the continuous functions $f_1, f_2 : [\frac{-\pi}{16}, \frac{\pi}{8}] \to [\frac{-\pi}{16}, \frac{\pi}{8}]$ defined by

$$f_1(x) := \begin{cases} -(x+\frac{\pi}{32}), & \text{if } x \in [\frac{-\pi}{16}, 0), \\ x - \frac{\pi}{32}, & \text{if } x \in [0, \frac{3\pi}{32}), \\ \frac{5\pi}{32} - x, & \text{if } x \in [\frac{3\pi}{32}, \frac{\pi}{8}], \end{cases}$$

and

$$f_2(x) := \begin{cases} \frac{\pi}{16} + \left| x \sin\left(\frac{\pi}{x}\right) \right|, & \text{if } x \in [\frac{-\pi}{16}, 0), \\ \frac{\pi}{16} - \frac{x}{2}, & \text{if } x \in [0, \frac{\pi}{8}]. \end{cases}$$

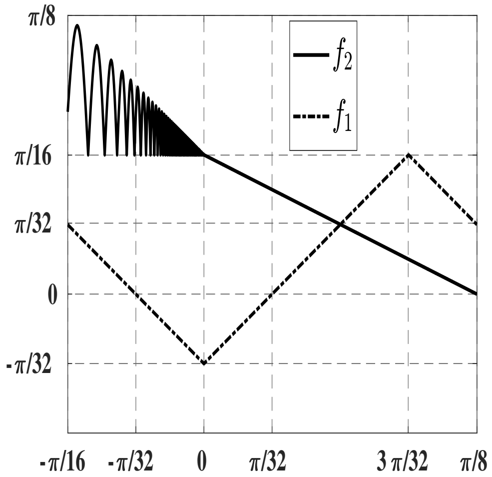


Figure 3.3 : $0 \in \Lambda_L^*(f_2)$ and $0 \notin \Lambda_L^*(f_2) \cap \Lambda_R^*(f_2)$

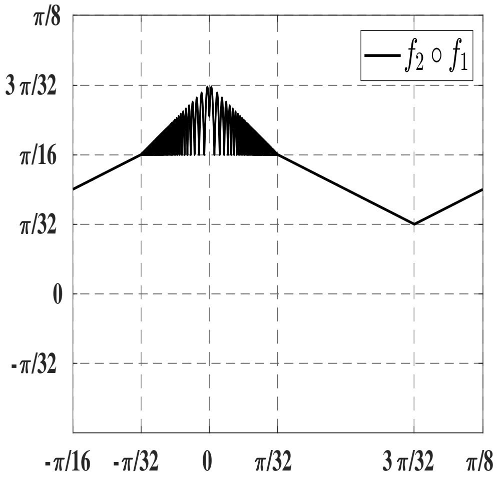


Figure 3.4 : $f_1^{-1}(\{0\}) \subseteq \Lambda^*(f_2 \circ f_1)$

Then

$$(f_2\circ f_1)(x)=\begin{cases}\frac{x}{2}+\frac{5\pi}{64}, & \text{if } x\in[\frac{-\pi}{16},\frac{-\pi}{32}],\\ \frac{\pi}{16}+\left|-(x+\frac{\pi}{32})\sin\left(\frac{\pi}{-(x+\frac{\pi}{32})}\right)\right|, & \text{if } x\in(\frac{-\pi}{32},0],\\ \frac{\pi}{16}+\left|(x-\frac{\pi}{32})\sin\left(\frac{\pi}{(x-\frac{\pi}{32})}\right)\right|, & \text{if } x\in[0,\frac{\pi}{32}),\\ \frac{5\pi}{64}-\frac{x}{2}, & \text{if } x\in[\frac{\pi}{32},\frac{3\pi}{32}],\\ \frac{x}{2}-\frac{\pi}{64}, & \text{if } x\in(\frac{3\pi}{32},\frac{\pi}{8}].\end{cases}$$

The point $0\in\Lambda^*(f_2)$,

$$f_1^{-1}(\{0\})=\left\{\frac{-\pi}{32},\frac{\pi}{32}\right\} \text{ and } \Lambda^*(f_2\circ f_1)=\left\{\frac{-\pi}{32},\frac{\pi}{32}\right\}.$$

However $0\notin\Lambda_L^*(f_2)\cap\Lambda_R^*(f_2)$ (see Figures 3.3 and 3.4).

3.1.2 Characterization of $\Lambda^*(f_2\circ f_1)$, $\Lambda(f_2\circ f_1)$ and $S(f_2\circ f_1)$

Now, let us capture the points of f_1, which change their monotonicity from monotone to isolated or non-isolated and isolated to non-isolated under composition with f_2.

Definition 3.1.6. *For $x\in S(f_2)$, define*

$$Q_x(f_2,f_1):=\{y\in\Lambda(f_2\circ f_1): f_1(y)=x \text{ and } y\notin\Lambda(f_1)\}$$

and

$$P_x(f_2,f_1):=\{y\in\Lambda^*(f_2\circ f_1): f_1(y)=x \text{ and } y\notin\Lambda^*(f_1)\}.$$

For the functions f_1,f_2 in Example 3.1.2, we have $P_0(f_2,f_1)=\{\frac{5\pi}{64}\}$.
We denote

$$P(f_2,f_1)=\bigcup_{x\in\Lambda^*(f_2)}P_x(f_2,f_1) \text{ and } Q(f_2,f_1)=\bigcup_{x\in\Lambda(f_2)}Q_x(f_2,f_1).$$

It is observed from Theorem 3.1.1 (v) that every non-isolated forts of f_1 are non-isolated forts of $f_2\circ f_1$. The set $P_x(f_2,f_1)$ is the collection of remaining non-isolated forts of $f_2\circ f_1$, which are the inverse images of a non-isolated fort x of f_2 under f_1.

The following theorem determines the set $P_x(f_2,f_1)$ for $x\in\Lambda_L^*(f_2)\cap\Lambda_R^*(f_2)$, $x\in\Lambda_L^*(f_2)\setminus\Lambda_R^*(f_2)$, and $x\in\Lambda_R^*(f_2)\setminus\Lambda_L^*(f_2)$.

Theorem 3.1.7. *Let $f_1 \in C(I_1, I_2), f_2 \in C(I_2, I_3)$ and $x \in \Lambda^*(f_2)$. Then the following hold:*

(i) *If $x \in \Lambda_L^*(f_2) \cap \Lambda_R^*(f_2)$, then $P_x(f_2, f_1) = \{y \in I_1 \setminus \Lambda^*(f_1) : f_1(y) = x\}$.*

(ii) *If $x \in \Lambda_L^*(f_2)$ and $x \notin \Lambda_R^*(f_2)$, then $P_x(f_2, f_1) = \{y \in I_1 \setminus \Lambda^*(f_1) :\ f_1(y) = x$ and $f_1(y') \leq x$ for all $y' \in (y-\delta, y]$ or $y' \in [y, y+\delta)$ for some $\delta > 0\}$.*

(iii) *If $x \in \Lambda_R^*(f_2)$ and $x \notin \Lambda_L^*(f_2)$, then $P_x(f_2, f_1) = \{y \in I_1 \setminus \Lambda^*(f_1) : f_1(y) = x$ and $f_1(y') \geq x$ for all $y' \in (y-\delta, y]$ or $y' \in [y, y+\delta)$ for some $\delta > 0\}$.*

Proof. (i) For $x \in \Lambda_L^*(f_2) \cap \Lambda_R^*(f_2)$, it follows from Definition 3.1.6 that

$$P_x(f_2, f_1) \subseteq \{y \in I_1 \setminus \Lambda^*(f_1) : f_1(y) = x\}.$$

The other inclusion of (i) follows from Lemma 3.1.5.

(ii) Let

$$A = \{y \in I_1 \setminus \Lambda^*(f_1) : f_1(y) = x \text{ and } f_1(y') \leq x \text{ for all } y' \in (y-\delta, y] \text{ or } y' \in [y, y+\delta)\}.$$

Let $y \in P_x(f_2, f_1)$. Then either $y \notin S(f_1)$ or $y \in \Lambda(f_1)$. If $y \notin S(f_1)$ with $y \in \mathrm{int}(I_1)$, there exists $\delta > 0$ such that f_1 is strictly monotone in $N_\delta(y)$ and

$$\text{either } f_1(y') \leq f_1(y) = x, \forall y' \in (y-\delta, y] \text{ or } f_1(y') \leq x, \forall y' \in [y, y+\delta).$$

Thus $y \in A$. If $y \in \Lambda(f_1)$ or y is an endpoint of I_1, then there is a $\delta > 0$ such that

$$\text{either } f_1(y') \leq x, \forall y' \in N_\delta(y) \text{ or } f_1(y') \geq x, \forall y' \in N_\delta(y).$$

Suppose that $f_1(y') \geq x$ for all $y' \in N_\delta(y)$. By the continuity of f_1 and the fact $x \in \Lambda_L^*(f_2)$ with $x \notin \Lambda_R^*(f_2)$, we have

$$f_1(N_{\delta'}(y)) \subseteq [x, x+\varepsilon),$$

and f_2 is strictly monotone in $[x, x+\varepsilon)$ for some $\varepsilon > 0$ and $\delta' \leq \delta$ (see Figure 3.5). This implies that

$$S(f_2 \circ f_1) \cap N_{\delta'}(y) \subseteq \{y\},$$

which contradicts the fact that $y \in \Lambda^*(f_2 \circ f_1)$. Therefore $f_1(y') \leq x$ for all $y' \in N_\delta(y)$. Hence $y \in A$.

To prove the other inclusion, let $y \in A$ and $\eta > 0$. Then the following two cases will occur:

Case 1. $y \notin S(f_1)$ and $y \in \mathrm{int}(I_1)$ and there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq f_1(N_\eta(y))$ and $f_1(N_\eta(y))$ is a neighborhood of x (see Figure 3.6).

Case 2: $y \in \Lambda(f_1)$ or y is an endpoint of I_1 with $(x-\varepsilon, x] \subseteq f_1(N_\eta(y))$ for some $\varepsilon > 0$ (see Figure 3.7).

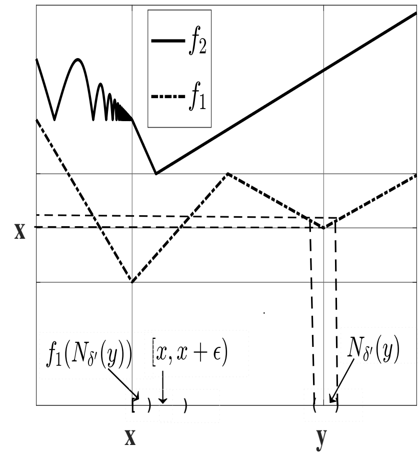


Figure 3.5 : $y \in \Lambda(f_1)$ and $f_1(y') \geq x$ for all $y' \in N_{\delta'}(y)$

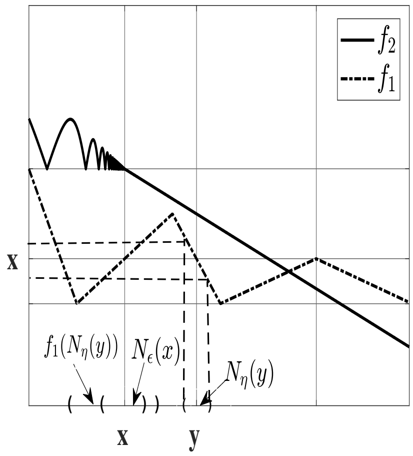


Figure 3.6 : Case 1, $y \notin S(f_1)$

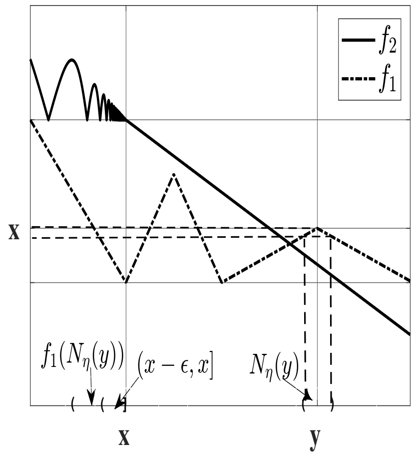


Figure 3.7 : Case 2, $y \in \Lambda(f_1)$

Since $x \in \Lambda_L^*(f_2) \setminus \Lambda_R^*(f_2)$, in both cases of y, by Lemma 3.1.4 (i), there exist distinct $x_1, x_2, x_3 \in (x-\varepsilon, x] \subseteq f_1(N_\eta(y))$ such that

$$f_2(x_1) = f_2(x_2) = f_2(x_3). \tag{3.1.4}$$

Choose $y_1, y_2, y_3 \in N_\eta(y)$ with $f_1(y_1) = x_1$, $f_1(y_2) = x_2$ and $f_1(y_3) = x_3$. Clearly,

y_1, y_2, y_3 are distinct, and by (3.1.4),

$$f_2(f_1(y_1)) = f_2(f_1(y_2)) = f_2(f_1(y_3)).$$

Thus by Lemma 2.1.3, we get $y \in \Lambda^*(f_2 \circ f_1)$. Hence $y \in P_x(f_2, f_1)$. This completes the proof of (ii).

The proof of (iii) is similar to that of (ii). □

Now, we characterize the sets $\Lambda^*(f_2 \circ f_1)$, $\Lambda(f_2 \circ f_1)$ and $S(f_2 \circ f_1)$ for any continuous functions $f_1 \in C(I_1, I_2)$ and $f_2 \in C(I_2, I_3)$ on an arbitrary intervals $I_1, I_2, I_3 \subseteq \mathbb{R}$.

Theorem 3.1.8. *Let $f_1 \in C(I_1, I_2)$ and $f_2 \in C(I_2, I_3)$. Then*

(i) $\Lambda^*(f_2 \circ f_1) = \Lambda^*(f_1) \bigcup P(f_2, f_1)$,

(ii) $\Lambda(f_2 \circ f_1) = (\Lambda(f_1) \setminus \Lambda^*(f_2 \circ f_1)) \bigcup Q(f_2, f_1)$,

(iii) $S(f_2 \circ f_1) = S(f_1) \bigcup P(f_2, f_1) \bigcup Q(f_2, f_1)$.

Proof. (i) From Theorem 3.1.1 (v) and Definition 3.1.6, we have

$$\Lambda^*(f_1) \subseteq \Lambda^*(f_2 \circ f_1) \text{ and } P(f_2, f_1) \subseteq \Lambda^*(f_2 \circ f_1).$$

For the other inclusion of (i), let $y \in \Lambda^*(f_2 \circ f_1)$ with $y \notin \Lambda^*(f_1)$ and $\varepsilon > 0$. By the continuity of f_1, there is a $\delta > 0$ such that f_1 is piecewise strictly monotone on $(y-\delta, y] \cup [y, y+\delta)$ and

$$f_1(N_\delta(y)) \subseteq N_\varepsilon(f_1(y)).$$

As $y \in \Lambda^*(f_2 \circ f_1)$, by Lemma 3.1.4 (i) and (ii), we get distinct

$$y_1, y_2, y_3 \in (y-\delta, y] \text{ (for } y \in \Lambda_L^*(f_2 \circ f_1))$$

or

$$y_1, y_2, y_3 \in [y, y+\delta) \text{ (for } y \in \Lambda_R^*(f_2 \circ f_1))$$

such that

$$f_2(f_1(y_1)) = f_2(f_1(y_2)) = f_2(f_1(y_3)). \tag{3.1.5}$$

The fact f_1 is piecewise strictly monotone on $(y-\delta, y] \cup [y, y+\delta)$ implies that the points $f_1(y_1), f_1(y_2), f_1(y_3)$ are distinct in $N_\varepsilon(f_1(y))$. By (3.1.5) and Lemma 2.1.3, $f_1(y) \in \Lambda^*(f_2)$. Hence $y \in P_{f_1(y)}(f_2, f_1)$.

(ii) It follows from Definition 3.1.6 that

$$Q(f_2, f_1) \subseteq \Lambda(f_2 \circ f_1).$$

Now, for each $y \in \Lambda(f_1)$ and $y \notin \Lambda^*(f_2 \circ f_1)$, we have $y \in \Lambda(f_2 \circ f_1)$ by Theorem 3.1.1 (i). To prove the other inclusion, let $y \in \Lambda(f_2 \circ f_1)$ and $y \notin \Lambda(f_1)$. By Theorem 3.1.1 (ii) and (v), $y \in f_1^{-1}(S(f_2))$ and $y \notin S(f_1)$. Thus by Theorem 3.1.1 (vi), $f_1(y) \in \Lambda(f_2)$ and then $y \in Q_{f_1(y)}(f_2, f_1)$.

(iii) The proof follows from the Facts 2.0.2 (i) and (ii), results (i) and (ii), and Theorem 3.1.1 (i). □

Corollary 3.1.9. *Let $f \in C(I)$. Then for any integer $k \geq 2$*

(i) $\Lambda^*(f^k) = \Lambda^*(f) \bigcup P(f^{k-1}, f)$,

(ii) $\Lambda(f^k) = (\Lambda(f) \setminus \Lambda^*(f^k)) \bigcup Q(f^{k-1}, f)$,

(iii) $S(f^k) = S(f) \cup P(f^{k-1}, f) \cup Q(f^{k-1}, f)$.

Proof. Take $f_1 = f$ and $f_2 = f^{k-1}$ in Theorem 3.1.8 for any integer $k \geq 2$. □

Corollary 3.1.10. *Let $f \in C(I)$. Then for any integer $k \geq 2$*

(i) $\Lambda^*(f^k) = \Lambda^*(f^{k-1}) \bigcup P(f, f^{k-1})$,

(ii) $\Lambda(f^k) = (\Lambda(f^{k-1}) \setminus \Lambda^*(f^k)) \bigcup Q(f, f^{k-1})$,

(iii) $S(f^k) = S(f^{k-1}) \cup P(f, f^{k-1}) \cup Q(f, f^{k-1})$.

Proof. Take $f_1 = f^{k-1}$ and $f_2 = f$ in Theorem 3.1.8 for any integer $k \geq 2$. □

The following theorem describes the cardinality of $\Lambda^*(f_2 \circ f_1)$ in terms of the cardinality of $\Lambda^*(f_1)$ and $\Lambda^*(f_2)$.

Theorem 3.1.11. *Let $f_1 \in C(I_1, I_2)$ and $f_2 \in C(I_2, I_3)$. If $\Lambda^*(f_1)$ and $\Lambda^*(f_2)$ are countable, then $\Lambda^*(f_2 \circ f_1)$ is countable.*

Proof. Suppose that $\Lambda^*(f_2 \circ f_1)$ is uncountable. Since $\Lambda^*(f_2)$ and $\Lambda^*(f_1)$ are countable, by Theorem 3.1.8 (i), $P_x(f_2, f_1)$ is uncountable for some $x \in \Lambda^*(f_2)$. If $P_x(f_2, f_1)$ contains an interval $I' \subseteq I_1$, then f_1 is constant on I'. By Lemma 2.1.3, $I' \subseteq \Lambda^*(f_1)$ a contradiction to the countability of $\Lambda^*(f_1)$.

Suppose that $P_x(f_2, f_1)$ does not contain any interval. By the continuity of f_1, it has a fort between any two points in $P_x(f_2, f_1)$. Since the interval I is second countable and $P_x(f_2, f_1)$ is an uncountable subset of I_1, uncountable many points of $P_x(f_2, f_1)$ are limit points of $P_x(f_2, f_1)$. Let $y \in P_x(f_2, f_1)$ be a limit point of $P_x(f_2, f_1)$. Then there is a sequence $\{y_n\}$ of distinct points in $P_x(f_2, f_1)$ such that $y_n < y_{n+1}$ or $y_n > y_{n+1}$ for all $n \in \mathbb{N}$ and $y_n \to y$. Now, choose $x_n \in S(f_1)$ with $y_n < x_n < y_{n+1}$ or $y_{n+1} < x_n < y_n$. Thus $x_n \to y$ as $y_n \to y$. Hence by Fact 2.0.2 (iii), $y \in \Lambda^*(f_1) \cap P_x(f_2, f_1)$, a contradiction to the fact $\Lambda^*(f_1) \cap P_x(f_2, f_1) = \emptyset$ for all $x \in \Lambda^*(f_2)$. Therefore $\Lambda^*(f_2 \circ f_1)$ is countable. □

3.2 MEASURE ZERO DENSE SET OF NON-ISOLATED FORTS

The most well-known and interesting example of an uncountable measure zero nowhere dense set in the real line $\mathbb{R}$ is the Cantor ternary set $\mathscr{C}$. Followed by Cantor, many authors studied the generalized construction of $\mathscr{C}$, each of these sets is called a Cantor set, which has positive or zero measure with all other properties of $\mathscr{C}$ (see Vallin (2013) and the references therein).

