DEGREE RESTRICTED DOMINATION IN GRAPHS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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July, 2021

Dedicated to

My family

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **DEGREE RESTRICTED DOMINATION IN GRAPHS** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **DEGREE RESTRICTED DOMINATION IN GRAPHS** submitted by **Mrs. Rashmi M**, (Register Number: 145047 MA14F01) as the record of the research work carried out by her is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

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ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude towards my research advisors, Prof. Shyam. S. Kamath and Dr. A. Senthil Thilak Department of Mathematical and Computational Sciences, for their patience, motivation and immense knowledge, I received throughout my Ph.D. study. Their valuable guidance and consistent encouragement helped me to improve myself a lot in various aspects of my research work.

I would like to thank the Research Progress Assessment Committee members, Dr. Manu Basavaraju, Dept. of CSE and Prof. S. M. Hegde, Dept. of MACS., National Institute of Technology Karnataka, for their insightful comments and encouragement.

I am thankful to the former HODs, Prof. Murulidhar N. N., Prof. Santhosh George, Prof. B. R. Shankar and the present HOD, Prof. Shyam. S. Kamath of Department of MACS, for the provision of a good working environment and all the facilities in the department.

I would like to thank all the professors of the MACS department for being kind towards me and helping me at various stages of my Ph.D. life. It is my pleasure to thank all my friends who supported me in the journey of my doctoral studies.

I am grateful to my parents, my husband Pradeep Kumar, my brother Rajesh M and family members, who encouraged and helped me at every stage of my life. Special thanks my uncle Rajkumar varier the one who motivated me to go through this wonderful journey.

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ABSTRACT

The thesis mainly involves the study of a new generalization of the domination parameter, k-part degree restricted domination, defined by imposing a restriction on the degree of the vertices in a dominating set.

A dominating set D of a graph G is a k-part degree restricted dominating set (k-DRD set), if for all $u \in D$, there exists a set $C_u \subseteq N(u) \cap (V-D)$ such that $|C_u| \leq \left\lceil \frac{d(u)}{k} \right\rceil$ and $\bigcup\limits_{u \in D} C_u = V - D$. The minimum cardinality of a k-part degree restricted dominating set of a graph G is the k-part degree restricted domination number of G. The thesis includes the detailed study of the k-part degree restricted domination and a particular case when $k = 2$. Bounds on the k-part degree restricted domination number in terms of covering and independence number. Relation between k-part degree restricted dominating set and some other domination invariants are discussed in the thesis.

Further, the complexity of k-part degree restricted domination problem is discussed in detail. The problem of finding minimum k-part degree restricted domination number is proved to be NP-complete for bipartite graphs, chordal graphs, undirected path graphs, chordal bipartite graphs, circle graphs, planar graphs and even when restricted to split graphs. Also, exhibit a polynomial time algorithm to find a minimum k-part degree restricted domination number of trees and an exponential time algorithm to find a minimum k-part degree restricted domination number of interval graphs. The critical aspects of the k-part degree restricted domination number is provided with respect to the removal of vertices and edges from the graph.

Keywords: Domination, degree, k-part degree restricted domination, k-domination, Covering number, Independence number, NP-complete.

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$G = (V, E)$	A graph G with vertex set V and edge set E
$\langle S \rangle$	The induced subgraph of a graph on vertex set S
$G - v$	The graph obtained from G by removing the vertex v of G
$G - e$	The graph obtained from G by removing the edge e of G
C_n	A cycle on n vertices
P_n	A path on n vertices
G^i	The i^{th} component of a disconnected graph G
$d(v)$	The degree of a vertex v
$d_G(v)$	The degree of a vertex v in a graph G
$N(v)$	The neighborhood of a vertex v
$N[v]$	The closed neighborhood of a vertex v
$\delta(G)$	The minimum degree of a graph G
$\Delta(G)$	The maximum degree of a graph G
K_n	The complete graph on n vertices
$K_{m,n}$	Complete bipartite graph
$K_{1,n}$	Star graph
$W_n = C_{n-1} + K_1$	Wheel graph
$B_{r,m}$	Bistar graph
$G \cong H$	The graph G is isomorphic to the graph H
$\alpha_0(G)$	Vertex covering number of a graph G
$\alpha_1(G)$	Edge covering number of a graph G