One of the challenging problems in real analysis is to find a set in $\mathbb{R}$, which is uncountable measure zero dense whose complement also uncountable and dense. Such a set was given as a dense G_δ subset of $\mathbb{R}$ (see Theorem 1.6 in Oxtoby (1980)). Recently, Ho and Zimmerman (2018) produced such sets, using the decimal expansion of real numbers, as a finite union of dense sets in $\mathbb{R}$.

In this section, we present such sets using the non-isolated forts. First we construct a continuous function T on $[0,1]$ such that

$$\Lambda^*(T) = \mathscr{C}$$

and $\bigcup_{i=1}^{\infty} \Lambda^*(T^i)$ is uncountable measure zero dense in $[0,1]$. Extend T periodically to obtain a function T_0 on $\mathbb{R}$ defined by $T_0(x+1) = T_0(x)$ for all $x \in \mathbb{R} \setminus [0,1]$. Then $\bigcup_{i=1}^{\infty} \Lambda^*(T_0^i)$ is uncountable measure zero dense in $\mathbb{R}$.

3.2.1 Construction of a function on the Cantor set

Let $D_n, D_{n,k}$ and C_n be as defined in Example 2.1.6. For $n \in \mathbb{N}$, define $T_n : [0,1] \to [0,1]$ by

$$T_n(x) := \begin{cases} 0, & \text{if } x \in C_n, \\ T_{(n-1),l}(x), & \text{if } x \in D_{n,k},\ k = 2l \text{ for some } l \in \{1, \dots, 2^{n-1}-1\}, \\ T_{n,k}(x), & \text{if } x \in D_{n,k},\ k \text{ odd}, \end{cases}$$

and

$$T_{n,k}(x) := \begin{cases} 2x - \frac{2x_{n,k}}{3^n}, & \text{if } x \in \left(\frac{x_{n,k}}{3^n}, y_{n,k}\right), \\ -2x + \frac{2(x_{n,k}+1)}{3^n}, & \text{if } x \in \left[y_{n,k}, \frac{x_{n,k}+1}{3^n}\right), \end{cases} \tag{3.2.1}$$

where $D_{n,k} = \left(\frac{x_{n,k}}{3^n}, y_{n,k}\right) \cup \left[y_{n,k}, \frac{x_{n,k}+1}{3^n}\right)$ and $y_{n,k} = \frac{2x_{n,k}+1}{2\cdot 3^n}$ is the midpoint of $D_{n,k}$.

Clearly, each T_n is well-defined and continuous on $[0,1]$. From (3.2.1), observe that for each $k \in \{1, 3, \ldots, 2^n - 1\}$, we have

$$T_{n,k}(y_{n,k}) = \frac{1}{3^n} \text{ and } T_{n,k}(x) < \frac{1}{3^n},\ \forall x \in D_{n,k} \text{ and } x \neq y_{n,k}. \tag{3.2.2}$$

Since $T_n = T_{n-1}$ on D_{n-1},

$$\max_{\substack{x \in D_{n,k} \\ (\text{odd } k)}} T_n(x) = \frac{1}{3^n}. \tag{3.2.3}$$

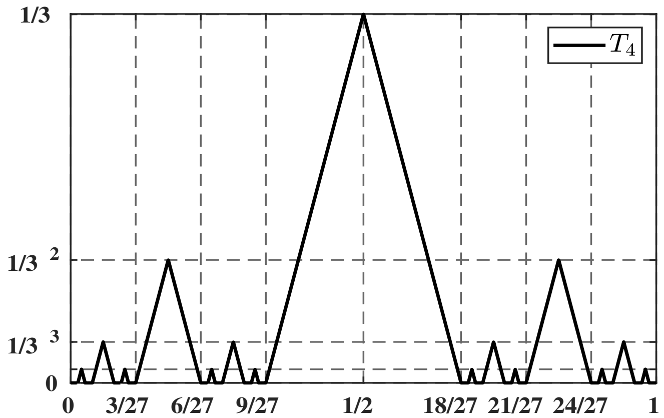


Figure 3.8 : T_4 on [0,1]

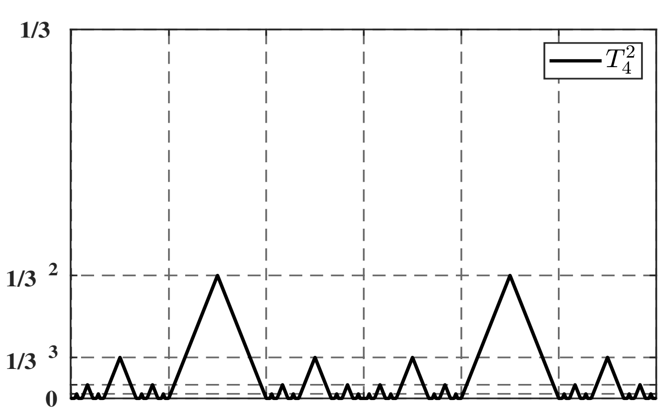


Figure 3.9 : T_4^2 on $[\frac{1}{3}, \frac{2}{3}]$

Define $T:[0,1]\to[0,1]$ by

$$T(x) := \begin{cases} 0, & \text{if } \quad x \in \mathscr{C}, \\ T_n(x), & \text{if } \quad x \in D_n. \end{cases} \tag{3.2.4}$$

Now, we discuss the properties of $\Lambda^*(T^i)$ and $\Lambda(T^i),\ i \in \mathbb{N}$.

Lemma 3.2.1. *Let T be the function as defined in* (3.2.4)*. Then the following holds:*

(i) *T is continuous on* $[0,1]$.

(ii) $\Lambda^*(T) = \mathscr{C}$.

(iii) $\Lambda^*(T^2) = \Lambda^*(T) \cup T^{-1}(\Lambda^*(T))$.

(iv) $\displaystyle\max_{x\in[0,1]} T^i(x) = \frac{1}{3^i}$ *for all $i \in \mathbb{N}$.*

(v) *For each $i \geq 1$, if $y \in \Lambda(T^i)$ and $y \notin \Lambda(T^{i-1})$, then $T^i(y) = \frac{1}{3^n}$ for some $n \in \mathbb{N}$.*

(vi) $\Lambda(T^i) \subseteq \Lambda^*(T^{i+1})$ *for all $i \in \mathbb{N}$.*

Proof. (i) Since $T_{n+1} = T_n$ on D_n, for each $x \in [0,1]$, $T_n(x)$ is an eventually constant sequence. Hence $T_n(x)$ converges to $T(x)$. Since $D_n \subsetneq D_{n+1}$, by (3.2.3),

$$\max_{x\in[0,1]} |T_n(x) - T(x)| = \max_{\substack{x\in D_{n+1,k} \\ (\text{odd } k)}} |T_n(x) - T_{n+1}(x)| = \frac{1}{3^{n+1}} \to 0 \text{ as } n \to \infty.$$

Therefore T_n converges uniformly to T. Hence T is continuous on $[0,1]$.

(ii) Let $x \in \mathscr{C}$ and $\varepsilon > 0$. By the fact every point in $\mathscr{C}$ is non-isolated and $T(\mathscr{C}) = \{0\}$, there exist distinct points $x_1, x_2, x_3 \in \mathscr{C} \cap N_\varepsilon(x)$ such that

$$T(x_1) = T(x_2) = T(x_3) = 0.$$

Thus by Lemma 2.1.3, $x \in \Lambda^*(T)$. On the other hand, since $T = T_{n,k}$ is piecewise strictly monotone on $D_{n,k}$, $\Lambda^*(T) \cap D_{n,k} = \emptyset$ for any $n \in \mathbb{N}$ and $k = 1, \ldots, 2^n - 1$. Hence $\Lambda^*(T) = \mathscr{C}$.

Now, for each $n \in \mathbb{N}$, let L_n be the set of all left endpoints of $D_{n,k}$ and R_n be the set of all right endpoints of $D_{n,k},\ k = 1, 2, \ldots, 2^{n-1}$. It follows from the fact $T = T_{n,k}$ is piecewise strictly monotone on $D_{n,k}$ that

$$\Lambda^*_L(T) \backslash \Lambda^*_R(T) = \bigcup_{n\in\mathbb{N}} L_n \cup \{1\},\ \Lambda^*_R(T) \backslash \Lambda^*_L(T) = \bigcup_{n\in\mathbb{N}} R_n \cup \{0\}$$

and $\Lambda_L^*(T)\cap\Lambda_R^*(T)=\mathscr{C}\setminus((\Lambda_L^*(T)\setminus\Lambda_R^*(T))\cup(\Lambda_R^*(T)\setminus\Lambda_L^*(T)))$. By (3.2.2),

$$\Lambda(T)=\{y_{n,k}:n\in\mathbb{N},\ k=1,2,\ldots,2^n-1\}.$$

(iii) It is clear that $T^{-1}(\{0\})=\Lambda^*(T)$. Now, let $x\in\Lambda_L^*(T)\cap\Lambda_R^*(T)$ or $x\in\Lambda_R^*(T)\setminus\Lambda_L^*(T)$ with $x\neq 0$. Note that $x\notin\{\frac{1}{3^n}:n\in\mathbb{N}\}$. From the construction of T on $D_{n,k}$, we get

$$T^{-1}(\{x\})\cap S(T)=\emptyset.$$

So, for each $y\in T^{-1}(\{x\})$, we have

$$T(y')\geq x,\ \forall y'\in(y-\delta,y]\text{ or }y'\in[y,y+\delta)$$

for some $\delta>0$. Thus by Theorem 3.1.7 (i) and (iii),

$$T^{-1}(\{x\})=P_x(T,T).$$

Suppose that $x\in\Lambda_L^*(T)\setminus\Lambda_R^*(T)$. Then for each $y\in T^{-1}(\{x\})$, either $y\notin S(T)$ or there is a $\delta>0$ such that $T(y')\leq x$ for all $y'\in N_\delta(y)$ (see Figure 3.8). Thus by Theorem 3.1.7 (ii),

$$T^{-1}(\{x\})=P_x(T,T).$$

The reverse inclusion follows from the definition of $P_x(T,T)$. Thus by Corollary 3.1.9 (i),

$$\Lambda^*(T^2)=\Lambda^*(T)\cup T^{-1}(\Lambda^*(T)). \tag{3.2.5}$$

(iv) It is clear from (3.2.2) and (3.2.3) that $T(x)\leq\frac{1}{3}$ for all $x\in[0,1]$ and $T(\frac{1}{2})=\frac{1}{3}$. Assume that $\frac{1}{3^{i-1}}$ is the maximum value of T^{i-1}. Take an element $x\in T^{-(i-1)}(\{y_{i,1}\})$. Since $T_{i,1}(y_{i,1})=\frac{1}{3^i}$,

$$T^i(x)=T(T^{i-1}(x))=T_{i,1}(y_{i,1})=\frac{1}{3^i}.$$

Suppose that $T^i(x_0)=T(T^{i-1}(x_0))>\frac{1}{3^i}$ for some $x_0\in[0,1]$. Then from (3.2.2), we get $T^{i-1}(x_0)\in D_{i-1,k}$ (i.e., $T^{i-1}(x_0)>\frac{1}{3^{i-1}}$), a contradiction.

(v) The result is trivial for $i=1$ by (3.2.2) (see Figure 3.8). Assume that for $i-1$. Now, for $y\in\Lambda(T^i)\setminus\Lambda(T^{i-1})$, by Corollary 3.1.9 (ii), we have $T(y)=x$ for some $x\in\Lambda(T^{i-1})$. From assumption ($T^{i-1}(x)=\frac{1}{3^n}$ for some $n\in\mathbb{N}$), we have

$$T^i(y)=T^{i-1}(T(y))=T^{i-1}(x)=\frac{1}{3^n}.$$

(vi) Let $y \in \Lambda(T^i)$. Then for each $\varepsilon > 0$, by results (iv) and (v), we get

$$T^i(N_\varepsilon(y)) \subseteq \left[0, \frac{1}{3^i}\right] \text{ and } T^i(N_\varepsilon(y)) \cap \mathscr{C} \neq \emptyset.$$

Since $T(\mathscr{C}) = \{0\}$, there exist three distinct points $x_1, x_2, x_3 \in T^i(N_\varepsilon(y)) \cap \mathscr{C}$ such that

$$T(x_1) = T(x_2) = T(x_3) = 0.$$

Choose $y_1, y_2, y_3 \in N_\varepsilon(y)$ such that $T^i(y_1) = x_1$, $T^i(y_2) = x_2$ and $T^i(y_3) = x_3$. Then we get,

$$T^{i+1}(y_1) = T^{i+1}(y_2) = T^{i+1}(y_3) = 0.$$

Thus by Lemma 2.1.3, $y \in \Lambda^*(T^{i+1})$. □

In the following theorem, we present an uncountable measure zero dense set of non-isolated forts in $[0,1]$.

Theorem 3.2.2. *The set* $\Gamma = \bigcup_{i=1}^{\infty} \Lambda^*(T^i)$ *is dense of measure zero and each* $\Lambda^*(T^i)$ *is a Cantor type set in* $[0,1]$. *Consequently,* Γ *is uncountable of type* F_σ.

Proof. Let μ be the Lebesgue measure on $[0,1]$. To prove $\mu(\Gamma) = 0$, by the countable sub-additive of μ, it suffices to prove that $\mu(\Lambda^*(T^i)) = 0$ for every $i \in \mathbb{N}$. For $i = 1$, we have

$$\mu(\Lambda^*(T)) = \mu(\mathscr{C}) = 0.$$

Assume that $\mu(\Lambda^*(T^{i-1})) = 0$ for $i \geq 2$. From Definition 3.1.6 and Corollary 3.1.9 (i), we have

$$\Lambda^*(T^i) = \Lambda^*(T) \bigcup P(T^{i-1}, T) \subseteq \Lambda^*(T) \bigcup T^{-1}(\Lambda^*(T^{i-1})). \tag{3.2.6}$$

In order to prove $\mu(\Lambda^*(T^i)) = 0$, it suffices to prove

$$\mu(T^{-1}(\Lambda^*(T^{i-1})) \cap D_{n,k}) = 0, \ \forall n \in \mathbb{N} \text{ and } i \geq 2$$

by the fact $[0,1] = (\cup_n D_n) \cup \mathscr{C}$ and (3.2.6). For $k = 1, 3, \ldots, 2^n - 1$, from (3.2.2) we have

$$T^{-1} : \left[0, \frac{1}{3^n}\right] \to \left[\frac{x_{n,k}}{3^n}, y_{n,k}\right]$$

is an affine map by the fact T is an affine map in $\left[\frac{x_{n,k}}{3^n}, y_{n,k}\right]$. From the assumption that

$\mu(\Lambda^*(T^{i-1})) = 0$ and the fact μ is translation invariant, we get

$$\mu\left(T^{-1}(\Lambda^*(T^{i-1}))\bigcap\left[\frac{x_{n,k}}{3^n}, y_{n,k}\right]\right) = 0.$$

Similarly, we can prove that

$$\mu\left(T^{-1}(\Lambda^*(T^{i-1}))\bigcap\left[y_{n,k}, \frac{x_{n,k}+1}{3^n}\right]\right) = 0.$$

Thus by (3.2.6),

$$\mu\left(T^{-1}(\Lambda^*(T^{i-1}))\bigcap D_{n,k}\right) = 0 \text{ and } \mu(\Lambda^*(T^i)) = 0.$$

To prove the denseness of Γ, let $I' \subseteq [0,1]$ be an open interval with $\mu(I') = \frac{1}{3^i}$, $i \in \mathbb{N}$. Suppose $S(T) \cap I' \neq \emptyset$, then

$$\Lambda^*(T^2) \cap I' \neq \emptyset$$

by (3.2.5) and Lemma 3.2.1 (vi). Assume that $S(T) \cap I' = \emptyset$. From (3.2.1) and Lemma 3.2.1 (v), for each $j \in \mathbb{N}$,

$$\mu(T^j(I')) = \frac{2^j}{3^i}, \quad \text{whenever } T^{j-1}(I') \cap S(T) = \emptyset.$$

Since $T^{i-1}(I') \subseteq [0, \frac{1}{3^{i-1}}]$, by the fact any interval in $[0, \frac{1}{3^{i-1}}]$ of measure greater than or equal to $\frac{2}{3^i}$ intersect with $\Lambda^*(T)$,

$$T^j(I') \cap \Lambda^*(T) \neq \emptyset \text{ for some } j \in \{1, \ldots, i-1\}. \tag{3.2.7}$$

If $S(T^j) \cap I' \neq \emptyset$, then

$$\emptyset \neq S(T^j) \cap I' \subseteq \Lambda^*(T^{j+1}) \cap I'$$

by Theorem 3.1.1 (v) and Lemma 3.2.1 (vi). In the other case (i.e., $S(T^j) \cap I' = \emptyset$), by Theorem 3.1.1 (vi) and (3.2.7), we get

$$\emptyset \neq T^{-j}(\Lambda^*(T)) \cap I' \subseteq \Lambda^*(T^{j+1}) \cap I'.$$

Thus $\Gamma \cap I' \neq \emptyset$. Hence Γ is dense in $[0,1]$. Since $\Lambda^*(T) = \mathscr{C}$ (Lemma 3.2.1 (ii)) and each $\Lambda^*(T^i)$ is closed in $[0,1]$ (Proposition 2.1.4), Γ is uncountable of type F_σ. □

More properties of the set Γ

(i) $\Lambda^*(T^{i+1}) \setminus S(T^i)$ is uncountable for every $i \in \mathbb{N}$.

(ii) $\Gamma \cap I'$ is uncountable for any non-empty open interval $I' \subseteq [0,1]$.

(iii) $\bigcup_{i=1}^{\infty} \Lambda^*(T^i) = \bigcup_{i=1}^{\infty} S(T^i)$.

(iv) $\lim_{i\to\infty} T^i(x) = 0$ for all $x \in [0,1]$, whereas $\Gamma \neq [0,1]$.

In general, for each measure zero Cantor set

$$\mathscr{C}(r_1,r_2) = \bigcap_{n=0}^{\infty} E_n, r_1 + r_2 < 1,$$

where E_n is a union of 2^n closed intervals (see (Coppel, 1983, p. 456)). Let

$$K_n = \{x \in [0,1] : x \notin E_n\}.$$

Then $K_n = \bigcup_{i=1}^{2^n-1} K_{n,i}$. Define F_n on $[0,1]$ similar to T_n with

$$\max_{\substack{x\in K_{n,i} \\ (\text{odd } i)}} F_n(x) = r_1^n \text{ and } F = \lim_{n\to\infty} F_n(x).$$

Then F is continuous on $[0,1]$, $\Lambda^*(F) = \mathscr{C}(r_1,r_2)$ and $\bigcup_{i=1}^{\infty} \Lambda^*(F^i)$ is uncountable measure zero and dense. Note that if $r_1 = r_2 = \frac{1}{3}$, then $\mathscr{C}(\frac{1}{3},\frac{1}{3}) = \mathscr{C}$.

CHAPTER 4

ITERATIVE ROOTS OF CONTINUOUS FUNCTIONS

For $f \in C(K)$, by Theorem 3.1.1 (i), we have

$$S(f) \subseteq S(f^2) \subseteq \ldots \subseteq S(f^k) \subseteq S(f^{k+1}) \subseteq \ldots. \tag{4.0.1}$$

Also, for $f \in PM(K)$ and $k \in \mathbb{N}$, from Corollary 3.1.10 (iii), we have

$$S(f^{k+1}) = S(f^k) \cup \{x \in (a,b) \setminus S(f^k) : f^k(x) \in S(f)\}. \tag{4.0.2}$$

In this chapter, we define an iteratively closed set in $C(K)$ and the non-monotonicity height for any continuous function using the characterization of $S(f^k)$, $k \in \mathbb{N}$, and study its properties. We prove the existence of continuous solutions of $f^n = F$ for a class of functions $F \in C(K)$ with the non-monotonicity height 1. Further, we discuss the non-existence of continuous solutions of $f^n = F$ for a class of continuous non-PM functions.

4.1 NON-MONOTONICITY HEIGHT OF CONTINUOUS FUNCTIONS

4.1.1 Iteratively closed set

For $f \in C(K)$, let $R(f^k) := [m_k, M_k]$, $Ch_{f^k} := [a_k, b_k]$, and $\mathrm{int}(R(f^k))$ be the interior of $R(f^k)$, $k \in \mathbb{N}$.

Definition 4.1.1. *A subset B of $C(K)$ is said to be iteratively closed in $C(K)$ if*

$$f^n \in B \textit{ for all } n \in \mathbb{N} \textit{ whenever } f^k \in B \textit{ for some } k \in \mathbb{N} \text{ and } f \in C(K).$$

The space $PM(K) \subseteq C(K)$ is iteratively closed in $C(K)$ (Corollary 2.3 in Zhang (1997)). The set of all constant functions on K is iteratively closed in $C(K)$.

In what follows, we try to get a subset of $C(K)$, which contains a class of PM and non-PM functions, and is iteratively closed in $C(K)$ to obtain continuous solutions of the iterative functional equation $f^n = F$.

Define

$$\mathcal{N}(K) := \{f \in C(K) : f(K) = f(R(f)),\ S(f) \neq \emptyset \text{ and } \operatorname{int}(R(f)) \neq \emptyset\}.$$

Lemma 4.1.2. *If $f \in \mathcal{N}(K)$, then $f^k \in \mathcal{N}(K)$ for all integers $k > 1$.*

Proof. As $R(f) = f([a,b]) = f(R(f))$, we get

$$R(f^k) = f^k([a,b]) = f^{k-1}(f([a,b])) = R(f). \tag{4.1.1}$$

Further

$$f^k(R(f^k)) = f^k(R(f)) = R(f) = R(f^k).$$

Therefore $f^k \in \mathcal{N}(K)$. □

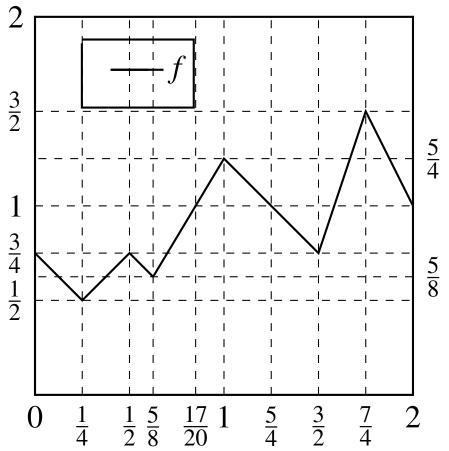
We remark that the space $\mathcal{N}(K)$ is not iteratively closed in $C(K)$. For instance, consider the continuous function $f : [0,2] \to [0,2]$ defined by

$$f(x) := \begin{cases} \frac{3}{4} - x, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ x + \frac{1}{4}, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ \frac{5}{4} - x, & \text{if } x \in \left[\frac{1}{2}, \frac{5}{8}\right], \\ \frac{5x}{3} - \frac{5}{12}, & \text{if } x \in \left[\frac{5}{8}, 1\right], \\ \frac{9}{4} - x, & \text{if } x \in \left[1, \frac{3}{2}\right], \\ 3x - \frac{15}{4}, & \text{if } x \in \left[\frac{3}{2}, \frac{7}{4}\right], \\ 5 - 2x, & \text{if } x \in \left[\frac{7}{4}, 2\right]. \end{cases}$$

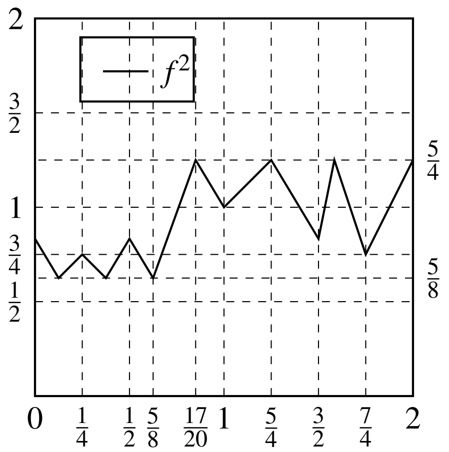
Here f attains its minimum $m_1 = \frac{1}{2}$ at $\frac{1}{4}$ and maximum $M_1 = \frac{3}{2}$ at $\frac{7}{4}$ and $\frac{1}{4}, \frac{7}{4} \notin R(f)$=$[\frac{1}{2}, \frac{3}{2}]$ (see Figure 4.1(a)). However, f^2 attains its minimum $m_2 = \frac{5}{8}$ at $\frac{5}{8}$ and maximum $M_2 = \frac{5}{4}$ at $\frac{17}{20}$ (see Figure 4.1(b)). Note that

$$\frac{5}{8}, \frac{17}{20} \in R(f^2) = \left[\frac{5}{8}, \frac{5}{4}\right].$$

Thus $f^2 \in \mathcal{N}([0,2])$ but $f \notin \mathcal{N}([0,2])$.



(a) f attains its extremum outside of $R(f)$



(b) f^2 attains its extremum on $R(f^2)$

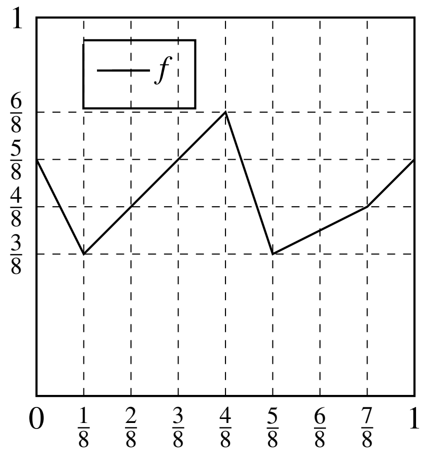
Figure 4.1 : The space $\mathscr{N}(K)$ is not iteratively closed in $C(K)$

We mention that for $f \in \mathscr{N}(K)$, we have $R(f) = R(f^k)$ for all $k \in \mathbb{N}$ by (4.1.1). But Ch_f need not be equal to Ch_{f^k} for some $k \in \mathbb{N}$. For example, consider the continuous function $f : [0,1] \to [0,1]$ defined by

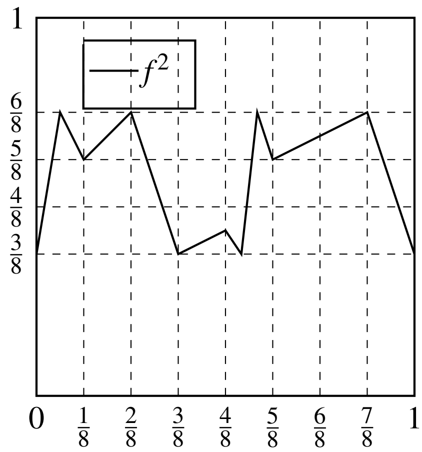
$$f(x) := \begin{cases} \frac{5}{8} - 2x, & \text{if } x \in \left[0, \frac{1}{8}\right], \\ x + \frac{2}{8}, & \text{if } x \in \left[\frac{1}{8}, \frac{4}{8}\right], \\ \frac{18}{8} - 3x, & \text{if } x \in \left[\frac{4}{8}, \frac{5}{8}\right], \\ \frac{x}{2} + \frac{1}{16}, & \text{if } x \in \left[\frac{5}{8}, \frac{7}{8}\right], \\ x - \frac{3}{8}, & \text{if } x \in \left[\frac{7}{8}, 1\right]. \end{cases}$$

It is easy to see that $S(f) \neq \emptyset$, $R(f) = [\frac{3}{8}, \frac{6}{8}]$ and $f(R(f)) = R(f)$ (see Figure 4.2(a)).

This implies $f \in \mathscr{N}([0,1])$, however $Ch_f = \left[\frac{1}{8}, 1\right] \neq \left[\frac{3}{8}, \frac{7}{8}\right] = Ch_{f^2}$ (see Figure 4.2(b)).



(a) $R(f) = \left[\frac{3}{8}, \frac{6}{8}\right]$ and $Ch_f = \left[\frac{1}{8}, 1\right]$



(b) $R(f^2) = [\frac{3}{8}, \frac{6}{8}]$ and $Ch_{f^2} = [\frac{3}{8}, \frac{7}{8}]$

Figure 4.2 : $f \in \mathscr{N}(K)$ but $Ch_f \neq Ch_{f^2}$

4.1.2 Continuous functions of height 1

Motivated from the concept of non-monotonicity height of PM functions, we define the non-monotonicity height for any continuous function $f \in C(K)$ and study its properties.

Proposition 4.1.3. *Let $f \in C(K)$ with $S(f) \neq \emptyset$. If $S(f^k) = S(f^{k+1})$ for some $k \in \mathbb{N}$, then $S(f^k) = S(f^{k+i})$, $\Lambda(f^k) = \Lambda(f^{k+i})$ and $\Lambda^*(f^k) = \Lambda^*(f^{k+i})$ for all $i \in \mathbb{N}$.*

Proof. From (4.0.1), we have

$$S(f^k) \subseteq S(f^{k+2}).$$

Let $y \in S(f^{k+2})$. By Proposition 2.1.1, for each $\varepsilon > 0$, there exist distinct $y_1, y_2 \in N_\varepsilon(y)$

such that

$$f^{k+1}(f(y_1)) = f^{k+1}(f(y_2)). \tag{4.1.2}$$

By the continuity of f, there exist $\varepsilon_1 > 0$ such that $f(N_\varepsilon(y)) \subseteq N_{\varepsilon_1}(f(y))$. Suppose $f(y_1) = f(y_2)$. Then $y \in S(f) \subseteq S(f^{k+1})$. Otherwise, $f(y) \in S(f^{k+1})$ by (4.1.2). Since $S(f^{k+1}) = S(f^k)$ and $y \notin S(f)$, there exist distinct $t_1, t_2 \in f(N_\varepsilon(y)) \subseteq N_{\varepsilon_1}(f(y))$ such that $f^k(t_1) = f^k(t_2)$. Take $z_1, z_2 \in N_\varepsilon(y)$ such that $f(z_1) = t_1$ and $f(z_2) = t_2$. This implies

$$f^{k+1}(z_1) = f^k(f(z_1)) = f^k(t_1) = f^k(t_2) = f^k(f(z_2)) = f^{k+1}(z_2).$$

Thus $y \in S(f^{k+1})$ and hence $S(f^{k+2}) = S(f^k)$. By applying the similar argument, we get $S(f^{k+i}) = S(f^k)$ for all $i \geq 3$.

To prove $\Lambda(f^k) = \Lambda(f^{k+i})$, let $x \in \Lambda(f^{k+i})$. Then $x \in S(f^k)$ by assumption. Suppose $x \in \Lambda^*(f^k)$, by Theorem 3.1.1(v), we have $x \in \Lambda^*(f^{k+i})$, a contradiction to Fact 2.0.2 (i). Therefore we get $\Lambda(f^{k+i}) \subseteq \Lambda(f^k)$. For the other inclusion, let $x \in \Lambda(f^k)$. Then

$$S(f^k) \cap N_\varepsilon(x) = \{x\}$$

for some $\varepsilon > 0$. Since $S(f^k) = S(f^{k+i})$, we get

$$S(f^{k+i}) \cap N_\varepsilon(x) = \{x\}.$$

Thus $\Lambda(f^{k+i}) = \Lambda(f^k)$. The equality $\Lambda^*(f^{k+i}) = \Lambda^*(f^k)$ follows from the Fact 2.0.2 (i) and (ii). $\square$

Here we remark that suppose $f \in C(K)$ with

$$\Lambda^*(f) \neq \emptyset,\ S(f^k) \neq S(f^{k+1}) \text{ and } \Lambda^*(f^k) = \Lambda^*(f^{k+1})$$

for some $k \in \mathbb{N}$, then it is *not necessarily true* that $\Lambda^*(f^k) = \Lambda^*(f^{k+i})$ for all $i \in \mathbb{N}$. Consider the continuous functions $f : [0, \frac{3\pi}{8}] \to [0, \frac{3\pi}{8}]$ defined by

$$f(x) := \begin{cases} \frac{7\pi}{32} + x - \left| |x - \frac{\pi}{32}| \sin\left(\frac{1}{|x-\frac{\pi}{32}|}\right) \right|, & \text{if } x \in [0, \frac{\pi}{32}), \\ \frac{\pi}{4} + \frac{32(z-\frac{\pi}{4})(x-\frac{\pi}{32})}{3\pi}, & \text{if } x \in [\frac{\pi}{32}, \frac{\pi}{8}], \\ \frac{\pi}{32} + \left| |\frac{\pi}{4} - x| \sin\left(\frac{1}{|\frac{\pi}{4}-x|}\right) \right|, & \text{if } x \in (\frac{\pi}{8}, \frac{\pi}{4}), \\ \frac{3x}{4} - \frac{5\pi}{32}, & \text{if } x \in [\frac{\pi}{4}, \frac{3\pi}{8}], \end{cases}$$

where $z = \frac{\pi}{32} + |\frac{\pi}{8} \sin(\frac{8}{\pi})|$.

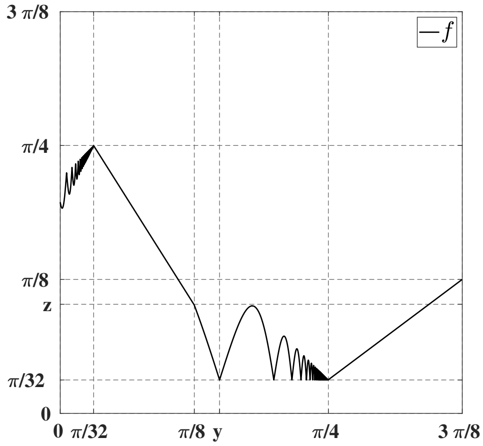


Figure 4.3 : $\frac{\pi}{32} \in \Lambda_L^*(f) \backslash \Lambda_R^*(f)$

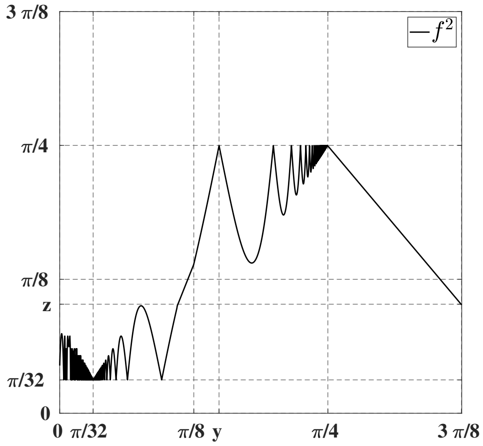


Figure 4.4 : $\frac{\pi}{32} \in \Lambda_L^*(f^2) \cap \Lambda_R^*(f^2)$

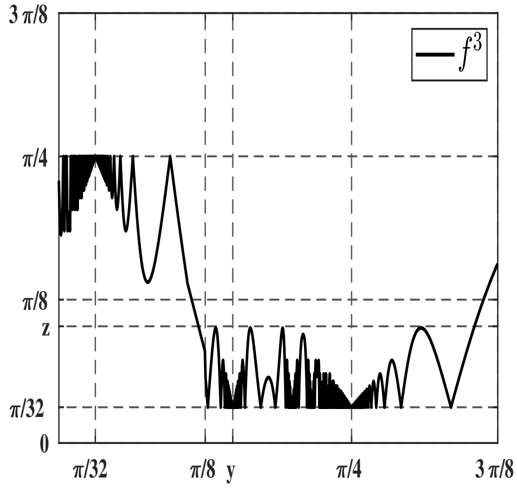


Figure 4.5 : $\Lambda^*(f) = \Lambda^*(f^2) \neq \Lambda^*(f^3)$

Here we have

$$\Lambda^*(f) = \left\{\frac{\pi}{32}, \frac{\pi}{4}\right\} = \Lambda_L^*(f) \backslash \Lambda_R^*(f),$$

f attains a local minimum at $y = \frac{\pi}{4} - \frac{1}{\pi} \in f^{-1}(\{\frac{\pi}{32}\})$, and $f^{-1}(\{\frac{\pi}{4}\}) = \{\frac{\pi}{32}\}$ (see Figure 4.3). By Theorem 3.1.7 (ii),

$$P_{\frac{\pi}{32}}(f,f) = \emptyset = P_{\frac{\pi}{4}}(f,f),$$

and by Corollary 3.1.9 (i), $\Lambda^*(f^2) = \Lambda^*(f)$ (see Figure 4.4). However, since the point $\frac{\pi}{32} \in \Lambda_L^*(f^2) \cap \Lambda_R^*(f^2)$, by Theorem 3.1.7 (i),

$$y \in P_{\frac{\pi}{32}}(f^2, f) \subseteq \Lambda^*(f^3).$$

Thus $\Lambda^*(f^2) \neq \Lambda^*(f^3)$ (see Figure 4.5). Note that $S(f) \neq S(f^2)$.

Definition 4.1.4. *Let $f \in C(K)$. The non-monotonicity height (or simply height) $H(f)$ of f is the least $k \in \mathbb{N} \cup \{0\}$ such that*

$$S(f^k) = S(f^{k+1})$$

if such k exists and $H(f) = \infty$, otherwise.

It is clear that $H(f) = 0$ if and only if f is strictly monotone. $H(f) = \infty$ if and only if $S(f^k) \subsetneq S(f^{k+1})$ for all $k \in \mathbb{N}$.

Note that for $f \in PM(K)$, Definition 4.1.4 and Definition 1.2.9 are equivalent.

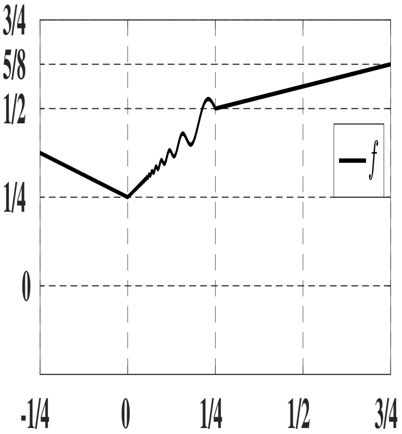
Example 4.1.5. *Consider the function $f : [\frac{-1}{4}, \frac{3}{4}] \to [\frac{-1}{4}, \frac{3}{4}]$ defined by*

$$f(x) := \begin{cases} \frac{1}{4} - \frac{x}{2}, & \text{if } x \in \left[\frac{-1}{4}, 0\right], \\ x + \frac{1}{4} + x^2 \sin(\frac{\pi}{x}), & \text{if } x \in \left(0, \frac{1}{4}\right), \\ \frac{7}{16} + \frac{x}{4}, & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right]. \end{cases}$$

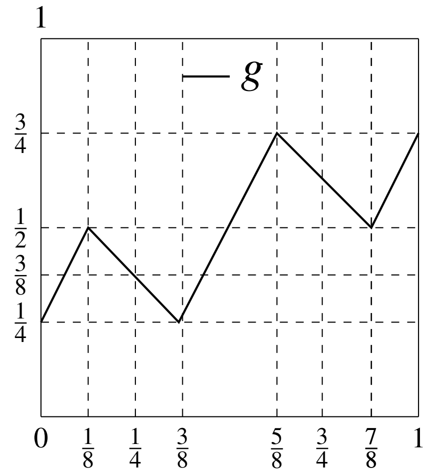
We can see that $R(f) = [\frac{1}{4}, \frac{5}{8}]$, f is strictly monotone on $R(f)$. This implies

$$P(f,f) = \emptyset = Q(f,f).$$

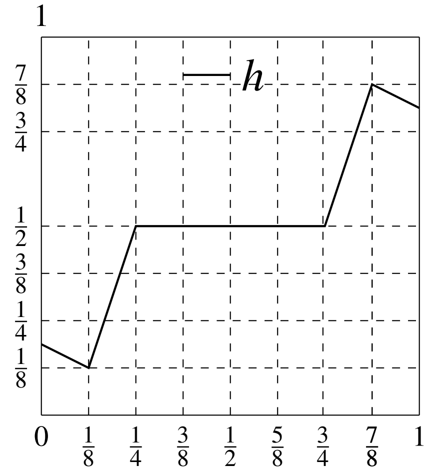
By Corollary 3.1.9 (iii), $S(f) = S(f^2)$ and hence $H(f) = 1$.



(a) $H(f) = 1$



(b) $H(g) > 1$



(c) $H(h) = \infty$

Figure 4.6 : Functions with different heights

Example 4.1.6. *For the continuous function* $g : [0,1] \to [0,1]$ *defined by*

$$g(x) := \begin{cases} 2x + \frac{1}{4}, & \text{if } x \in [0, \frac{1}{8}), \\ \frac{5}{8} - x, & \text{if } x \in [\frac{1}{8}, \frac{3}{8}), \\ 2x - \frac{4}{8}, & \text{if } x \in [\frac{3}{8}, \frac{5}{8}), \\ \frac{11}{8} - x, & \text{if } x \in [\frac{5}{8}, \frac{7}{8}), \\ 2x - \frac{5}{4}, & \text{if } x \in [\frac{7}{8}, 1], \end{cases}$$

we have $\frac{3}{8} \in \Lambda(g)$, $\frac{1}{4} \in Q_{\frac{3}{8}}(g,g)$ *and* $\frac{1}{4} \notin S(f)$ *(see Figure 4.6(b)). This imply*

$$S(g) \subsetneq S(g^2).$$

Thus $H(g) > 1$.