$\beta_0(G)$	Independence number of a graph G
$\beta_1(G)$	Edge Independence number or matching number of a graph G
$\gamma(G)$	Domination number of a graph G
$\lfloor x \rfloor$	Floor value of x
$\lceil x \rceil$	Ceiling value of x
$i(G)$	Independent domination number of a graph G
$\gamma_t(G)$	Total domination number of a graph G
$\gamma_c(G)$	Connected domination number of a graph G
$\gamma_k(G)$	k-domination number of a graph G
$\gamma_{\frac{d}{k}}(G)$	k-part degree restricted domination number of a graph G
$\overline{G}$	The complement of the graph G

CHAPTER 1

INTRODUCTION

Graph theory is a branch of mathematics having its applications in several areas such as computer science, information technology, biosciences and operation research, to name a few. The study of graph theory perhaps initiated from the problem of the Königsberg bridge in 1735. The paper written by Leonhard Euler (published in 1736) on Seven Bridges of Königsberg is considered as the first paper in the context of graph theory. The term "graph" was introduced by Sylvester in a paper 'Chemistry and Algebra,' published in 1878 in Nature, Sylvester (1878). The first book on Graph Theory was written by Dénes König and published in 1936. Many books have published on Graph Theory in the later years, to quote a few, introductory books by Ore (1962), Berge (1962), Harary (1969), West (2001), Bondy and Murty (2008), etc. Graph coloring and domination are two significant areas that are well studied in graph theory.

1.1 SOME BASIC DEFINITIONS AND TERMINOLOGIES

Graphs are the mathematical structures used to model pairwise relations between objects or a pictorial representation of a set of objects where a link connects some pairs of objects. The interacting objects are called points, vertices, or nodes and the relationships that connect the objects are called lines, edges or arcs. The formal description of a graph is given as follows:

Definition 1.1.1. *A graph $G = (V,E)$ consists of a finite nonempty set $V = V(G)$ of vertices together with a set $E = E(G)$ of unordered pairs $e = \{u,v\}$ of distinct elements of V.*

In a graph $G = (V,E)$, number of elements in V or the cardinality of V is called the *order* of G and number of elements in E or the cardinality of E is called the *size* of G, the order of a graph is usually denoted as n and the size of G is denoted as m.

Every element of V is called a *vertex* and every element of E is called an *edge*. For an edge $e = uv$, vertex u, vertex v are adjacent vertices and also are neighbors; the edge e and the vertex u (or v) are incident with each other. For each edge $e = uv$, vertices u, v are called *end vertices*. A *loop* is an edge $e = uv$ whose end vertices are same or $u = v$, *multiple edges* are set of edges having same pair of end vertices. A *simple graph* is a graph having no loops or multiple edges. In this thesis, we consider only a finite, undirected graph with no loops or multiple edges.