Example 4.1.7. *Let $h : [0,1] \to [0,1]$ defined by*

$$h(x) := \begin{cases} \frac{3}{16} - \frac{x}{2}, & \text{if } x \in [0, \frac{1}{8}], \\ 3x - \frac{2}{8}, & \text{if } x \in [\frac{1}{8}, \frac{1}{4}], \\ \frac{1}{2}, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ 3x - \frac{7}{4}, & \text{if } x \in [\frac{3}{4}, \frac{7}{8}], \\ \frac{21}{16} - \frac{x}{2}, & \text{if } x \in [\frac{7}{8}, 1]. \end{cases}$$

Then $\Lambda^*(h) = [\frac{1}{4}, \frac{3}{4}]$, and $\frac{1}{8}$, $\frac{7}{8}$ are fixed points of h (see Figure 4.6(c)). Therefore for each $x_k \in (\frac{1}{4}, \frac{1}{2}), k \in \mathbb{N}$, there exists $y_k \in (\frac{1}{8}, \frac{1}{4})$ such that $y_k \in P_{x_k}(h^{k-1}, h)$. This implies

$$S(h^k) \subsetneq S(h^{k+1}),\ \forall k \in \mathbb{N}$$

by Corollary 3.1.9 (iii). Thus $H(h) = \infty$.

Now, we discuss the properties of $H(f)$ for $f \in C(K)$.

Proposition 4.1.8. *Let $f \in \mathcal{N}(K)$. If $f \in PM(K)$ and $S(f) \cap \operatorname{int}(R(f)) \neq \emptyset$, then $H(f) = \infty$.*

Proof. It follows from (4.1.1) that

$$[m_1, M_1] = [m_k, M_k],\ \forall k \in \mathbb{N}. \tag{4.1.3}$$

Let $y_0 \in S(f) \cap (m_1, M_1)$. Then $y_0 \in (m_k, M_k)$ by (4.1.3). Since $y_0 \in (m_k, M_k)$, we have $(f^k)^{-1}((y_0, M_k))$ and $(f^k)^{-1}((m_k, y_0))$ are non-empty open sets in K. Choose open intervals J_1 and J_2 such that

$$J_1 \subseteq (f^k)^{-1}((y_0, M_k)),\ J_2 \subseteq (f^k)^{-1}((m_k, y_0)) \text{ and } \operatorname{cl}(J_1) \cap \operatorname{cl}(J_2) = \{x^*\}$$

for some $x^* \in (a,b)$, where $\operatorname{cl}(J_1)$ and $\operatorname{cl}(J_2)$ are the closure of J_1 and J_2 respectively. Clearly, $f^k(x^*) = y_0$ by the continuity of f^k. Since $f^k \in PM(K)$ and every forts of f^k are points of local extremum of f^k, f^k is strictly monotone in some neighborhood of x^*. By (4.0.2), we have

$$x^* \in S(f^{k+1}) \setminus S(f^k)$$

and hence $H(f) = \infty$. □

The next example shows that Proposition 4.1.8 is *not necessarily true* for non-PM functions.

Example 4.1.9. *Define* $f : [a,b] \to [a,b]$ *by*

$$f(x) := \begin{cases} f_0(x), & \text{if } x \in [c,d], \\ x, & \text{otherwise,} \end{cases}$$

where $a < c < d < b$, *and* f_0 *is a continuous nowhere differentiable function on* $[c,d]$ *such that* $f_0(c) = c$ *and* $f_0(d) = d$. *Clearly,* f *is well-defined and continuous on* $[a,b]$. *Also,*

$$R(f) = [a,b], \text{ and } S(f) = [c,d].$$

As f *is strictly monotone and a self-map on* $[a,c) \cup (d,b]$, *we get*

$$f^{-1}(S(f)) = S(f) \text{ and } P(f,f) = Q(f,f) = \emptyset.$$

By Corollary 3.1.9 (iii), $S(f^2) = S(f)$. *Thus*

$$f \in \mathscr{N}([a,b]) \text{ and } S(f) \cap (m,M) \neq \emptyset$$

but $H(f) = 1$.

Lemma 4.1.10. *Let* $f \in C(K)$ *with* $S(f) \neq \emptyset$. *If* f *is strictly monotone on* $R(f)$, *then* $H(f) = 1$.

Proof. From (4.0.1), we have $S(f) \subseteq S(f^2)$. Since f is strictly monotone on $[m,M]$, we have

$$S(f) \cap [m,M] \subseteq \{m,M\}.$$

This implies

$$f^{-1}(S(f)) \subseteq S(f) \cup \{a,b\},$$

and f is strictly monotone on $f(N_\varepsilon(m))$ and $f(N_\delta(M))$ for some $\varepsilon, \delta > 0$. Thus

$$P(f,f) = \emptyset = Q(f,f).$$

This implies $S(f^2) = S(f)$ by Corollary 3.1.9 (iii). Hence $H(f) = 1$. □

Here it is worth to remark that the converse of Lemma 4.1.10 is *not necessarily* true. Consider the continuous non-PM functions g, f given in Figures 4.7 and 4.8.

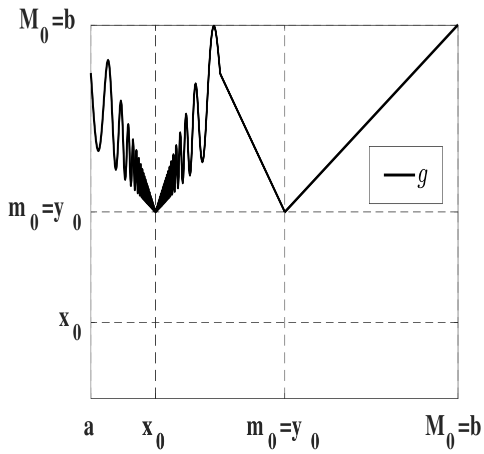


Figure 4.7 : $H(g) = 1$ and g is strictly monotone on $R(g)$

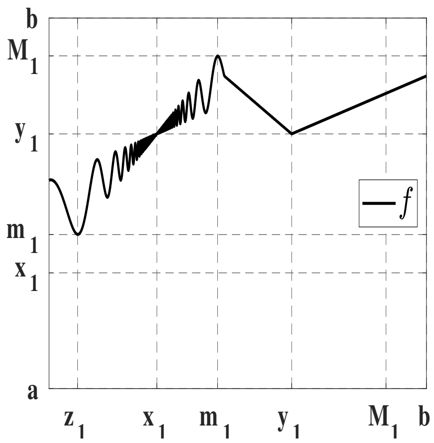


Figure 4.8 : $H(f) = 1$ and f is not strictly monotone on $R(f)$

Note that $[m_0, M_0] \cap S(g) = \{y_0\}$ and $[m_1, M_1] \cap S(f) = \{m_1, y_1\}$. Also,

$$g^{-1}(\{y_0\}) = \{x_0, y_0\} \subseteq S(g) \text{ and } f^{-1}(\{m_1, y_1\}) = \{z_1, x_1, y_1\} \subseteq S(f).$$

This implies

$$P(g,g) = Q(g,g) = \emptyset = P(f,f) = Q(f,f).$$

By Corollary 3.1.9 (iii), we have

$$S(g^2) = S(g) \text{ and } S(f^2) = S(f).$$

Thus $H(g) = 1 = H(f)$. Here g is strictly monotone on $R(g)$ (see Figure 4.7), whereas f is not strictly monotone on $R(f)$ (see Figure 4.8).

It is clear from Figures 4.7 and 4.8 that $y_0 \notin \text{int}(R(g))$, $x_0 \in g^{-1}(\{y_0\})$ and g *attains* *a local extremum at* x_0. On the other hand, $y_1 \in \operatorname{int}(R(f)) \cap S(f)$, $x_1 \in f^{-1}(\{y_1\})$ and f *does not attain a local extremum at* x_1. This explains the crucial role of non-isolated forts in the above remark.

From the above observation, we obtain a class of continuous functions f such that f does not attain a local extremum at some $x \in f^{-1}(S(f))$.

Definition 4.1.11. *Let $f \in C(K)$. f is called a locally constant function on K if there exists an open interval $I' \subsetneq K$ such that $f(x) = c$ for all $x \in I'$ and for some $c \in \mathbb{R}$.*

The function h defined in Example 4.1.7 is a locally constant continuous function on $[0,1]$ with constant $c = \frac{1}{2}$ (see Figure 4.6(c)).

Theorem 4.1.12. *Let $f \in C(K)$ with $\operatorname{int}(R(f)) \neq \emptyset$. If f is a locally constant function on $R(f)$, then f does not attain a local extremum at some $x \in f^{-1}(S(f))$.*

Proof. Let

$$C := \left\{c \in K : f^{-1}(\{c\}) \text{ contains an open interval } J_c \subsetneq R(f) = [m,M]\right\}.$$

Clearly, $C \neq \emptyset$, J_c's are pairwise disjoint and $J_c \subseteq S(f)$. Since $[m,M]$ has a countable dense subset, C is countable. Let $y_0 \in S(f) \cap (m,M)$ with $y_0 \notin C$. Note that

$$\text{there is no open interval } I' \subseteq K \text{ such that } f(x) = y_0 \text{ for all } x \in I'. \tag{4.1.4}$$

Let $x_0 \in f^{-1}(\{y_0\})$. If f does not attain a local extremum at x_0, then we are done. Suppose that f attains a local minimum at x_0. Since $y_0 \in (m,M)$, there exists $x_1, x_2 \in K$ such that

$$f(x_1) > f(x_0) = y_0 > f(x_2).$$

Without loss of generality, we assume that $x_0 < x_1 < x_2$. Define

$$A_{y_0} := \{x \in [x_0, x_2] : f(x) = y_0 \text{ and } f \text{ attains a local mimimum at } x\},$$

and $x^* := \sup A_{y_0}$. Clearly, $f(x^*) = y_0$ and there is no $x \in (x^*, x_2)$ such that $f(x) = y_0$ and f attains a local minimum at x.

Case 1: f does not attain a local minimum at x^*.

Claim 1: f does not attain a local extremum at x^*.

Let $\varepsilon > 0$. By the fact x^* is not a point of local minimum of f, we have

$$f(x_\varepsilon) < f(x^*)$$

for some $x_\varepsilon \in N_\varepsilon(x^*)$. Since x^* is a limit point of A_{y_0}, there exist $x_3 \in A_{y_0}$ and $\eta > 0$ with $N_\eta(x_3) \subseteq N_\varepsilon(x^*)$. As f attains a local minimum at x_3 and by (4.1.4), there exists a point $y_\varepsilon \in N_\eta(x_3)$ such that

$$f(y_\varepsilon) > f(x^*).$$

Therefore x^* is not a point of local maximum of f and hence f does not attain a local extremum at x^*.

Case 2: f attains a local minimum at x^*.

Claim 2: If there is no $x \in (x^*, x_2)$ with the property that $f(x) = y_0$ and f attains a local maximum at x, then f does not attain a local extremum at some $z^* \in (x^*, x_2)$.

Since x^* is a point of local minimum of f, there exists $\delta' > 0$ and $s \in N_{\delta'}(x^*)$ with

$$x^* < s < x_2 \text{ such that } f(s) > y_0.$$

By the Intermediate Value Theorem (IMVT), $f(z^*) = y_0$ for some $z^* \in (s, x_2)$, and for each $\delta > 0$, there exist $x_\delta, y_\delta \in N_\delta(z^*)$ such that

$$f(y_\delta) > f(z^*) > f(x_\delta).$$

On the other hand, suppose that f is as in Case 2 and there is $x \in (x^*, x_2) \cap f^{-1}(\{y_0\})$ such that x is a point of local maximum of f. Then define

$$B_{y_0} := \{y \in [x^*, x_2] : f(y) = y_0 \text{ and } f \text{ attains a local maximum at } y\},$$

and $y^* := \inf B_{y_0}$. Note that $B_{y_0} \neq \emptyset$ and $f(y^*) = y_0$. If f does not attain a local maximum at y^*, then f does not attain a local extremum at y^* by a similar argument as in Claim 1.

Suppose that f attains a local maximum at y^*. By (4.1.4), we have

$$x^* < y^* \text{ and } y_0 \in \operatorname{int}(f([x^*, y^*])),$$

and there is no $x \in (x^*, y^*)$ with $f(x) = y_0$ and f attains a local extremum at x. Now, since x^* is a point of local minimum of f and y^* is a point of local maximum of f, there exist $s \in N_{\varepsilon'}(x^*)$ and $r \in N_{\varepsilon''}(y^*)$ such that

$$x^* < s < r < y^* \text{ and } f(s) > y_0 > f(r)$$

for some $\varepsilon', \varepsilon'' > 0$. By IMVT, there exists $w^* \in (s, r)$ such that $f(w^*) = y_0$ and f does not attain a local extremum at w^*.

The proof of the case when f attains a local maximum at x_0 is similar. $\square$

Note that the converse of Theorem 4.1.12 is not necessarily true (see Figure 4.8).

From the observation in Figures 4.7 and 4.8, and Theorem 4.1.12, we define a subset of $\mathcal{N}(K)$ and prove it is iteratively closed in $C(K)$. Let

$$\mathcal{N}_1(K) := \{f \in \mathcal{N}(K) : f \text{ attains a local extremum at every } x \in f^{-1}(S(f))\}. \quad (4.1.5)$$

Theorem 4.1.13. *Let $f \in \mathcal{N}(K)$. Then the following hold:*

(i) *$f \in \mathcal{N}_1(K)$ if and only if f is strictly monotone on $R(f)$.*

(ii) *$\mathcal{N}_1(K)$ is iteratively closed in $C(K)$.*

(iii) *If $f \in \mathcal{N}_1(K)$, then $H(f) = 1$.*

Proof. (i) Let $f \in \mathcal{N}_1(K)$. Clearly, f is not a locally constant function on $R(f)$ by Theorem 4.1.12. Suppose $y_0 \in S(f) \cap (m,M)$, then by a similar argument as in the proof of Theorem 4.1.12, f does not attain a local extremum at x for some $x \in f^{-1}(\{y_0\})$, a contradiction to $f \in \mathcal{N}_1(K)$. Thus

$$S(f) \cap (m,M) = \emptyset.$$

Hence f is strictly monotone on $[m,M]$. Conversely, suppose $f \in \mathcal{N}(K)$ is strictly monotone on $[m,M]$. As $f([m,M]) = [m,M]$, we get

$$S(f) \cap [m,M] \subseteq \{m,M\}.$$

This implies that for each $x \in f^{-1}(S(f))$, either $f(x) = m$ or $f(x) = M$. Therefore f attains a local extremum at every $x \in f^{-1}(S(f))$. Hence $f \in \mathcal{N}_1(K)$.

(ii) Assume that $f^k \in \mathcal{N}_1(K)$ for some $k \in \mathbb{N}$. Then f^k is strictly monotone on $R(f^k)$ by (i). This implies f is strictly monotone on $R(f)$ by (4.0.1). Therefore from (4.1.1), we have

$$R(f^k) = R(f^n) \text{ and } S(f^n) \cap (m_n, M_n) = \emptyset, \ \forall n \in \mathbb{N}.$$

Hence f^n is strictly monotone on $R(f^n)$ as f is strictly monotone on $R(f)$. By (i), $f^n \in \mathcal{N}_1(K)$ for all $n \in \mathbb{N}$.

Proof of (iii) follows form (i) and Lemma 4.1.10. □

Remark 4.1.14. *For $f \in \mathcal{N}_1(K)$, by Theorem 4.1.13 (i) and (ii), we have*

$$Ch_{f^k} = R(f^k), \ \forall k \in \mathbb{N}.$$

4.2 EXISTENCE OF ITERATIVE ROOTS

In this section, using the extension method given in Zhang (1997), we prove the existence of continuous solutions of $f^n = F$ for $F \in \mathscr{N}_1(K)$.

Theorem 4.2.1. *Let $F \in \mathscr{N}_1(K)$.*

(i) *Suppose that F is strictly increasing on Ch_F. Then $f^n = F$ has infinitely many continuous solutions $f \in \mathscr{N}_1(K)$ for any $n \geq 2$.*

(ii) *Suppose that F is strictly decreasing on Ch_F. Then $f^n = F$ has infinitely many continuous solutions $f \in \mathscr{N}_1(K)$ for only odd $n \geq 3$.*

Proof. Let $F_0 = F|_{Ch_F}$ and $n > 1$. Since $F \in \mathscr{N}_1(K)$, $F_0 : Ch_F \to Ch_F$ is a bijection by Theorem 4.1.13 (i). Suppose that $f_0 : Ch_F \to Ch_F$ is a continuous solution of $f^n = F$ on Ch_F. By (4.0.1) and (4.1.1), f_0 is strictly monotone and onto. Define

$$f(x) := (F_0^{-1} \circ f_0 \circ F)(x), \ \forall x \in K. \tag{4.2.1}$$

Clearly, f is well-defined and continuous on K by the continuity of F, F_0^{-1}, and f_0. For each $x \in Ch_F$, we have

$$f(x) = (F_0^{-1} \circ f_0 \circ F)(x) = f_0^{-n}(f_0^{n+1}(x)) = f_0(x).$$

Now, for any $x \in K$,

$$\begin{aligned} f^n(x) &= (F_0^{-1} \circ f_0 \circ F)^n(x) \\ &= (F_0^{-1} \circ f_0^n \circ F)(x) \\ &= F(x). \end{aligned} \tag{4.2.2}$$

Thus f is a continuous solution of $f^n = F$, and $f \in \mathscr{N}_1(K)$ by Theorem 4.1.13 (ii).

(i) Since F_0 is strictly increasing on Ch_F, $f^n = F$ has a continuous solution f_0 on Ch_F for any $n \geq 2$ by Theorem 1.2.5. Therefore f defined in (4.2.1) is a continuous solution of $f^n = F$ for any $n \geq 2$.

(ii) As F_0 is strictly decreasing on Ch_F, from Theorem 1.2.7, we get a continuous map $f_0 : Ch_F \to Ch_F$ such that

$$f_0^n(x) = F_0(x), \ \forall x \in Ch_F$$

only for odd $n \geq 3$. Hence $f^n = F$ has a continuous solution f defined in (4.2.1) for only odd $n \geq 3$.

The solution $f \in \mathscr{N}_1(K)$ of $f^n = F$ depends on a solution f_0 of $f^n = F$ on Ch_F. From Theorem 1.2.5 and Theorem 1.2.7, we have infinitely many f_0. Hence $f^n = F$ has infinitely many solutions in $\mathscr{N}_1(K)$. □

Clearly, $\mathscr{N}_1(K)$ contains a class of PM functions. We derive Theorem 1.2.15 partially from Theorem 4.2.1.

Corollary 4.2.2. *Let $F \in PM(K)$ with $S(F) \neq \emptyset$. If F is strictly monotone on the characteristic interval $Ch_F = [a', b']$ and $F(\{a', b'\}) = \{a', b'\}$, then $f^n = F$ has a solution $f \in \mathscr{N}_1(K) \cap PM(K)$ for any $n \geq 2$.*

Proof. As F is strictly monotone on $[a', b']$ and $F(\{a', b'\}) = \{a', b'\}$, we get

$$[a', b'] = [m, M] \text{ and } F([m, M]) = [m, M].$$

From Theorem 4.1.13 (i), we have $F \in \mathscr{N}_1(K)$. By Theorem 4.2.1, there exists a function $f \in \mathscr{N}_1(K)$ such that

$$f^n(x) = F(x), \ \forall x \in K.$$

Since $F \in PM(K)$ and $PM(K)$ is iteratively closed in $C(K)$, $f \in PM(K)$. □

Let $p, q \in (0, 1)$ with $p < q$. As in Lin (2014), a function $\phi \in C([0, 1])$ is called a sickle-like function if one of the following conditions is satisfied:

(A1) ϕ is constant on $[0, p]$, ϕ is strictly decreasing on $[p, q]$ and strictly increasing on $[q, 1]$ (see Figure 4.9(a));

(A2) ϕ is constant on $[0, p]$, ϕ is strictly increasing on $[p, q]$ and strictly decreasing on $[q, 1]$ (see Figure 4.9(b));

(A3) ϕ is constant on $[q, 1]$, ϕ is strictly increasing on $[0, p]$ and strictly decreasing on $[p, q]$ (see Figure 4.9(c));

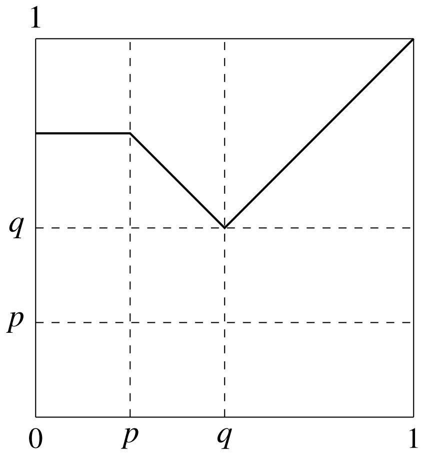
(A4) ϕ is constant on $[q, 1]$, ϕ is strictly decreasing on $[0, p]$ and strictly increasing on $[p, q]$ (see Figure 4.9(d)).

Let B_1 (resp. B_2, B_3, B_4) be set of all $\phi \in C([0, 1])$ satisfying (A1) (resp. (A2), (A3), (A4)). Note that for $\phi \in B_1 \cup B_2$, we have

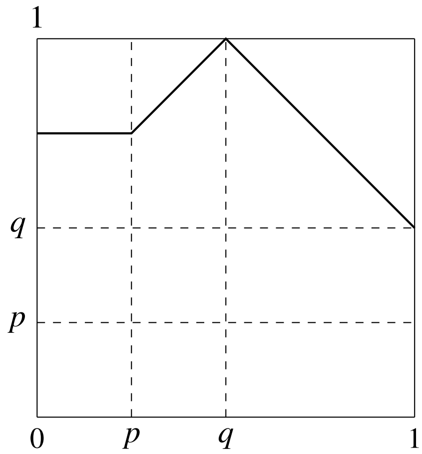
$$\Lambda^*(\phi) = [0, p] \text{ and } \Lambda(\phi) = \{q\},$$

and for $\phi \in B_3 \cup B_4$,

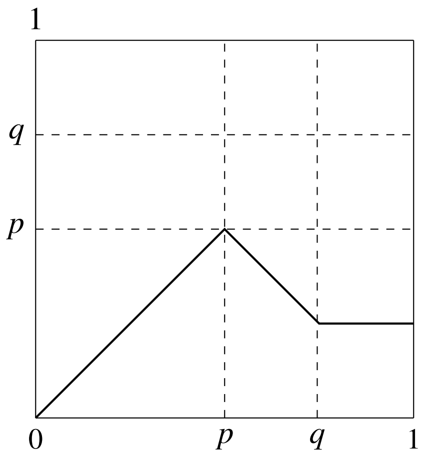
$$\Lambda^*(\phi) = [q, 1] \text{ and } \Lambda(\phi) = \{p\}.$$



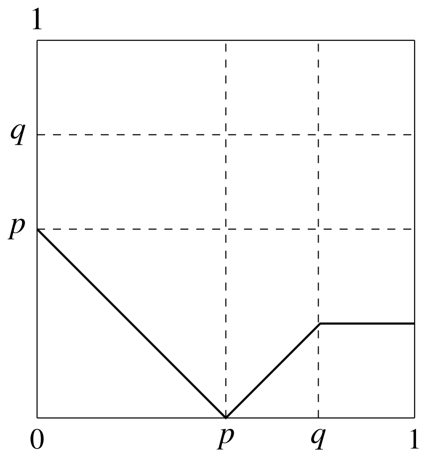
(a) A sickle-like function in B_1



(b) A sickle-like function in B_2



(c) A sickle-like function in B_3



(d) A sickle-like function in B_4

Figure 4.9 : Sickle-like functions

Special cases of Theorem 3.1 (i), Theorem 3.2 (ii), Theorem 4.1, and Theorem 4.2 (i) in Lin (2014) follow as corollaries of Theorem 4.2.1.

Corollary 4.2.3. (i) *Let $F \in B_1$ (resp. $F \in B_3$). If $F([0,1]) \subseteq [q,1]$ and $F(1) = 1$, $F(q) = q$ (resp. $F([0,1]) \subseteq [0,p]$ and $F(0) = 0$, $F(p) = p$), then $f^n = F$ has infinitely many continuous solutions of all $n \geq 2$ on $[0,1]$.*

(ii) *Let $F \in B_2$ (resp. $F \in B_4$). If $F([0,1]) \subseteq [q,1]$ and $F(q) = 1$, $F(1) = q$ (resp. $F([0,1]) \subseteq [0,p]$ and $F(0) = p$, $F(p) = 0$), then $f^n = F$ has infinitely many continuous solutions of only odd $n \geq 3$ on $[0,1]$.*

Proof. (i) Let $F \in B_1$ (resp. $F \in B_3$). As $F([0,1]) = [q,1] = F([q,1])$ (resp. $F([0,1]) = [0,p] = F([0,p])$), $F \in \mathscr{N}([0,1])$. Since F is strictly increasing on $R(F)$, $F \in \mathscr{N}_1([0,1])$ by Theorem 4.1.13 (i) (see Figures 4.9(a) and 4.9(c)). Therefore $f^n = F$ has infinitely many continuous solutions in $\mathscr{N}_1([0,1])$ for any $n \geq 2$ by Theorem 4.2.1 (i).

(ii) The proof of (ii) is similar to that of (i). □

Corollary 4.2.4. (i) *Let $F \in B_1 \cup B_3$. If $F([0,1]) \subseteq [p,q]$ and $F(p) = q$, $F(q) = p$, then $f^n = F$ has infinitely many continuous solutions of only odd $n \geq 3$ on $[0,1]$.*

(ii) *Let $F \in B_2 \cup B_4$. If $F([0,1]) \subseteq [p,q]$ and $F(p) = p$, $F(q) = q$, then $f^n = F$ has infinitely many continuous solutions of all $n \geq 2$ on $[0,1]$.*

Proof. (i) It follows from the fact $F([0,1]) = [p,q] = F([p,q])$ that $F \in \mathscr{N}([0,1])$. As F is strictly decreasing on $R(F) = [p,q]$, $F \in \mathscr{N}_1([0,1])$ by Theorem 4.1.13 (i). Thus $f^n = F$ has infinitely many strictly decreasing solutions $f \in \mathscr{N}_1([0,1])$ for only odd $n \geq 3$ by Theorem 4.2.1 (ii).

(ii) The proof of (ii) is similar to that of (i). □

In the following corollaries, we discuss the solutions of $f^n = F$ for a class of clenched single-plateau functions F, which was studied in Lin et al. (2017).

Corollary 4.2.5. *Let $F \in C([0,1])$ and $p,q \in (0,1)$ with $p < q$.*

(i) *If $F([0,1]) = [0,p] = F([0,p])$ and F is strictly increasing on $[0,p]$ (resp. F is strictly decreasing on $[0,p]$) and F is constant on $[p,1]$, then $f^n = F$ has infinitely many continuous solutions for all $n \geq 2$ (resp. $f^n = F$ has infinitely many continuous solutions of only odd $n \geq 3$) on $[0,1]$.*

(ii) *If $F([0,1]) = [q,1] = F([q,1])$, F is constant on $[0,q]$ and F is strictly increasing on $[q,1]$ (resp. F is strictly decreasing on $[q,1]$), then $f^n = F$ has infinitely many continuous solutions for all $n \geq 2$ (resp. $f^n = F$ has infinitely many continuous solutions of only odd $n \geq 3$) on $[0,1]$.*

Proof. (i) Suppose that $F([0,1]) = [0,p] = F([0,p])$ and F is strictly increasing on $[0,p]$ (resp. F is strictly decreasing on $[0,p]$). Then $F \in \mathscr{N}_1([0,1])$ by Theorem 4.1.13 (i). Therefore $f^n = F$ has infinitely many solutions in $\mathscr{N}_1([0,1])$ by Theorem 4.2.1.

(ii) The proof of (ii) is similar to that of (i). □

Corollary 4.2.6. *Let $F \in C([0,1])$ and $p,q \in (0,1)$ with $p < q$.*

(i) *If $F([0,1]) = [0,p] = F([0,p])$, F is strictly increasing on $[0,p]$ (resp. F is strictly decreasing on $[0,p]$), F is constant on $[p,q]$ and F is strictly decreasing on $[q,1]$ (resp. F is strictly increasing on $[q,1]$), then $f^n = F$ has infinitely many continuous solutions for all $n \geq 2$ (resp. $f^n = F$ has infinitely many continuous solutions of only odd $n \geq 3$) on $[0,1]$.*

(ii) *If* $F([0,1]) = [q,1] = F([q,1])$*,* F *is strictly increasing on* $[0,p]$ *(resp.* F *is strictly decreasing on* $[0,p]$*),* F *is constant on* $[p,q]$ *and* F *is strictly decreasing on* $[q,1]$ *(resp.* F *is strictly increasing on* $[q,1]$*), then* $f^n = F$ *has infinitely many continuous solutions of only odd* $n \geq 3$ *(resp.* $f^n = F$ *has infinitely many continuous solutions for all* $n \geq 2$*) on* $[0,1]$.

Proof. (i) Since $F([0,1]) = [0,p] = F([0,p])$, $F \in \mathscr{N}([0,1])$. As F is strictly monotone on $R(F) = [0,p]$, $F \in \mathscr{N}_1([0,1])$ by Theorem 4.1.13 (i). Thus $f^n = F$ has infinitely many continuous solutions $f \in \mathscr{N}_1([0,1])$ by Theorem 4.2.1 (ii).

(ii) The proof of (ii) is similar to that of (i). □

In Liu and Zhang (2011), it was proved that every continuous solution of $f^n = F$ for $F \in PM(K)$ with $H(F) \leq 1$ is an extension (l-extension) from the solution of $f^n = F$ of the same $n \in \mathbb{N}$ on the characteristic interval of F (Theorem 1 and (2.9) in Liu and Zhang (2011)). In the following theorem, we prove that every solution $f \in \mathscr{N}_1(K)$ of $f^n = F$ for $F \in \mathscr{N}_1(K)$ is of the form given in (4.2.1) (1-extension).

Theorem 4.2.7. *Let* $F \in \mathscr{N}_1(K)$ *and* $n \geq 2$*. Every continuous solution* $f \in \mathscr{N}_1(K)$ *of* $f^n = F$ *is a* 1*-extension from a solution of* $f^n = F$ *on* Ch_F*.*

Proof. Let $f \in C(K)$ be a solution of $f^n = F$. By Theorem 4.1.13 (ii),

$$f \in \mathscr{N}_1(K) \text{ and } R(f) = Ch_f = Ch_F = R(F).$$

Let $f_0 = f|_{Ch_F}$ and $F_0 = F|_{Ch_F}$. Now, for $x \in K \setminus Ch_F$, we have

$$\begin{aligned} F(x) &= (f^{n-1} \circ f)(x), \\ (f_0 \circ F)(x) &= (f_0^n \circ f)(x), \\ (F_0^{-1} \circ f_0 \circ F)(x) &= f(x). \end{aligned}$$

Therefore

$$f(x) = \begin{cases} f_0(x), & \text{if } x \in Ch_F, \\ (F_0^{-1} \circ f_0 \circ F)(x), & \text{if } x \in K \setminus Ch_F. \end{cases}$$

□

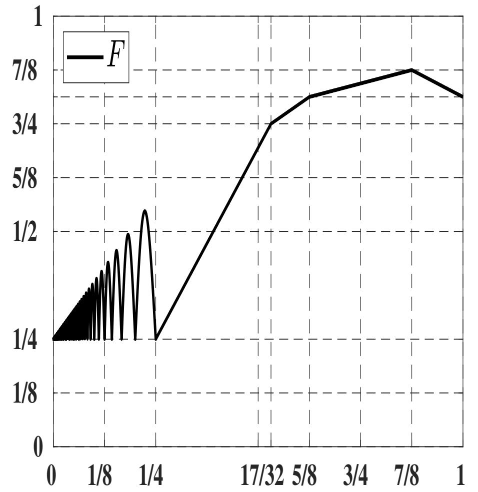


Figure 4.10 : A non-PM function $F \in \mathcal{N}_1([0,1])$

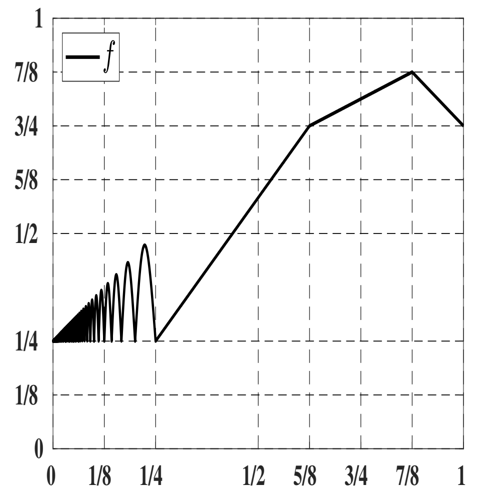


Figure 4.11 : A continuous solution $f \in \mathcal{N}_1([0,1])$ of $f^2 = F$

Now, we illustrate Theorem 4.2.1 by the following example.

Example 4.2.8. *Consider the function* $F : [0,1] \to [0,1]$ *defined by*

$$F(x) = \begin{cases} \frac{1}{4}, & \text{if } x = 0, \\ \frac{1}{4} + \frac{4}{3}\left|x \sin\left(\frac{\pi}{x}\right)\right|, & \text{if } x \in \left(0, \frac{1}{4}\right), \\ \frac{16x}{9} - \frac{7}{36}, & \text{if } x \in \left[\frac{1}{4}, \frac{17}{32}\right), \\ \frac{4x}{6} + \frac{19}{48}, & \text{if } x \in \left[\frac{17}{32}, \frac{5}{8}\right], \\ \frac{x}{4} + \frac{21}{32}, & \text{if } x \in \left(\frac{5}{8}, \frac{7}{8}\right], \\ \frac{21}{16} - \frac{x}{2}, & \text{if } x \in \left(\frac{7}{8}, 1\right]. \end{cases}$$

It is clear that F *is continuous on* $[0,1]$, F *attains its minimum* $\frac{1}{4}$ *and maximum* $\frac{7}{8}$ *on* $R(F)=[\frac{1}{4},\frac{7}{8}]=Ch_F$, *and* $F_0=F|_{Ch_F}$ *is strictly increasing on* Ch_F *(see Figure 4.10). By Theorem 4.1.13 (i),* $F\in\mathscr{N}_1([0,1])$. *Let* $f_0:[\frac{1}{4},\frac{7}{8}]\to[\frac{1}{4},\frac{7}{8}]$ *be defined by*

$$f_0(x):=\begin{cases}\frac{4x}{3}-\frac{1}{12}, & \text{if } x\in[\frac{1}{4},\frac{5}{8}],\\ \frac{x}{2}+\frac{7}{16}, & \text{if } x\in(\frac{5}{8},\frac{7}{8}].\end{cases} \tag{4.2.3}$$

It is easy to verify that f_0 *is a continuous solution of* $f^2=F_0$ *on* $Ch_F=[\frac{1}{4},\frac{7}{8}]$ *(see Figure 4.11). The function* $F_0^{-1}:\left[\frac{1}{4},\frac{7}{8}\right]\to\left[\frac{1}{4},\frac{7}{8}\right]$ *is computed as*

$$F_0^{-1}(x)=\begin{cases}\frac{9x}{16}+\frac{7}{64}, & \text{if } x\in\left[\frac{1}{4},\frac{3}{4}\right],\\ \frac{6x}{4}-\frac{57}{96}, & \text{if } x\in\left(\frac{3}{4},\frac{13}{16}\right],\\ 4x-\frac{21}{8}, & \text{if } x\in\left(\frac{13}{16},\frac{7}{8}\right].\end{cases}$$

Now, from (4.2.1), we compute $f(x)=(F_0^{-1}\circ f_0\circ F)(x)$ *for all* $x\in[0,1]$. *For each* $x\in\left(0,\frac{1}{4}\right)$, *we have*

$$F(x)\in\left[\frac{1}{4},\frac{5}{8}\right], \text{ and } f_0\left(\left[\frac{1}{4},\frac{5}{8}\right]\right)\subseteq\left[\frac{1}{4},\frac{3}{4}\right].$$

This implies

$$\begin{aligned}(F_0^{-1}\circ f_0\circ F)(x) &= F_0^{-1}\left(f_0\left(\frac{1}{4}+\frac{4}{3}\left|x\sin\left(\frac{\pi}{x}\right)\right|\right)\right)\\ &= F_0^{-1}\left(\frac{3}{12}+\frac{16}{9}\left|x\sin\left(\frac{\pi}{x}\right)\right|\right)\\ &= \frac{1}{4}+\left|x\sin\left(\frac{\pi}{x}\right)\right|.\end{aligned}$$

Therefore $f:[0,1]\to[0,1]$ *is computed as*

$$f(x)=(F_0^{-1}\circ f_0\circ F)(x)=\begin{cases}\frac{1}{4}, & \text{if } x=0,\\ \frac{1}{4}+\left|x\sin\left(\frac{\pi}{x}\right)\right|, & \text{if } x\in\left(0,\frac{1}{4}\right),\\ f_0(x), & \text{if } x\in\left[\frac{1}{4},\frac{7}{8}\right],\\ \frac{7}{4}-x, & \text{if } x\in\left[\frac{7}{8},1\right].\end{cases} \tag{4.2.4}$$

It is easy to verify that $f^2(x)=F(x)$ *for all* $x\in\{0\}\cup\left[\frac{1}{4},1\right]$. *Now, for* $x\in\left(0,\frac{1}{4}\right)$, *we* *have* $f(x) \in \left[\frac{1}{4}, \frac{5}{8}\right]$. *Thus*

$$f^2(x) = \frac{4}{3}\left(\frac{1}{4} + \left|x\sin\left(\frac{\pi}{x}\right)\right|\right) - \frac{1}{12} = \frac{1}{4} + \frac{4}{3}\left|x\sin\left(\frac{\pi}{x}\right)\right| = F(x).$$

Therefore f *defined in* (4.2.4) *is a continuous solution of* $f^2 = F$ *on* $[0,1]$.

4.3 NON-EXISTENCE OF ITERATIVE ROOTS

In this section, we discuss the non-existence of continuous solutions of $f^n = F$ for a class of continuous functions on an arbitrary interval I.

Let

$$\mathscr{F} := \{\phi \in C(I) : \Lambda^*(\phi) \cap \phi(\Lambda^*(\phi)) = \emptyset \text{ and } \emptyset \neq \Lambda^*(\phi) \subseteq \text{int}(R(\phi))\}. \qquad (4.3.1)$$

Theorem 4.3.1. *If* $F \in \mathscr{F}$ *and* $\Lambda^*(F)$ *is finite, then* $f^n = F$ *has no continuous solution* $f \in C(I)$ *of any* $n \geq 2$ *with* $\Lambda^*(f) = \Lambda^*(F)$.