In a graph $G = (V, E)$, the *open* and the *closed neighborhood* of a vertex $v \in V$ are denoted by $N(v)$ and $N[v]$, respectively, where $N(v) = \{u : uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. For a set $B \subseteq V$, the open neighborhood $N(B)$ of B is $\cup_{v \in B} N(v)$ and the closed neighborhood of B is $N[B] = N(B) \cup B$. For a subset $S \subseteq V$ and $u \in S$, a vertex v is a *private neighbor* of u with respect to S if $N[v] \cap S = \{u\}$. The private neighbor set of u with respect to S is $P_n[u, S] = \{v \in V : N[v] \cap S = \{u\}\}$. The *degree* of a vertex v is $|N(v)|$ and is denoted by $d_G(v)$ or simply $d(v)$. The minimum degree of a graph G is $\min\{d_G(v) : v \in V\}$ and is denoted by $\delta(G)$. The maximum degree of a graph is $\max\{d_G(v) : v \in V\}$ is denoted by $\Delta(G)$. For any graph G, $0 \leq \delta(G) \leq \Delta(G) \leq n - 1$. If $\delta(G) = \Delta(G) = r$, then G is called a *regular graph* of degree r. If $d_G(v) = 1$, then v is called *pendant vertex* and the *support vertex* of v is the unique vertex $u \in V(G)$ such that $uv \in E(G)$. A support vertex with exactly one adjacent pendant vertex is called a *weak support* and a support vertex with at least two adjacent pendant vertices is called a *strong support*. If $d_G(v) = 0$, then v is called an *isolated vertex*. A *walk* in a graph G is a finite non-null sequence $W = w_0, e_1, w_1, e_2, \ldots, w_{n-1}, e_n, w_m$, whose terms are alternately vertices and edges ($e_i = w_{i-1}w_i$), beginning and ending with vertices. Here, W is a walk from w_0 to w_m or w_0-w_m walk. The length of a walk is the number of edges in it. If $w_0 = w_n$, then W is a *closed walk*, otherwise it is an *open walk*. A *trail* is a walk with no repeated edges. A graph is *Eulerian* if it has a closed trail spanning all the edges. A *path* is a walk having all distinct vertices. A path on n vertices is denoted by P_n. A closed path is a *cycle* and cycle on n vertices is denoted by C_n. A graph G is *connected* if for every pair of vertices $\{u, v\}$ in V there exists a u-v path; otherwise graph is *disconnected*.

1.2 SOME SPECIAL CLASSES OF GRAPHS

A *subgraph* H of a graph G is a graph having all of its vertices and edges in G. That is, $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and G is called a *supergraph* of H. A maximal connected subgraph of a graph G is called a *component* of G. If a subgraph H contains all the vertices in G, then H is a *spanning subgraph* of G. If a graph G has a spanning cycle, then graph G is a *Hamiltonian graph*. For a subset $S \subseteq V$, *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S. The removal of a vertex v from a graph G results in a maximal subgraph $G - v = \langle V - \{v\} \rangle$. Similarly, the removal of an edge e results in a maximal subgraph $G - e = (V(G), E(G) - \{e\})$.

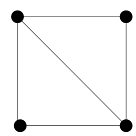
Several graph classes are obtained from a graph by applying specified graph operations on it. A few of them are given below.
The *union* $G = G_1 \cup G_2$ of two graphs G_1 and G_2 with vertex sets V_1 and V_2 and edge sets E_1 and E_2, is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The *join* $G = G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2, is the graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The *cartesian product* $G = G_1 \Box G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \Box G_2) = V(G_1) \times V(G_2)$ and any two vertices (x,u), (y,v) are adjacent in $G_1 \Box G_2$ if and only if $x = y$ and $uv \in E(G_2)$ or $xy \in E(G_1)$ and $u = v$. The *corona* $G = G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. Let $\mathcal{A}$ be a family of nonempty sets. The *intersection graph* is a graph obtained from $\mathcal{A}$ by representing each set in $\mathcal{A}$ by a vertex and connecting two vertices by an edge if and only if their corresponding sets intersect. The *subdivision* of an edge is an operation. An edge $e = uv$ is said to be subdivided, when it is deleted and replaced by a path of length two connecting its ends. An *isomorphism* from a graph G to a graph H is a bijection $f : V(G) \to V(H)$ such that $v,u \in E(G)$ if and only if $f(v), f(u) \in E(H)$. If there is an isomorphism from graph G to H, then graph G is isomorphic to H and denoted by $G \cong H$. The *complement* $\overline{G}$ of a graph G has $V(G)$ as its vertex set and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Some important classes of graphs are listed below