Proof. To the contrary, suppose that there is a function $f \in C(I)$ such that

$$f^n(x) = F(x),\ \forall x \in I \text{ and } \Lambda^*(f) = \Lambda^*(F).$$

Since $\Lambda^*(F) \subseteq \text{int}(R(F))$, by Theorem 3.1.1 (v), we have

$$\emptyset \neq \Lambda^*(f^i) \subseteq \Lambda^*(F) \subseteq \text{int}(R(F)) \subseteq \text{int}(R(f)),\ i = 1, \ldots, n-1. \qquad (4.3.2)$$

As $\Lambda^*(f) = \Lambda^*(F)$, by Corollary 3.1.9 (i),

$$P_x(f^i, f) = \emptyset,\ \forall x \in \Lambda^*(f^i),\ i = 1, \ldots, n-1. \qquad (4.3.3)$$

This implies

$$\Lambda^*(f) = \Lambda^*(f^i),\ i = 1, \ldots, n-1.$$

By (4.3.2) and the fact every isolated forts of f are points of local extremum of f, for each $x \in \Lambda^*(f)$, there exists $y \in f^{-1}(\{x\})$ such that $y \in \text{int}(I)$ and $y \notin \Lambda(f)$. Suppose $y \notin S(f)$, by Theorem 3.1.1 (vi), $y \in \Lambda^*(f^2)$ and hence $y \in P_x(f,f)$, a contradiction to (4.3.3). Therefore $y \in \Lambda^*(f)$ and

$$f^{-1}(\{x\}) \cap \Lambda^*(f) \neq \emptyset,\ \forall x \in \Lambda^*(f). \qquad (4.3.4)$$

It follows from (4.3.4) and the fact $\Lambda^*(f) = \Lambda^*(F)$ is finite that

$$f^{-1}(\Lambda^*(f)) \cap \Lambda^*(f) = \Lambda^*(f). \tag{4.3.5}$$

This implies $f(\Lambda^*(f)) \subseteq \Lambda^*(f)$. Thus for any $x \in \Lambda^*(F)$, we have

$$F(x) = f^n(x) \in \Lambda^*(F),$$

a contradiction to $F(\Lambda^*(F)) \cap \Lambda^*(F) = \emptyset$. This completes the proof. $\square$

Corollary 4.3.2. *(Cho et al. (2018)) Let $F \in C(K)$. Suppose $\Lambda^*(F) = \{x\}$, $F(x) \neq x$ and $x \in \operatorname{int}(R(F))$, then $f^n = F$ has no solution in $C(K)$ for any $n \geq 2$.*

Proof. Clearly, $F \in \mathscr{F}$ and F satisfies the hypotheses of Theorem 4.3.1. Thus by Theorem 4.3.1, and (4.3.2), $f^n = F$ has no continuous solution f in $C(K)$ for any $n \geq 2$. $\square$

Example 4.3.3. *Consider the function $F : [0, \frac{\pi}{2}] \to [0, \frac{\pi}{2}]$ defined by*

$$F(x) := \begin{cases} \frac{\pi}{32} + \left| |x - \frac{\pi}{8}| \sin\left(\frac{1}{|x-\frac{\pi}{8}|}\right) \right|, & \text{if } x \in [0, \frac{\pi}{8}), \\ \frac{11x}{4} - \frac{5\pi}{16}, & \text{if } x \in [\frac{\pi}{8}, \frac{\pi}{4}], \\ \frac{3\pi}{8} - \left| |x - \frac{\pi}{4}| \sin\left(\frac{1}{|x-\frac{\pi}{4}|}\right) \right|, & \text{if } x \in (\frac{\pi}{4}, \frac{\pi}{2}]. \end{cases}$$

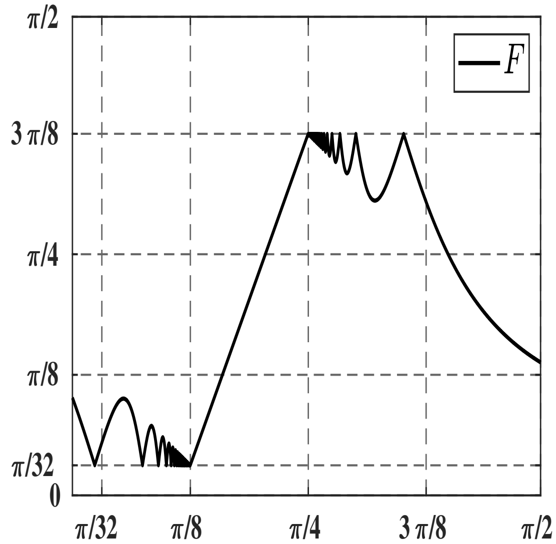


Figure 4.12 : A continuous function $F \in \mathscr{F}$

Clearly, we have $F \in C\left([0, \frac{\pi}{2}]\right)$*,*

$$\Lambda^*(F) = \left\{\frac{\pi}{8}, \frac{\pi}{4}\right\} \subseteq \operatorname{int}(R(F)) = \left(\frac{\pi}{32}, \frac{3\pi}{8}\right) \text{ and } F\left(\left\{\frac{\pi}{8}, \frac{\pi}{4}\right\}\right) = \left\{\frac{\pi}{32}, \frac{3\pi}{8}\right\}$$

(see Figure 4.12). Therefore by Theorem 4.3.1, $f^n = F$ *has no continuous solution* f *for any* $n \geq 2$ *with* $\Lambda^*(f) = \left\{\frac{\pi}{8}, \frac{\pi}{4}\right\}$*.*

CHAPTER 5

HYERS-ULAM STABILITY

In this chapter, we study the Hyers-Ulam stability of $f^n = F$ on K for strictly increasing homeomorphisms (strictly increasing, continuous and onto) and for a class of continuous functions in $\mathscr{N}(K)$.

5.1 FUNCTIONS WITH HEIGHT 0

Let $l, L > 0$ and define

$$\mathscr{C}_L(K) := \{\phi \in C(K) : |\phi(x) - \phi(y)| \le L|x-y|,\ \forall x, y \in K\} \tag{5.1.1}$$

and

$$\mathscr{D}_l(K) := \{\phi \in C(K) : l|x-y| \le |\phi(x) - \phi(y)|,\ \forall x, y \in K\}. \tag{5.1.2}$$

To study the Hyers-Ulam stability of $f^n = F$ on K for an arbitrary strictly increasing homeomorphism, it is enough to study for a strictly increasing homeomorphism F with either $F(x) < x$ or $F(x) > x$ for all $x \in (a,b)$. Indeed, let

$$\mathbb{F} := \{x \in K : F(x) = x\},$$

the set of all fixed points of F and (c,d) are pairwise disjoint intervals with $c, d \in \mathbb{F}$ or $c = a$ or $d = b$. Clearly,

$$K = \mathbb{F} \bigcup \left(\bigcup_{c,d \in \mathbb{F}} (c,d) \right),$$

$F|_{(c,d)} : (c,d) \to (c,d)$ is a strictly increasing homeomorphism, and either

$$c < F(x) < x < d,\ \forall x \in (c,d) \text{ or } c < x < F(x) < d,\ \forall x \in (c,d).$$

Let $g \in C(K)$ such that $|g^n(x) - F(x)| \le \delta$ for all $x \in K$. Suppose that there exists a strictly increasing function $f_{c,d} \in C([c,d])$ such that $f^n_{c,d}(x) = F(x)$ for all $x \in [c,d]$ and

$$|g(x) - f_{c,d}(x)| \leq \varepsilon_{c,d}, \forall x \in [c,d],\ c,d \in \mathbb{F}.$$

Then the function $f : K \to K$ defined by

$$f(x) := \begin{cases} f_{c,d}(x), & \text{if} \quad x \in (c,d), \\ x, & \text{if} \quad x \in \mathbb{F}, \end{cases}$$

is a strictly increasing homeomorphism on K such that

$$f^n(x) = F(x),\ \forall x \in K$$

and

$$|f(x) - g(x)| \leq \sup_{c,d \in \mathbb{F}} \{\varepsilon_{c,d}\},\ \forall x \in K$$

provided the set $\{\varepsilon_{c,d} : c,d \in \mathbb{F}\}$ is bounded. So, the problem is reduced to study the Hyers-Ulam stability of $f^n = F$ on a compact interval $[c,d]$.

Theorem 5.1.1. *Let $F \in C(K)$ be a strictly increasing homeomorphism and $F(x) < x$ for all $x \in (a,b)$ and $n \geq 2$. Suppose that $g \in \mathscr{C}_L(K) \cap \mathscr{D}_l(K)$ is a strictly increasing homeomorphism for fixed constants $L, l > 0$ and satisfies the following conditions:*

(i) *there exists $x_0 \in (a,b)$ such that $g(x_0) < x_0$ and x_0 is a fixed point of $F^{-1} \circ g^n$ and $g^{-1} \circ F \circ (g^{-1})^{n-1}$,*

(ii) *$g \in \mathscr{C}_{L_1}([a,x_0])$ and $g^{-1} \in \mathscr{C}_{L_2}([g(x_0),b])$ for some fixed constants $L_1, L_2 > 0$, and*

(iii) *$|g^n(x) - F(x)| \leq \delta$ for all $x \in K$, for some constant $\delta > 0$.*

Then there exists a strictly increasing homeomorphism f on K such that $f(x) < x$ for all $x \in (a,b)$,

$$f^n(x) = F(x),\ \forall x \in K$$

and

$$|f(x) - g(x)| \leq \max\left\{\frac{1}{(1-r_1)}, \frac{L}{(1-r_2)l^n}\right\}\delta,\ \forall x \in K,$$

where

$$r_1 := \sum_{j=1}^{n-1} L_1^j < 1 \text{ and } r_2 := \sum_{i=1}^{n-1} L_2^i < 1.$$

To prove the above theorem, we need the following results.

Lemma 5.1.2. *Let $f, g \in C(K)$ be homeomorphisms. Suppose $g \in \mathscr{C}_L(K) \cap \mathscr{D}_l(K)$ for some constants $l, L > 0$. Then*

(i) $g^{-1} \in \mathscr{C}_{\frac{1}{l}}(K) \cap \mathscr{D}_{\frac{1}{L}}(K)$,

(ii) $g^i \in \mathscr{C}_{L^i}(K) \cap \mathscr{D}_{l^i}(K)$ *for all* $i \in \mathbb{N}$.

Proof. (i) Let $x, y \in K$. As $g \in \mathscr{C}_L(K) \cap \mathscr{D}_l(K)$, we have

$$\frac{1}{L}|g(g^{-1}(x)) - g(g^{-1}(y))| \leq |g^{-1}(x) - g^{-1}(y)|$$

and

$$|g^{-1}(x) - g^{-1}(y)| \leq \frac{1}{l}|g(g^{-1}(x)) - g(g^{-1}(y))|.$$

This implies $g^{-1} \in \mathscr{C}_{\frac{1}{l}}(K) \cap \mathscr{D}_{\frac{1}{L}}(K)$.
(ii) Assume that $g^{i-1} \in \mathscr{C}_{L^{i-1}}(K) \cap \mathscr{D}_{l^{i-1}}(K)$ for $i \geq 2$. Then for each $x, y \in K$,

$$\begin{aligned} |g^i(x) - g^i(y)| &= |g(g^{i-1}(x)) - g(g^{i-1}(y))| \\ &\leq L|g^{i-1}(x) - g^{i-1}(y)| \leq L^i|x - y|. \end{aligned}$$

Similarly, we get $l^i|x - y| \leq |g^i(x) - g^i(y)|$ for all $x, y \in K$. Thus $g^i \in \mathscr{C}_{L^i}(K) \cap \mathscr{D}_{l^i}(K)$ for all $i \in \mathbb{N}$. □

Lemma 5.1.3. *Let $f \in C(K)$ and $g \in \mathscr{C}_L(K)$ for some constant $L > 0$. Then for each $x \in K$,*

(i) $|g(x) - f(x)| \leq L|f^{-1}(y) - g^{-1}(y)|$, *where* $y = f(x)$.

(ii)
$$|g^i(x) - f^i(x)| \leq \sum_{j=0}^{i-1} L^j |g(f^{i-1-j}(x)) - f(f^{i-1-j}(x))|, \ \forall i \in \mathbb{N}. \tag{5.1.3}$$

Proof. (i) Since $g \in \mathscr{C}_L(K)$, for $x \in K$, it is easy to see that

$$\begin{aligned} |g(x) - f(x)| &= |g(x) - g \circ g^{-1} \circ f(x)| \\ &\leq L|x - g^{-1} \circ f(x)|. \end{aligned}$$

Let $y = f(x)$. Since f is a homeomorphism on K,

$$|g(x) - f(x)| \leq L|x - g^{-1} \circ f(x)| = L|f^{-1}(y) - g^{-1}(y)|.$$

(ii) The result is trivial for $i=1$. Assume that (5.1.3) is true for $i=k$. Then

$$\begin{aligned}|g^{k+1}(x)-f^{k+1}(x)| &= |g(g^k(x))-g(f^k(x))+g(f^k(x))-f(f^k(x))| \\ &\leq L|g^k(x)-f^k(x)|+|g(f^k(x))-f(f^k(x))| \\ &\leq L\sum_{j=0}^{k-1}L^j|g(f^{k-1-j}(x))-f(f^{k-1-j}(x))|+|g(f^k(x))-f(f^k(x))| \\ &= \sum_{j=0}^{k}L^j|g(f^{k-j}(x))-f(f^{k-j}(x))|.\end{aligned}$$

Hence (5.1.3) is proved by induction on i. □

Lemma 5.1.3 is a generalization of the equation (3.7) of Xu and Zhang (2007) by allowing g to be Lipschitz on K (not necessarily contraction).

The proof of Theorem 5.1.1 on the Hyers-Ulam stability of $f^n=F$ on K for a strictly increasing homeomorphism F is a generalization of Theorem 3.1 of Xu and Zhang (2007) by dividing K into two intervals. In the following proof, first we prove the stability of $\psi^n=G$ on $[g(x_0),b]$ using $h=g^{-1}$, where $G=F^{-1}$ and ψ is unknown. Next, we prove the stability of $\phi^n=F$ on $[a,x_0]$ using g, where ϕ is unknown. Then the map $f:[a,b]\to[a,b]$ defined by

$$f(x):=\begin{cases}\phi(x), & \text{if } x\in[a,x_0],\\ \psi^{-1}(x), & \text{if } x\in(x_0,b],\end{cases} \tag{5.1.4}$$

satisfies

$$f^n(x)=F(x),\ \forall x\in K$$

and

$$|f(x)-g(x)|\leq \max\left\{\frac{1}{(1-r_1)},\frac{L}{(1-r_2)l^n}\right\}\delta,\ \forall x\in K.$$

Proof of Theorem 5.1.1. Let

$$h(x):=g^{-1}(x) \text{ and } G(x):=F^{-1}(x),\ \forall x\in K.$$

Since g is a strictly increasing homeomorphism on K and $g\in\mathscr{C}_L(K)\cap\mathscr{D}_l(K)$, h^n is a strictly increasing homeomorphism on K and $h^n\in\mathscr{C}_{\frac{1}{l^n}}(K)\cap\mathscr{D}_{\frac{1}{L^n}}(K)$ by Lemma 5.1.2 (i) and (ii). From Lemma 5.1.3 (i) and hypothesis (iii), for each $x\in K$, we have

$$|h^n(x)-G(x)|\leq\frac{1}{l^n}|F(y)-g^n(y)|\leq\frac{\delta}{l^n}. \tag{5.1.5}$$

As g is strictly increasing and $g(x_0) < x_0$, h is strictly increasing and

$$g(x_0) < h(g(x_0)) < h(y), \ \forall y \in (g(x_0), b).$$

This implies $h([g(x_0), b]) = [x_0, b] \subseteq [g(x_0), b]$. It follows from the hypothesis (i) that

$$g^n(x_0) = F(x_0) \text{ and } (g^{-1})^n(g(x_0)) = F^{-1}(g(x_0)). \tag{5.1.6}$$

Let $y_k := h^k(g(x_0))$ for $k = 0, 1, \ldots, n$, and $y_{k+n} := G(y_k)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By (5.1.6) and the monotonicity of G and h on (a, b), we get

$$y_0 < y_1 < \cdots < y_{n-1} < y_n = G(g(x_0)) < y_{n+1} < \cdots < b \tag{5.1.7}$$

and $\lim\limits_{k\to\infty} y_k = b$. This implies $[g(x_0), b) = \bigcup_{k=0}^{\infty}[y_k, y_{k+1}]$.

Define $J_k := [y_k, y_{k+1}]$, $k \in \mathbb{N}_0$. For each $y \in J_k$, $k \in \mathbb{N}_0$,

$$\psi_k(y) := \begin{cases} h(y), & \text{if } k = 0, 1, \ldots, n-2, \\ G \circ \psi_{k-n+1}^{-1} \circ \cdots \circ \psi_{k-1}^{-1}(y), & \text{if } k \geq n-1. \end{cases} \tag{5.1.8}$$

It is easy to see that for each $k \in \mathbb{N}_0$, $\psi_k : J_k \to J_{k+1}$ is strictly increasing, $\psi_k(y_k) = y_{k+1}$, and $\psi_k(y_{k+1}) = y_{k+2}$. Define ψ on $[y_0, b]$ by

$$\psi(y) := \begin{cases} \psi_k(y), & \text{if } y \in J_k, \\ b, & \text{if } y = b. \end{cases} \tag{5.1.9}$$

It is clear that $\psi([y_0, b]) = [y_1, b] \subseteq [y_0, b]$, and for each $y \in J_k$,

$$\begin{aligned} \psi^n(y) &= \psi_{k+n-1} \circ \psi_{k+n-2} \circ \cdots \circ \psi_{k+1} \circ \psi_k(y) \\ &= G \circ \psi_k^{-1} \circ \psi_{k+1}^{-1} \circ \cdots \circ \psi_{k+n-2}^{-1} \circ \psi_{k+n-2} \circ \cdots \circ \psi_{k+1} \circ \psi_k(y) \\ &= G(y). \end{aligned}$$

Thus ψ defined in (5.1.9) is a strictly increasing continuous solution of $\psi^n = G$ on $[y_0, b]$. Moreover, since $h([y_0, b]) \subseteq [y_0, b]$, by Lemma 5.1.3 (ii), for each $y \in [y_0, b]$, we have

$$|h^i(y) - \psi^i(y)| \leq \sum_{j=0}^{i-1} L_2^j |h(\psi^{i-1-j}(y)) - \psi(\psi^{i-1-j}(y))|, \ \forall i \in \mathbb{N}. \tag{5.1.10}$$

Claim 1: For each $y \in J_k,\ k \in \mathbb{N}_0$,

$$|h(y) - \psi(y)| \leq \frac{\delta}{(1-r_2)l^n}. \tag{5.1.11}$$

The above inequality is trivial for $k = 0, \ldots, n-2$ by (5.1.8). Assume (5.1.11) for $y \in J_k$, for all $k \leq m$, where $m \geq n-2$. Let $y \in J_{m+1}$ and define

$$\alpha = \alpha_{m-n+2} := \psi_{m-n+2}^{-1} \circ \psi_{m-n+3}^{-1} \circ \cdots \circ \psi_{m-1}^{-1} \circ \psi_m^{-1}(y). \tag{5.1.12}$$

Observe that $\alpha \in J_{m-n+2}$ and $J_{m-n+2} \subseteq [y_0, b]$. Then for $i = 1, \ldots, n-1$, we have

$$\begin{aligned} \psi^i(\alpha) &= \underbrace{\psi \circ \cdots \circ \psi}_{i \text{ times}} \circ \psi_{m-n+2}^{-1} \circ \psi_{m-n+3}^{-1} \circ \cdots \circ \psi_m^{-1}(y) \\ &= \psi_{i+m-n+1} \circ \cdots \circ \psi_{m-n+2} \circ \psi_{m-n+2}^{-1} \circ \psi_{m-n+3}^{-1} \circ \cdots \circ \psi_m^{-1}(y). \end{aligned}$$

So,

$$\psi^i(\alpha) \in J_{i+m-n+2} \subseteq [y_0, b],\ i = 1, \ldots, n-1 \tag{5.1.13}$$

and

$$y = \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_{m-n+2}(\alpha). \tag{5.1.14}$$

From (5.1.8) and (5.1.14), we have

$$\begin{aligned} |h(y) - \psi(y)| &= |h(\psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_{m-n+2}(\alpha)) \\ &\quad - \psi_{m+1}(\psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_{m-n+2}(\alpha))| \\ &= |h(\psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_{m-n+2}(\alpha)) \\ &\quad - G \circ \psi_{m-n+2}^{-1} \circ \cdots \circ \psi_m^{-1}(\psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_{m-n+2}(\alpha))| \\ &= |h(\psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_{m-n+2}(\alpha)) - G(\alpha)| \\ &= |h(\underbrace{\psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_{m-n+2}}_{n-1 \text{ times}}(\alpha)) - h^n(\alpha) + h^n(\alpha) - G(\alpha)| \\ &\leq |h(\psi^{n-1}(\alpha)) - h(h^{n-1}(\alpha))| + |h^n(\alpha) - G(\alpha)|. \end{aligned}$$

Since $\psi^{n-1}(\alpha), h^{n-1}(\alpha) \in [y_0, b]$ and $h \in \mathscr{C}_{L_2}([y_0, b])$,

$$|h(\psi^{n-1}(\alpha)) - h(h^{n-1}(\alpha))| \leq L_2 |\psi^{n-1}(\alpha) - h^{n-1}(\alpha)|.$$

As $\alpha \in [y_0, b]$, from (5.1.10) and by (5.1.5), we get

$$\begin{aligned} L_2|h^{n-1}(\alpha) - \psi^{n-1}(\alpha)| &+ |h^n(\alpha) - G(\alpha)| \\ &\le L_2 \sum_{i=0}^{n-2} L_2^i |h(\psi^{n-2-i}(\alpha)) - \psi(\psi^{n-2-i}(\alpha))| + \frac{\delta}{l^n} \\ &= \sum_{i=1}^{n-1} L_2^i |h(\psi^{n-1-i}(\alpha)) - \psi(\psi^{n-1-i}(\alpha))| + \frac{\delta}{l^n}. \end{aligned}$$

Note that $h(\psi^{n-1-i}(\alpha)) \in J_{m-i+1},\ i = 1, \ldots, n-1$ by (5.1.13). Thus by assumption,

$$\begin{aligned} \sum_{i=1}^{n-1} L_2^i |h(\psi^{n-1-i}(\alpha)) - \psi(\psi^{n-1-i}(\alpha))| + \frac{\delta}{l^n} &\le \frac{r_2\delta}{(1-r_2)l^n} + \frac{\delta}{l^n} \\ &= \frac{\delta}{(1-r_2)l^n}. \end{aligned}$$

This implies

$$|h(y) - \psi(y)| \le \frac{\delta}{(1-r_2)l^n},\ \forall y \in [y_0, b]. \tag{5.1.15}$$

Now, to prove the stability of $\phi^n = F$ on $[a, x_0]$, let $x_k := g^k(x_0)$, $k = 0, 1, \ldots, n$ and $x_{k+n} := F(x_k)$, $k \in \mathbb{N}_0$. Observe that the sequence $\{x_k\}_{k \in \mathbb{N}_0}$ is strictly decreasing, and $\lim\limits_{k\to\infty} x_k = a$. This implies $(a, x_0] = \bigcup_{k=0}^{\infty} [x_{k+1}, x_k]$.

Let $I_k := [x_{k+1}, x_k]$, $k \in \mathbb{N}_0$. For each $x \in I_k$, $k \in \mathbb{N}_0$, define

$$\phi_k(x) := \begin{cases} g(x), & \text{if } k = 0, 1, \ldots, n-2, \\ F \circ \phi_{k-n+1}^{-1} \circ \cdots \circ \phi_{k-1}^{-1}(x), & \text{if } k \ge n-1. \end{cases} \tag{5.1.16}$$

Clearly, for each $k \in \mathbb{N}_0$, $\phi_k : I_k \to I_{k+1}$ is strictly increasing and continuous on I_k, $\phi_k(x_k) = x_{k+1}$, and $\phi_k(x_{k+1}) = x_{k+2}$. Define a function ϕ on $[a, x_0]$ by

$$\phi(x) := \begin{cases} a, & \text{if } x = a, \\ \phi_k(x), & \text{if } x \in I_k. \end{cases} \tag{5.1.17}$$

Note that $\phi([a, x_0]) \subseteq [a, x_0]$. Then for each $x \in I_k$, $k \in \mathbb{N}_0$, from (5.1.16), we get

$$\begin{aligned} \phi^n(x) &= \phi_{k+n-1} \circ \phi_{k+n-2} \circ \cdots \circ \phi_{k+1} \circ \phi_k(x) \\ &= F \circ \phi_k^{-1} \circ \phi_{k+1}^{-1} \circ \cdots \circ \phi_{k+n-2}^{-1} \circ \phi_{k+n-2} \circ \cdots \circ \phi_{k+1} \circ \phi_k(x) \\ &= F(x). \end{aligned}$$

Thus ϕ defined in (5.1.17) is a strictly increasing continuous solution of $\phi^n = F$ on $[a,x_0]$. Moreover, since $g([a,x_0]) \subseteq [a,x_0]$, from Lemma 5.1.3 (ii), for each $x \in [a,x_0]$, we have

$$|g^i(x) - \phi^i(x)| \le \sum_{j=0}^{i-1} L_1^j |g(\phi^{i-1-j}(x)) - \phi(\phi^{i-1-j}(x))|, \ \forall i \in \mathbb{N}. \tag{5.1.18}$$

Claim 2: For each $x \in I_k$, $k \in \mathbb{N}_0$,

$$|g(x) - \phi(x)| \le \frac{\delta}{(1-r_1)}. \tag{5.1.19}$$

For $x \in I_k$, $k = 0,1,\ldots,n-2$, the above inequality is trivial by (5.1.16). Assume (5.1.19) for $x \in I_k$, for all $k \le m$, where $m \ge n-2$. Let $x \in I_{m+1}$ and define

$$\beta = \beta_{m-n+2} := \phi_{m-n+2}^{-1} \circ \phi_{m-n+3}^{-1} \circ \cdots \circ \phi_m^{-1}(x) \in I_{m-n+2} \subseteq [a,x_0].$$

Then for $j = 1,\ldots,n-1$,

$$\phi^j(\beta) \in I_{j+m-n+2} \tag{5.1.20}$$

and

$$x = \phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_{m-n+2}(\beta). \tag{5.1.21}$$

It follows from (5.1.16) and (5.1.21) that

$$\begin{aligned}
|g(x) - \phi(x)| &= |g(\phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_{m-n+2}(\beta)) - \phi_{m+1}(\phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_{m-n+2}(\beta))| \\
&= |g(\phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_{m-n+2}(\beta)) \\
&\quad - F \circ \phi_{m-n+2}^{-1} \circ \cdots \circ \phi_m^{-1}(\phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_{m-n+2}(\beta))| \\
&= |g(\phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_{m-n+2}(\beta)) - F(\beta)| \\
&= |g(\underbrace{\phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_{m-n+2}}_{n-1 \text{ times}}(\beta)) - g^n(\beta) + g^n(\beta) - F(\beta)|.
\end{aligned}$$

As $g \in \mathscr{C}_{L_1}([a,x_0])$, and $\phi^{n-1}(\beta)$, $g^{n-1}(\beta) \in [a,x_0]$, we get

$$|g(\phi^{n-1}(\beta)) - g(g^{n-1}(\beta))| \le L_1 |\phi^{n-1}(\beta) - g^{n-1}(\beta)|.$$

Since $\beta \in [a,x_0]$, from (5.1.18) and by hypothesis (iii),

$$L_1|\phi^{n-1}(\beta) - g^{n-1}(\beta)| \quad + \quad |g^n(\beta) - F(\beta)|$$

$$
\begin{aligned}
&\leq L_1 \sum_{j=0}^{n-2} L_1^j |g(\phi^{n-2-j}(\beta)) - \phi(\phi^{n-2-j}(\beta))| + \delta \\
&= \sum_{j=1}^{n-1} L_1^j |g(\phi^{n-1-j}(\beta)) - \phi(\phi^{n-1-j}(\beta))| + \delta.
\end{aligned}
$$

From the assumption in (5.1.19),

$$
\begin{aligned}
\sum_{j=1}^{n-1} L_1^j |h(\phi^{n-1-j}(\beta)) - \phi(\phi^{n-1-j}(\beta))| + \delta &\leq r_1(1-r_1)^{-1}\delta + \delta \\
&= (1-r_1)^{-1}\delta.
\end{aligned}
$$

Thus

$$
|g(x) - \phi(x)| \leq (1-r_1)^{-1}\delta, \ \forall x \in [a, x_0]. \tag{5.1.22}
$$

Note that $\psi([g(x_0), b]) = [x_0, b]$. Define $f : [a,b] \to [a,b]$ by

$$
f(x) := \begin{cases} \phi(x), & \text{if } x \in [a, x_0], \\ \psi^{-1}(x), & \text{if } x \in (x_0, b]. \end{cases}
$$

Clearly, f is well-defined and strictly increasing homeomorphism on K, $f(x) < x$ for all $x \in (a,b)$ and

$$
f^n(x) = F(x), \ \forall x \in K.
$$

Since $h^{-1} \in \mathscr{C}_L(K)$, for each $y \in (x_0, b]$, we have

$$
\begin{aligned}
|g(y) - f(y)| &= |h^{-1}(y) - \psi^{-1}(y)| \\
&= |h^{-1}(y) - h^{-1}(h(\psi^{-1}(y)))| \\
&\leq L|y - h(\psi^{-1}(y))|.
\end{aligned}
$$

As $\psi^{-1}(y) = x \in [y_0, b]$,

$$
|g(y) - f(y)| \leq L|\psi(x) - h(x)| \leq \frac{L\delta}{(1-r_2)l^n}, \ \forall y \in (x_0, b] \tag{5.1.23}
$$

by (5.1.15). Thus from (5.1.22) and (5.1.23), we get

$$
|g(x) - f(x)| \leq \max\left\{\frac{1}{(1-r_1)}, \frac{L}{(1-r_2)l^n}\right\}\delta, \ \forall x \in K.
$$

□

Theorem 5.1.4. *Let $F \in C(K)$ be a strictly increasing homeomorphism and $F(x) > x$ for all $x \in (a,b)$ and $n \geq 2$. Suppose that $g \in \mathscr{C}_L(K) \cap \mathscr{D}_l(K)$ is a strictly increasing homeomorphism for fixed constants $L, l > 0$ and satisfies the following conditions:*

(i) *there exists $x_0 \in (a,b)$ such that $g(x_0) > x_0$ and x_0 is a fixed point of $F^{-1} \circ g^n$ and $g^{-1} \circ F \circ (g^{-1})^{n-1}$,*

(ii) *$g^{-1} \in \mathscr{C}_{L_1}([a, g(x_0)])$ and $g \in \mathscr{C}_{L_2}([x_0, b])$ for some fixed constants $L_1, L_2 > 0$, and*

(iii) *$|g^n(x) - F(x)| \leq \delta$ for all $x \in K$, for some constant $\delta > 0$.*

Then there exists a strictly increasing homeomorphism $f \in C(K)$ such that $f(x) > x$ for all $x \in (a,b)$,

$$f^n(x) = F(x), \forall x \in K$$

and

$$|f(x) - g(x)| \leq \max\left\{\frac{L}{(1-r_1)l^n}, \frac{1}{(1-r_2)}\right\}\delta, \ \forall x \in K,$$

where

$$r_1 := \sum_{j=1}^{n-1} L_1^j < 1 \text{ and } r_2 := \sum_{i=1}^{n-1} L_2^i < 1.$$

Proof. The proof is similar to that of Theorem 5.1.1. Here we obtain a continuous function $\psi : [a, g(x_0)] \to [a, g(x_0)]$ such that

$$\psi^n(x) = G(x), \ \forall x \in [a, g(x_0)]$$

and

$$|\psi(x) - h(x)| \leq \frac{L\delta}{(1-r_1)l^n}, \ \forall x \in [a, g(x_0)],$$

where $h = g^{-1}$ and $G = F^{-1}$ on K. Also, we obtain $\phi : [x_0, b] \to [x_0, b]$ such that

$$\phi^n(y) = F(y), \ \forall y \in [x_0, b]$$

and

$$|\phi(y) - g(y)| \leq \frac{\delta}{(1-r_2)}, \ \forall y \in [x_0, b].$$

Then the function $f : [a,b] \to [a,b]$ defined by

$$f(x) := \begin{cases} \psi^{-1}(x), & \text{if } x \in [a, x_0), \\ \phi(x), & \text{if } x \in [x_0, b], \end{cases}$$

is a strictly increasing homeomorphism on K, and satisfies $f(x) > x$ for all $x \in (a,b)$,

$$f^n(x) = F(x),\ \forall x \in K$$

and

$$|f(x) - g(x)| \leq \max\left\{\frac{L}{(1-r_1)l^n}, \frac{1}{(1-r_2)}\right\}\delta,\ \forall x \in K.$$

$\square$

Example 5.1.5. *Consider the continuous functions $F, g : [0,1] \to [0,1]$ defined by*

$$F(x) := \begin{cases} \frac{x^2}{2}, & \text{if } x \in [0, \frac{1}{2}], \\ \frac{3x-1}{4}, & \text{if } x \in [\frac{1}{2}, \frac{2}{3}], \\ \frac{9x-5}{4}, & \text{if } x \in [\frac{2}{3}, 1], \end{cases} \text{ and } g(x) := \begin{cases} \frac{x}{2}, & \text{if } x \in [0, \frac{1}{2}], \\ \frac{3x-1}{2}, & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, F and g are strictly increasing homeomorphisms on $[0,1]$ (see Figures 5.1 and 5.2) and

$$\frac{1}{2}|x-y| \leq |g(x) - g(y)| \leq \frac{3}{2}|x-y|,\ \forall x, y \in [0,1].$$

This implies that $g \in \mathscr{C}_{\frac{3}{2}}([0,1]) \cap \mathscr{D}_{\frac{1}{2}}([0,1])$, and we have

$$g^{-1}(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{4}), \\ \frac{2x+1}{3}, & \text{if } x \in [\frac{1}{4}, 1]. \end{cases}$$

It is easy to verify that

$$g\left(\frac{1}{2}\right) = \frac{1}{4} < \frac{1}{2},\ g^2\left(\frac{1}{2}\right) = F\left(\frac{1}{2}\right) = \frac{1}{8},$$

and

$$g^{-2}\left(\frac{1}{4}\right) = F^{-1}\left(\frac{1}{4}\right) = \frac{2}{3}.$$

Also, we have

$$|g(x) - g(y)| \leq \frac{1}{2}|x-y|,\ \forall x, y \in \left[0, \frac{1}{2}\right]$$

and

$$|g^{-1}(x) - g^{-1}(y)| \leq \frac{2}{3}|x-y|,\ \forall x, y \in \left[\frac{1}{4}, 1\right].$$

Moreover,

$$|g^2(x) - F(x)| \leq \left|g^2\left(\frac{1}{4}\right) - F\left(\frac{1}{4}\right)\right| = \frac{1}{32} = \delta,\ \forall x \in [0,1]$$

with $r_1 = \frac{1}{2}$ *and* $r_2 = \frac{2}{3} < 1$.

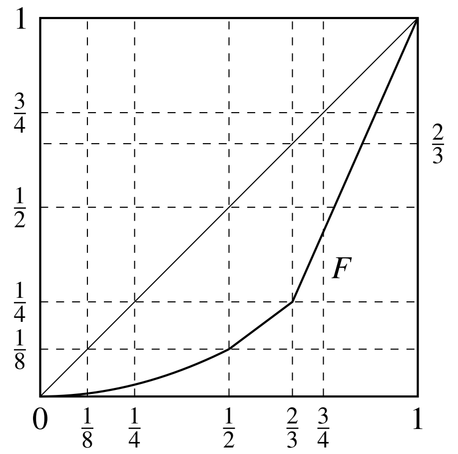


Figure 5.1 A strictly increasing homeomorphism F with $F(x) < x$ on $[0,1]$

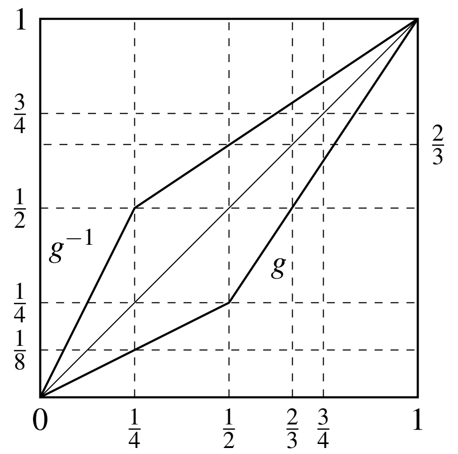


Figure 5.2 An approximate solution g of $f^2 = F$ on $[0,1]$

Note that $F \notin R_{0,0}([0,1]) \cup R_{1,0}([0,1])$*. So,* F *does not satisfy the conditions of Corollaries 4.2 and 4.3 of Xu and Zhang (2007). By Theorem 5.1.1, there exists a strictly increasing homeomorphism* f *on* $[0,1]$ *such that*

$$f^2(x) = F(x),\ \forall x \in [0,1]$$

and

$$|f(x) - g(x)| \le \max\left\{\frac{1}{(1-r_1)}, \frac{L}{(1-r_2)l^2}\right\}\delta = \frac{9}{16},\ \forall x \in [0,1].$$

In the following corollaries, we discuss the Hyers-Ulam stability of $f^n = F$ on K for $F \in R_{a,0}(K)$ or $F \in R_{b,0}(K)$.

Corollary 5.1.6. *Let $F \in R_{a,0}(K)$ and $n \geq 2$. Suppose that $g \in \mathscr{C}_{L_1}(K)$ is strictly increasing on K for some fixed constant $L_1 > 0$ and satisfies the following conditions:*

(i) *$g(b) < b$ and $g^n(b) = F(b)$,*

(ii) *$|g^n(x) - F(x)| \leq \delta$ for all $x \in K$, for some constant $\delta > 0$.*

Then $f^n = F$ has a strictly increasing continuous solution f on K such that $f(x) < x$ for all $x \in (a,b]$ and

$$|f(x) - g(x)| \leq \frac{\delta}{(1-r_1)}, \ \forall x \in K,$$

where

$$r_1 := \sum_{j=1}^{n-1} L_1^j < 1.$$

Proof. The proof follows from the stability result of $\phi^n = F$ on $[a, x_0]$ in Theorem 5.1.1. □

Corollary 5.1.7. *Let $F \in R_{b,0}(K)$ and $n \geq 2$. Suppose that $g \in \mathscr{C}_{L_2}(K)$ is strictly increasing on K for some fixed constant $L_2 > 0$ and satisfies the following conditions:*

(i) *$g(a) > a$ and $g^n(a) = F(a)$,*

(ii) *$|g^n(x) - F(x)| \leq \delta$ for all $x \in K$, for some constant $\delta > 0$.*

Then $f^n = F$ has a strictly increasing continuous solution f on K such that $f(x) > x$ for all $x \in [a,b)$ and

$$|f(x) - g(x)| \leq \frac{\delta}{(1-r_2)}, \ \forall x \in K,$$

where

$$r_2 := \sum_{j=1}^{n-1} L_2^j < 1.$$

Proof. The proof follows from the stability result of $\phi^n = F$ on $[x_0, b]$ in Theorem 5.1.4. □

Example 5.1.8. *Consider the continuous functions $F, g : [0,1] \to [0,1]$ defined by*

$$F(x) := \begin{cases} \frac{84x}{512} + \frac{485}{512}, & \text{if } x \in [0, \frac{1}{4}], \\ \frac{x}{64} + \frac{63}{64}, & \text{if } x \in (\frac{1}{4}, 1], \end{cases} \text{ and } g(x) := \frac{3x}{8} + \frac{5}{8}, \ x \in [0,1].$$

Clearly, F and g are strictly increasing and continuous, $F(x) > x$ for all $x \in [0,1)$ (see Figures 5.3 and 5.4), and

$$|g(x) - g(y)| \leq \frac{3}{8} |x-y|, \ \forall x, y \in [0,1].$$

This implies $g \in \mathscr{C}_{\frac{3}{8}}([0,1])$. *It is easy to verify that*

$$g^3(x) = \frac{27x}{512} + \frac{485}{512},\ \forall x \in [0,1],$$

and

$$g(0) = \frac{5}{8} > 0,\ g^3(0) = F(0) = \frac{485}{512}.$$

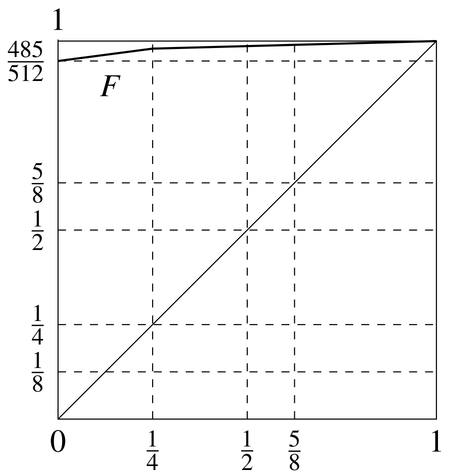


Figure 5.3 A strictly increasing function F with $F(x) > x$ on $[0,1]$

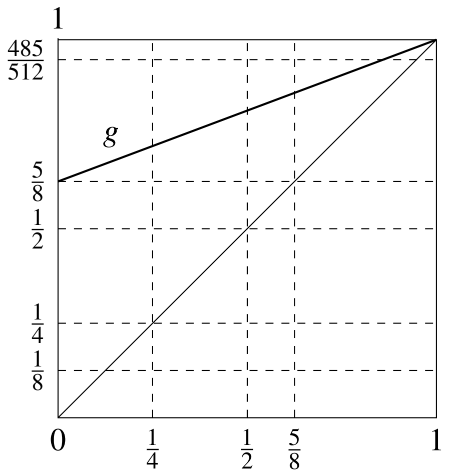


Figure 5.4 An approximate solution g of $f^3 = F$ on $[0,1]$

Moreover, for each $x \in [0,1]$, *we have*

$$|g^3(x) - F(x)| \le \left| g^3\left(\frac{1}{4}\right) - F\left(\frac{1}{4}\right) \right| = \frac{57}{2048} = \delta$$

with $r_2 = \frac{33}{64} < 1$. *Therefore by Corollary 5.1.7, there exists a strictly increasing continuous function* f *on* $[0,1]$ *such that* $f(x) > x$ *for all* $x \in [0,1)$,

$$f^3(x) = F(x), \ \forall x \in [0,1],$$

and for each $x \in [0,1]$,

$$|g(x) - f(x)| \le \frac{\delta}{(1-r_2)} = \frac{57}{992}.$$

5.2 FUNCTIONS WITH HEIGHT 1

Let $l_0, L_0 > 0$ and $l_0 < L_0$. Define

$$\mathscr{N}_1(K, l_0, L_0) := \left\{\phi \in \mathscr{N}_1(K) : l_0|x-y| \le |\phi(x) - \phi(y)| \le L_0|x-y|, \forall x, y \in Ch_\phi\right\},$$

where $\mathscr{N}_1(K)$ as in (4.1.5). Let $F \in \mathscr{N}_1(K, l_0, L_0)$ and $l, L > 0$ with $l < L$. Define

$$\mathscr{N}_F(K, l, L) := \{\phi \in \mathscr{N}_1(K, l, L) : Ch_\phi = Ch_F\}.$$

Theorem 5.2.1. *Let* $F \in \mathscr{N}_1(K, l_0, L_0)$ *and* $n \ge 1$. *Suppose* $g \in \mathscr{N}_F(K, l, L)$ *and*

(i) *Equation* $f^n = F$ *has the Hyers-Ulam stability on* Ch_F,

(ii) $|g^n(x) - F(x)| \le \delta$ *for all* $x \in K$, *for a constant* $\delta > 0$.