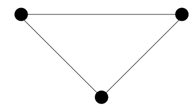
A *tree* T is a connected graph with no cycles. If each component of a graph G is a tree, then G is called a *forest*. Any non-trivial tree has at least two pendant vertices and any two vertices of a tree are connected by a unique path. A *caterpillar* is a tree in which the removal of all pendant vertices results in a path. A *rooted* tree T is a tree with one vertex $r \in V(T)$ chosen as root. For each vertex $v \in V(T)$, let $P(v)$ be the unique v-r path. The *parent* of $v \in V(T)$ is its neighbor on $P(v)$; its *children* are its other neighbors. The *leaves* are vertices with no children. A graph in which each pair of distinct vertices are joined by an edge is called a *complete graph* and denoted as K_n. A *bipartite graph* $G = (V,E)$ is a graph, whose vertex set V can be partitioned into two sets V_1 and V_2 such that, every edge of G has one end vertex in V_1 and the other in V_2. If every vertex of V_1 is joined with every vertex of V_2, then G is called a *complete bipartite graph* and is denoted by $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$. Hence, $K_{m,n} = \overline{K}_m + \overline{K}_n$. The complete bipartite graph $K_{1,n}$ is called a *star graph*. A *galaxy* is a forest in which each component is a star. A *wheel graph* is a graph obtained by the join of two graphs K_1 and C_{n-1} and is denoted by W_n. That is, $W_n = K_1 + C_{n-1}$. A graph G obtained from the cartesian product of C_n and K_2 is called *prism graph*. That is, $G = C_n \Box K_2$. A *bistar graph* $B_{n,m}$ is the graph obtained from K_2 by joining m pendant vertices to one end and n pendant vertices to other end of K_2. A graph G is *chordal* if every cycle of G of length greater than three has a chord, that is an edge between two nonconsecutive vertices of the cycle. A bipartite graph G is *chordal bipartite* if each cycle in G of length at least 6 has a chord. A *split graph* $G = (V,E)$ is a graph, whose vertices can be partitioned into two sets V_1 and V_2, where the vertices in V_1 forms a complete graph and the vertices in V_2 are independent. A graph is said to be *planar* or embeddable in the plane, if it can be drawn in the plane so that its edges intersect only at their end vertices. A *plane graph* is the one which is already drawn in a plane so that no two edges intersect. A graph G is a *circle graph* if G is the intersection graph of chords in a circle. The graph G is an *undirected path graph* if G is the intersection graph of paths in a tree. A graph $G = (V,E)$ is an *interval graph*, if every vertex in the graph can be associated with an interval in the real line so that two vertices are adjacent in the graph if and only if the two corresponding intervals intersects that is, interval graphs are the intersection graphs of sets of intervals on the real line. The names for some graphs derived from graph drawing, some of them are mentioned in Figure 1.1.

We can also represent a finite graph by a matrix. Let G be a loopless graph with vertex set $V = \{v_1, v_2, \ldots v_n\}$ and edge set $E = \{e_1, e_2, \ldots, e_m\}$. The *adjacency matrix* $A(G) = [a_{i,j}]$ of G is the n-by-n matrix in which entry $a_{i,j}$ is the number of edges joining v_i and v_j in G. The *incidence matrix* $M(G) = [m_{i,j}]$ of G is the n-by-n matrix in which entry $m_{i,j}$ is 1 if and only if v_i is incident with edge e_j and otherwise 0.

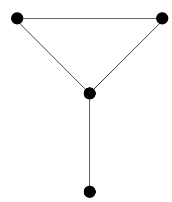
For any real number x, $\lfloor x \rfloor$ is the largest integer not greater than x, called the *floor value* of x and $\lceil x \rceil$ is the smallest integer not less than x, called the *ceiling value* of x. For any positive integer x, $\lfloor \frac{x}{2} \rfloor + \lceil \frac{x}{2} \rceil = x$.