Then $f^n = F$ *has a solution* $f \in \mathscr{N}_1(K)$ *such that for each* $x \in K$,

$$|g(x) - f(x)| \le \frac{\left(L^n l_0^{-1} \varepsilon_\delta^0 + (1+L)\delta\right)}{l^n},$$

where ε_δ^0 *depends only on* δ.

Proof. Let $G(x) = g^n(x)$ for all $x \in K$. Since $g \in \mathscr{N}_F(K, l, L)$, $G \in \mathscr{N}_F(K, l^n, L^n)$ and $Ch_g = Ch_G = Ch_F$. By Theorem 4.2.7, g is of the form

$$g(x) = \begin{cases} g_0(x), & \text{if } x \in Ch_G, \\ (G_0^{-1} \circ g_0 \circ G)(x), & \text{if } x \in K \setminus Ch_G, \end{cases}$$

where $G_0 = G|_{Ch_G}$ and $g_0 = g|_{Ch_G}$. Since $|g_0^n(x) - F_0(x)| \le \delta$ for all $x \in Ch_F$, by assumption (i), there exists $f_0 : Ch_F \to Ch_F$ satisfying $f^n(x) = F(x)$ for all $x \in Ch_F$ and

$$|f_0(x) - g_0(x)| \le \varepsilon_\delta^0, \ \forall x \in Ch_F \tag{5.2.1}$$

for some ε_δ^0 depends only on δ. Observe that $\varepsilon_\delta^0 \le L^n l_0^{-1} \varepsilon_\delta^0$, by the fact $L, l_0^{-1} \ge 1$. Let

$$f(x) := \begin{cases} f_0(x), & \text{if } x \in Ch_F, \\ (F_0^{-1} \circ f_0 \circ F)(x), & \text{if } x \in K \setminus Ch_F. \end{cases}$$

By (4.2.2) and Theorem 4.1.13 (ii), f is a solution of $f^n = F$ on K and $f \in \mathscr{N}_1(K)$.

From the fact $g \in \mathscr{N}_F(K, l, L)$ and hypothesis (ii), for each $x \in K \setminus Ch_F$, we have

$$|(g_0 \circ F)(x) - (g_0 \circ G)(x)| \le L|F(x) - G(x)| \le L\delta.$$

Since $G \in \mathscr{N}_F(K, l^n, L^n)$,

$$\begin{aligned} |(g_0 \circ F)(x) - (g_0 \circ G)(x)| &= |(G_0 \circ G_0^{-1} \circ g_0 \circ F)(x) - (G_0 \circ G_0^{-1} \circ g_0 \circ G)(x)| \\ &\ge l^n |(G_0^{-1} \circ g_0 \circ F)(x) - (G_0^{-1} \circ g_0 \circ G)(x)|. \end{aligned}$$

Thus

$$|(G_0^{-1} \circ g_0 \circ F)(x) - (G_0^{-1} \circ g_0 \circ G)(x)| \le L\delta l^{-n}, \forall x \in K \setminus Ch_F. \tag{5.2.2}$$

Now, for $x \in K \setminus Ch_F$, let $u = f_0 \circ F(x)$ and $v = g_0 \circ F(x)$. It follows from (5.2.1) that

$$|u - v| = |f_0(F(x)) - g_0(F(x))| \le \varepsilon_\delta^0. \tag{5.2.3}$$

Then

$$\begin{aligned} |(F_0^{-1} \circ f_0 \circ F)(x) &- (G_0^{-1} \circ g_0 \circ F)(x)| = |F_0^{-1}(u) - G_0^{-1}(v)| \\ &= |(G_0^{-1} \circ G_0 \circ F_0^{-1})(u) - (G_0^{-1} \circ F_0 \circ F_0^{-1})(v)|. \end{aligned}$$

As $F^{-1} \in \mathscr{N}_1(K, L_0^{-1}, l_0^{-1})$ and $G^{-1} \in \mathscr{N}_F(K, L^{-n}, l^{-n})$, for each $x, y \in Ch_F = Ch_G$,

$$|F_0^{-1}(x) - F_0^{-1}(y)| \le \frac{1}{l_0}|x - y| \text{ and } |G_0^{-1}(x) - G_0^{-1}(y)| \le \frac{1}{l^n}|x - y|. \tag{5.2.4}$$

It follows from the fact $G \in \mathscr{N}_F(K, l^n, L^n)$ and hypothesis (ii) that

$$\begin{aligned} |(G_0^{-1} \circ G_0 \circ F_0^{-1})(u) - (G_0^{-1} \circ F_0 \circ F_0^{-1})(v)| &\le \frac{1}{l^n}|(G_0 \circ F_0^{-1})(u) - (F_0 \circ F_0^{-1})(v)| \\ &\le \frac{1}{l^n}(|(G_0 \circ F_0^{-1})(u) - (G_0 \circ F_0^{-1})(v) \\ &+ (G_0 \circ F_0^{-1})(v) - (F_0 \circ F_0^{-1})(v)|) \\ &\le \frac{1}{l^n}(L^n|F_0^{-1}(u) - F_0^{-1}(v)| + \delta). \end{aligned}$$

From (5.2.3) and (5.2.4), we have

$$\begin{aligned}\frac{1}{l^n}(L^n|F_0^{-1}(u)-F_0^{-1}(v)|+\delta) &= \frac{1}{l^n}(L^n l_0^{-1}|u-v|+\delta)\\ &\leq \frac{1}{l^n}(L^n l_0^{-1}\varepsilon_\delta^0+\delta).\end{aligned}$$

This implies

$$|(F_0^{-1}\circ f_0\circ F)(x)-(G_0^{-1}\circ g_0\circ F)(x)| \leq \frac{1}{l^n}(L^n l_0^{-1}\varepsilon_\delta^0+\delta), \forall x\in K\setminus Ch_F. \qquad (5.2.5)$$

For each $x\in K$, we have

$$\begin{aligned}|f(x)-g(x)| &= |(F_0^{-1}\circ f_0\circ F)(x)-(G_0^{-1}\circ g_0\circ G)(x)|\\ &\leq |(F_0^{-1}\circ f_0\circ F)(x)-(G_0^{-1}\circ g_0\circ F)(x)\\ &+ (G_0^{-1}\circ g_0\circ F)(x)-(G_0^{-1}\circ g_0\circ G)(x)|.\end{aligned}$$

Thus from (5.2.1), (5.2.2), (5.2.5), and the fact $\varepsilon_\delta^0 \leq L^n l_0^{-1}\varepsilon_\delta^0$ that, we get

$$|f(x)-g(x)| \leq \frac{1}{l^n}(L^n l_0^{-1}\varepsilon_\delta^0+\delta)+\frac{L\delta}{l^n} = \frac{\left(L^n l_0^{-1}\varepsilon_\delta^0+(1+L)\delta\right)}{l^n}, \ \forall x\in K.$$

□

Example 5.2.2. *Let* $F,g:[0,2]\to[0,2]$ *be defined by*

$$F(x) := \begin{cases} \frac{1}{64}(x^4+6x^3+21x^2+36x), & \text{if } x\in[0,1],\\ \frac{3}{2}-\frac{x}{2}, & \text{if } x\in\left(1,\frac{3}{2}\right],\\ \frac{3}{4}-12\left(x-\frac{2n-1}{n}\right)\left(x-\frac{2n+1}{n+1}\right), & \text{if } x\in\left[\frac{2n-1}{n},\frac{2n+1}{n+1}\right),\ (n\geq 2),\\ \frac{3}{4}, & \text{if } x=2,\end{cases}$$

and

$$g(x) := \begin{cases} \frac{1}{8}(x^2+7x), & \text{if } x\in[0,1],\\ 1+12\left(x-\frac{n+1}{n}\right)\left(x-\frac{n+2}{n+1}\right), & \text{if } x\in\left(\frac{n+2}{n+1},\frac{n+1}{n}\right],\ (n\geq 2),\\ \frac{11}{8}-\frac{x}{4}, & \text{if } x\in\left[\frac{3}{2},2\right].\end{cases}$$

It is easy to see that $Ch_F = Ch_g = [0,1]$*,* $F(Ch_F) = [0,1] = g(Ch_g)$*, and* F *and* g *are strictly increasing continuous on the characteristic interval of* F *(see Figure 5.5 and*

Figure 5.6). Clearly, $F, g \in \mathscr{N}_1([0,2])$ by Theorem 4.1.13 (i). Also, F and g satisfy

$$\frac{9}{16}|x-y| \leq |F(x)-F(y)| \leq \frac{25}{16}|x-y|, \ \forall x,y \in [0,1],$$

and

$$\frac{7}{8}|x-y| \leq |g(x)-g(y)| \leq \frac{9}{8}|x-y|, \ \forall x,y \in [0,1].$$

Thus $F \in \mathscr{N}_1\left([0,2], \frac{9}{16}, \frac{25}{16}\right)$ and $g \in \mathscr{N}_F\left([0,2], \frac{7}{8}, \frac{9}{8}\right)$. Moreover, we have

$$g_0^2(x) = \frac{1}{512}(x^4 + 14x^3 + 105x^2 + 392x)$$

and

$$\left|g_0^2(x) - F_0(x)\right| \leq \frac{1}{16}, \ \forall x \in [0,1].$$

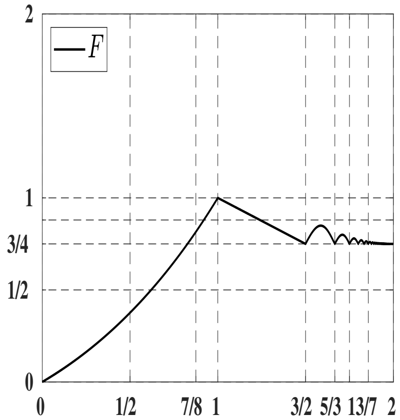


Figure 5.5 : $F \in \mathscr{N}_1\left([0,2], \frac{9}{16}, \frac{25}{16}\right)$

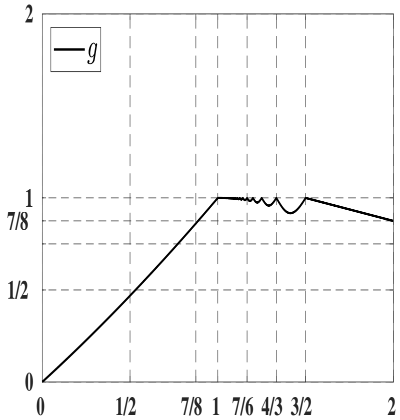


Figure 5.6 : $g \in \mathscr{N}_F\left([0,2], \frac{7}{8}, \frac{9}{8}\right)$

Let $f_0(x) = \frac{1}{4}(x^2 + 3x)$. *It is easy to verify that* $f_0^2(x) = F_0(x)$ *for all* $x \in [0,1]$ *and*

$$|g_0(x) - f_0(x)| \leq \left| g_0\left(\frac{1}{2}\right) - f_0\left(\frac{1}{2}\right) \right| = \frac{1}{32} = \varepsilon_\delta^0, \ \forall x \in [0,1].$$

Since $g^2([1,2]) \subseteq \left[\frac{3}{4}, 1\right] = F([1,2])$, *for each* $x \in [0,2]$,

$$\left|g^2(x) - F(x)\right| \leq \left| g^2\left(\frac{3}{2}\right) - F\left(\frac{3}{2}\right) \right| = \frac{1}{4} = \delta.$$

Therefore by Theorem 5.2.1, there exists $f \in \mathscr{N}_1([0,2])$ *such that*

$$f^2(x) = F(x), \ \forall x \in [0,2] \text{ and } |g(x) - f(x)| \leq \frac{77}{98}, \ \forall x \in [0,2].$$

CHAPTER 6

CONCLUSIONS AND FUTURE SCOPE

6.1 CONCLUSIONS

The present work is focused on the study of the set of non-isolated forts of nowhere constant continuous functions and characterizes the sets of isolated and non-isolated forts of iterates of a continuous self-map on an arbitrary interval I to study the existence of continuous solutions and Hyers-Ulam stability of $f^n = F$.

We generalized the concept of forts of functions in $C(K)$ into functions in $C(I,J)$ and shown how large and complicated can be the set of non-isolated forts by obtaining a continuous function f on $[0,1]$ as a limit of the sequence of continuous functions $f_n,\ n \in \mathbb{N}$ with finitely many isolated forts on $[0,1]$ such that

$$S(f) = \Lambda^*(f) = \mathscr{C}.$$

Also, we discussed the difference between forts and non-differentiable points of a continuous function. Moreover, we proved that the continuous nowhere differentiable functions have the whole domain as the set of non-isolated forts, and such functions are dense in $C(K)$.

The analysis of the non-monotone behavior of isolated and non-isolated forts under composition has been studied in detail, in particular, for a continuous function f, it is observed that a point $x \in f^{-1}(\{x_0\}),\ x_0 \in \Lambda^*(f)$ is not necessarily a non-isolated fort of f^2. A characterization for the sets $\Lambda(f^k)$, $\Lambda^*(f^k)$, and $S(f^k),\ f \in C(I),\ k \in \mathbb{N}$ on an arbitrary interval I is obtained. Further, an uncountable measure zero dense set of non-isolated forts in the real line is constructed as a countable union of nowhere dense sets of non-isolated forts using the characterization of $\Lambda^*(f^k)$.

The concept of iteratively closed set in $C(K)$ and the non-monotonicity height of continuous functions are introduced. We proved that continuous non-PM functions with non-monotonicity height 1 is not necessarily strictly monotone on its range unlike

PM functions. The existence of continuous solutions of $f^n = F$ is studied for F in the class $\mathscr{N}_1(K)$, where $\mathscr{N}_1(K)$ is defined in (4.1.5). Also, we discussed the non-existence of continuous solutions of $f^n = F$ for $F \in \mathscr{F}$, where $\mathscr{F}$ is defined in (4.3.1).

The Hyers-Ulam stability of $f^n = F$ has been studied for strictly increasing continuous functions F. We also studied the Hyers-Ulam stability of $f^n = F$ for $F \in \mathscr{N}_1(K)$ with $H(F) = 1$.

6.2 FUTURE SCOPE

The existence of continuous solutions of $f^n = F$ is still an unsolved problem for many classes of continuous functions, in particular, continuous functions f with $H(f) = 1$ and f is not strictly monotone on its range, and continuous functions of non-monotonicity height greater than 1.

In Theorem 4.3.1, we proved that $f^n = F$ has no continuous solutions f with the property that $\Lambda^*(f) = \Lambda(F)$ for $F \in \mathscr{F}$ and $\Lambda^*(F)$ is finite. The problem of the existence of continuous solutions of $f^n = F$ for $F \in \mathscr{F}$ with infinitely many non-isolated forts is still unsolved.

In Section 5.1, we discussed the Hyers-Ulam stability of $f^n = F$ for strictly increasing homeomorphisms F by taking the approximate solution g of $f^n = F$ as a strictly increasing homeomorphism. Taking g as a strictly decreasing homeomorphism, the Hyers-Ulam stability of $f^n = F$ for strictly increasing homeomorphism F is still unsolved. Also, Hyers-Ulam stability of $f^n = F$ for strictly decreasing homeomorphism F is still an unsolved problem.

Many authors used fixed point theorems to prove the Hyers-Ulam stability for different kinds of iterative functional equations. There is no such study on the existence of continuous solutions and Hyers-Ulam stability of $f^n = F$, in particular, for strictly increasing homeomorphisms.

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LIST OF SYMBOLS

$\mathbb{N}$	:	Set of natural numbers
$\mathbb{Z}$	:	Set of integers
$\mathbb{R}$	:	Set of real numbers
I,J	:	Intervals in $\mathbb{R}$
K	:	Compact interval in $\mathbb{R}$
$\mathrm{cl}(I)$	:	Closure of I
$\mathrm{int}(I)$	:	Interior of I
$C(I,J)$	:	Set of continuous functions from I into J
$C(I)$	:	Set of continuous self-maps on I
$PM(K)$	:	Set of piecewise monotone function in $C(K)$
$R(f)$	:	Range of f
Ch_f	:	Characteristic interval of f
$S(f)$	:	Set of forts of f
$\Lambda(f)$	:	Set of isolated forts of f
$\Lambda^*(f)$	:	Set of non-isolated forts of f
$\Lambda_L^*(f)$	:	$\{x \in \Lambda^*(f) : x = \lim_{n\to\infty} x_n,$ where $x_n \in S(f)$ and $x_n < x,\ \forall n \in \mathbb{N}\}$
$\Lambda_R^*(f)$	:	$\{x \in \Lambda^*(f) : x = \lim_{n\to\infty} x_n,$ where $x_n \in S(f)$ and $x_n > x,\ \forall n \in \mathbb{N}\}$
$Q_x(f_2,f_1)$	:	$\{y \in \Lambda(f_2 \circ f_1) : f_1(y) = x$ and $y \notin \Lambda(f_1)\}$
$P_x(f_2,f_1)$	:	$\{y \in \Lambda^*(f_2 \circ f_1) : f_1(y) = x$ and $y \notin \Lambda^*(f_1)\}$
$P(f_2,f_1)$	:	$\bigcup_{x\in\Lambda^*(f_2)} P_x(f_2,f_1)$
$Q(f_2,f_1)$	:	$\bigcup_{x\in\Lambda(f_2)} Q_x(f_2,f_1)$
$\mathscr{N}(K)$	:	$\{f \in C(K) : f(K) = f(R(f)),\ S(f) \neq \emptyset$ and $\mathrm{int}(R(f)) \neq \emptyset\}$
$\mathscr{N}_1(K)$	:	$\{f \in \mathscr{N}(K) : f$ attains a local extremum at every $x \in f^{-1}(S(f))\}$
$\mathscr{F}$	:	$\{\phi \in C(I) : \Lambda^*(\phi) \cap \phi(\Lambda^*(\phi)) = \emptyset$ and $\emptyset \neq \Lambda^*(\phi) \subseteq \mathrm{int}(R(\phi))\}$
$\mathscr{C}_L(K)$	:	$\{\phi \in C(K) : \lvert\phi(x)-\phi(y)\rvert \leq L\lvert x-y\rvert,\ \forall x,y \in K\}$
$\mathscr{D}_l(K)$	:	$\{\phi \in C(K) : l\lvert x-y\rvert \leq \lvert\phi(x)-\phi(y)\rvert,\ \forall x,y \in K\}$

$R_{a,0}(\|a,b\|)$	:	$\{\phi \in C(\|a,b\|) : \phi$ is strictly increasing and $\phi(x) < x,\ \forall x \in \|a,b\|,\ x \neq a\}$
$R_{b,0}(\|a,b\|)$	:	$\{\phi \in C(\|a,b\|) : \phi$ is strictly increasing and $\phi(x) > x,\ \forall x \in \|a,b\|,\ x \neq b\}$
$\mathcal{N}_1(K,l_0,L_0)$	:	$\left\{\phi \in \mathcal{N}_1(K) : l_0\|x-y\| \leq \|\phi(x)-\phi(y)\| \leq L_0\|x-y\|, \forall x,y \in Ch_\phi\right\}$
$\mathcal{N}_F(K,l,L)$	:	$\{\phi \in \mathcal{N}_1(K,l,L) : Ch_\phi = Ch_F\}$

PUBLICATIONS

1. Veerapazham Murugan and Rajendran Palanivel, *Non-isolated non-strictly monotone points of iterates of continuous functions*, Real Analysis Exchange, 46(1):1-31, 2021. DOI: 10.14321/realanalexch.46.1.0001

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