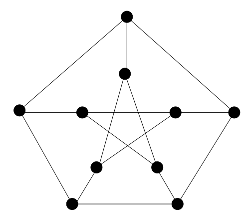
(a) Kite



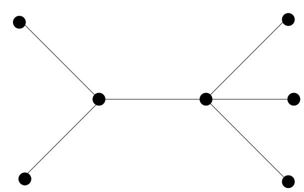
(b) Triangle



(c) Paw



(d) Petersen Graph



(e) Bistar $B_{2,3}$

Figure 1.1 Examples of some well known graphs

1.3 CONCEPT OF DOMINATION IN GRAPHS

The mathematical study of domination in graphs started around 1960 although there are some references to domination-related problems about 100 years prior. That is, in 1862, when de Jaenisch attempted to determine the minimum number of queens required to cover an $n \times n$ chessboard. Berge (1962) wrote a book on Graph Theory, in which he defined for the first time the domination number of a graph, he called this number the "*coefficient of external stability*". Ore (1962) published his book on Graph Theory, in which he used, for the first time, the terminologies '*dominating set*' and '*domination number*' and used the notation $d(G)$ for the domination number of a graph. A decade later, Cockayne and Hedetniemi (1977) published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of a graph G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic. The formal definition of domination is given below.

Definition 1.3.1. *A set $D \subseteq V$ of vertices in a graph G is called a dominating set of G, if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. The minimum cardinality of a dominating set is the domination number of graph G and is denoted by $\gamma(G)$.*

Example 1.3.2. *For the graph in Figure 1.2, $\{v_1, v_4\}$, $\{v_2, v_7\}$ are some minimum dominating sets, $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_6\}$ are some minimal dominating sets of the graph and its domination number is* 2.

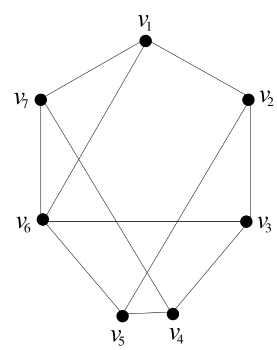


Figure 1.2 An illustration for the minimum and minimal dominating sets

Any superset of a dominating set is also a dominating set. Hence, for any graph G, $V(G)$ itself is a dominating set and every dominating set contains a minimal dominating set. In addition to this, $V(G)$ is the unique maximal dominating set for any graph G and contains all the dominating sets of G. The number of vertices in the graph is an obvious upper bound on the domination number. Since it takes at least one vertex to dominate a graph, $1 \leq \gamma(G) \leq n$ for any graph of order n. The first ever result on minimal dominating sets was stated by Ore (1962) as given below.

Theorem 1.3.3. *A dominating set D is a minimal dominating set of a graph $G = (V, E)$ if and only if for each vertex $v \in D$ one of the following conditions hold.*

- *v is not adjacent to any vertex in D.*
- *There is a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$.*

Theorem 1.3.4. *If G is a graph with no isolated vertices, then the complement $V - D$ of every minimal dominating set D is a dominating set.*

The immediate consequence of the above theorem is the following bound.

Theorem 1.3.5. *If a graph G order n has no isolated vertices, then $\gamma(G) \leq \frac{n}{2}$.*

Graphs with no isolated vertices having domination number exactly half of their order is identified by Fink J. F. et al. (1985).

Theorem 1.3.6. *For a graph G with even order n and no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

Several bounds are obtained for domination numbers in terms of various graph theoretical parameters. The inequality chain of parameters has become one of the main objectives of the study of domination. This chain was first illustrated in a paper by Cockayne et al. (1978). Some graph theoretical parameters and relations between them are given below.

Definition 1.3.7. *A vertex and an edge are said to cover each other if they are incident. A set $S \subseteq V$ of vertices of a graph G is said to be a vertex cover if it covers all the edges in G. A set $S^* \subseteq E$ of edges is said to be an edge cover if it covers all the vertices in G. The minimum cardinality of a vertex cover of a graph G is denoted by $\alpha_0(G)$ and the minimum cardinality of an edge cover is denoted by $\alpha_1(G)$.*

Definition 1.3.8. *A set $S \subseteq V$ of vertices of a graph G is called independent if no two vertices of S are joined by an edge. A set $S^* \subseteq E$ of edges of a graph G is called edge independent set if no two edges in S^* are adjacent. The number of vertices in a largest independent set is the independence number of G and is denoted by $\beta_0(G)$. The number of edges in a largest independent edge set is the edge independence number of G and is denoted by $\beta_1(G)$.*

The edge independent set is also known as matching and edge independence number as matching number. A graph is said to have a perfect matching if $\beta_1(G) = \frac{n}{2}$. Some straightforward inequalities are given below.

Proposition 1.3.9. *For any graph G with no isolated vertices,*

$$\begin{aligned} \gamma(G) &\leq \alpha_0(G), \\ \gamma(G) &\leq \alpha_1(G), \\ \gamma(G) &\leq \beta_0(G), \\ \gamma(G) &\leq \beta_1(G). \end{aligned}$$

In 1959 Gallai presented some classical theorem, involving the vertex covering number $\alpha_0(G)$, the vertex independence number $\beta_0(G)$, the edge covering number $\alpha_1(G)$ and the edge independence number $\beta_1(G)$ Haynes et al. (1998).

Theorem 1.3.10. *For any graph G of order n,*

$$\alpha_0(G) + \beta_0(G) = n.$$

Theorem 1.3.11. *For any graph G of order n with no isolated vertices,*

$$\alpha_1(G) + \beta_1(G) = n.$$

A *Nordhaus-Gaddum-type result* is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. In 1972, Jaeger and Payan have given the first Nordhaus-Gaddum type results on domination Haynes et al. (1998).

Theorem 1.3.12. *For any graph G,*

$$\begin{aligned} \gamma(G) + \gamma(\overline{G}) &\leq n+1, \\ \gamma(G)\gamma(\overline{G}) &\leq n. \end{aligned}$$

Joseph and Arumugam (1995) improved the upper bound on the sum of the domination numbers of a graph and its complement.

Theorem 1.3.13. *If graph G and $\overline{G}$ has no isolated vertices, then*

$$\gamma(G) + \gamma(\overline{G}) \leq \left\lfloor \frac{n}{2} \right\rfloor + 2.$$

1.4 CONDITIONS ON THE DOMINATING SET

Many domination parameters are formed by combining domination with some graph theoretical properties P. There are certain parameters identified by imposing a further restriction on the dominating set. Haynes et al. (1998) defined the conditional domination number as the smallest cardinality of a dominating set $D \subseteq V$ such that the subgraph $\langle D \rangle$ induced by D satisfies property P. Number of different types of domination were introduced by B.D. Acharya, E. Sampathkumar, S.T. Hedetniemi, S. Arumugam, H.B. Walikar and many others. Some of them are mentioned below.

The idea of an independent dominating set arose in chessboard problems. In 1862, de Jaenisch posed the problem of finding the minimum number of mutually non-attacking queens that can be placed on a chessboard so that each square of the chessboard is attacked by at least one of the queens. The theory of independent domination was formalized by Berge (1962) and Ore (1962).

Definition 1.4.1. *A dominating set D of a graph $G = (V,E)$ is said to be an independent dominating set if the subgraph $\langle D \rangle$ induced by D has no edges. The minimum cardinality of an independent dominating set is called the independent domination number of the graph and is denoted by $i(G)$.*

A solution to the famous Five Queens Problem inspired Cockayne et al. (1980) to introduce total domination.

Definition 1.4.2. *A dominating set D of a graph $G = (V,E)$ is said to be a total dominating set if every vertex of G is adjacent to at least one vertex of D. The minimum cardinality of a total dominating set of G is the total domination number of G and is denoted by $\gamma_t(G)$.*

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Definition 1.4.3. *A connected dominating set D of a graph $G = (V,E)$ is a dominating set D whose induced subgraph $\langle D \rangle$ is connected. The minimum cardinality of a connected dominating set is the connected domination number and is denoted by $\gamma_c(G)$.*

Sampathkumar and Walikar (1979) defined the connected dominating set. Since any nontrivial connected dominating set is also a total dominating set,

$$\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G).$$

Some domination parameters are defined by applying the conditions on the dominating set D, or on $V - D$, or on V, or on the method by which vertices in $V - D$ are dominated. For example efficient domination Bange et al. (1988) and k-domination Fink and Jacobson (1985).

Definition 1.4.4. *A dominating set D of a graph G is called an efficient dominating set, if for every vertex of $v \in V$, $|N[v] \cap D| = 1$.*

Definition 1.4.5. *For a positive integer k, a dominating set D of a graph G is called a k-dominating set, if every vertex in $V - D$ is adjacent to at least k vertices in D. The k-domination number of G is the minimum cardinality of a k-dominating set in G and is denoted by $\gamma_k(G)$.*

1.5 CHANGING AND UNCHANGING DOMINATION NUMBER OF A GRAPH

The structural properties of a graph are determined by its adjacency relation and preserved by isomorphism. Some graph properties remain the same even though two graphs are non-isomorphic. For example, two non-isomorphic graphs may have same maximum degree or same minimum degree. The graph $G-x$ obtained from a graph G by removing a vertex or an edge can not be isomorphic to graph G but some properties may be similar. Removal of an edge or a vertex from a graph may affects the domination number of some graph or may not bring any change. The change in the domination number by removing edge or vertex is studied as the changing and unchanging domination by Julie Carrington (1991). Terminology "changing and unchanging" was first suggested by Harary. The following six classes of graphs are defined depending on the changes in the domination number by removing a vertex or an edge or by adding an edge. Julie Carrington (1991) surveyed the problem of characterizing the graphs among these six classes. Commonly used acronyms to denote the following classes of graphs are C represents changing; U : unchanging; V : vertex; E : edge; R : removal; A : addition.

$$\begin{aligned}
(CVR)\ \gamma(G-v) &\neq \gamma(G), \text{ for all } v \in V.\\
(CEA)\ \gamma(G+e) &\neq \gamma(G), \text{ for all } e \in E(\overline{G}).\\
(CER)\ \gamma(G-e) &\neq \gamma(G), \text{ for all } e \in E(G).\\
(UVR)\ \gamma(G-v) &= \gamma(G), \text{ for all } v \in V.\\
(UEA)\ \gamma(G+e) &= \gamma(G), \text{ for all } e \in E(\overline{G}).\\
(UER)\ \gamma(G-e) &= \gamma(G), \text{ for all } e \in E(G).
\end{aligned}$$

Bauer et al. (1983) characterized the vertices for which $\gamma(G-v) > \gamma(G)$.

Theorem 1.5.1. *For any tree T with $n \geq 2$, there exists a vertex $v \in V$, such that*

$$\gamma(T-v) = \gamma(T).$$

Theorem 1.5.2. *For a vertex $v \in V$, $\gamma(G-v) > \gamma(G)$ if and only if the following conditions hold,*

- *Vertex v is in every γ-set of G and v is not an isolated vertex.*

- *No subset $D \subseteq V - N[v]$ with cardinality $\gamma(G)$ dominates $G - v$.*

Theorem 1.5.3. *A graph $G \in CER$ if and only if G is a galaxy.*

Later, Sampathkumar and Neeralagi (1992) characterized the vertices in a graph G for which $\gamma(G - v) < \gamma(G)$.

Theorem 1.5.4. *For a vertex $v \in V$, $\gamma(G - v) < \gamma(G)$ if and only if $pn[v, D] = \{v\}$, for some γ-set D containing v.*

Many graph theorists approached this problem independently. Sampathkumar and Neeralagi (1992) classified the vertices according to whether they belong to all, or at least one but not all, or none of the minimum dominating sets. They defined the critical aspect in the following way.

Definition 1.5.5. *Let t be any parameter defined on the graph G and an element of G be either vertex or an edge of graph G. Then, the element x is said to be*

1. *t-critical if $t(G - x) \neq t(G)$.*
2. *t^+-critical if $t(G - x) > t(G)$.*
3. *t^--critical if $t(G - x) < t(G)$.*
4. *t-redundant if $t(G - x) = t(G)$.*
5. *t-fixed if x belongs to every t-set.*
6. *t-free if x belongs to some t-sets but not all t-sets.*
7. *t-totally free if x belongs to no t-set.*

1.6 ALGORITHMIC PRELIMINARIES

As several bounds on $\gamma(G)$ are obtained, some started to study the problems involved in computing $\gamma(G)$ and finding γ-sets for any given graph G. Since for any graph G the domination number $\gamma(G)$ lies in between 1 and n, there are only a finite number of γ-sets. We can calculate $\gamma(G)$ of any graph G by finding all 2^n subsets of V and arranging them in the increasing order of their cardinality. Starting from the least cardinality subset $D \subseteq V$, check whether D is a dominating set or not. If it is a dominating set of G, then cardinality of D is the domination number of G. Otherwise, check for the next subset. By this procedure, we can find a dominating set of minimum cardinality. In worst case, this type of algorithm requires $O(2^n)$ steps, which is exponential time complexity. So, the study started to find whether an algorithm could determine the value of $\gamma(G)$ for an arbitrary graph G significantly faster. Later, the theory of NP-completeness proved that the construction of polynomial time algorithm to compute $\gamma(G)$ is not possible. A given instance of a computational problem is represented by a set of inputs. In the theory of NP-completeness, we restrict our attention to the class of problems called decision problems. These are problems where, every instance of which can be stated in such a way that the answer is either a yes or no. For example, for a given graph G an algorithm which decides whether G has a dominating set of size $\leq k$.

The formal definitions of algorithmic preliminaries are given below and the references used are Ausiello et al. (1999), Cormen et al. (2009) and Rosen et al. (1999).

Definition 1.6.1. *A problem $\mathcal{P}$ is called **decision problem** if the set of all instances of $\mathcal{P}$ denoted by $I_{\mathcal{P}}$ is partitioned into a set of positive instances $Y_{\mathcal{P}}$ and a set of negative instances $N_{\mathcal{P}}$ and the problem asks, for any instance $x \in I_{\mathcal{P}}$, to verify whether $x \in Y_{\mathcal{P}}$.*

Definition 1.6.2. *Given a decision problem $\mathcal{P}$, a **non-deterministic algorithm** $\mathcal{A}$ solves $\mathcal{P}$ if, for any instance $x \in I_{\mathcal{P}}$, $\mathcal{A}$ halts for any possible guess sequence and $x \in Y_{\mathcal{P}}$ if and only if there exists at least one sequence of guesses which leads the algorithm to return the value YES.*

Definition 1.6.3. *A non-deterministic algorithm $\mathcal{A}$ solves a decision problem $\mathcal{P}$ in **time complexity** $t(n)$ if, for any instance $x \in Y_{\mathcal{P}}$ with $|x| = n$, $\mathcal{A}$ halts for any possible guess sequence and $x \in Y_{\mathcal{P}}$ if and only if there exists at least one sequence of guesses which leads the algorithm to return the value YES in time at most $t(n)$.*

Definition 1.6.4. ***P** is the class of all decision problems for which there exists an algorithm to solve any instance of a given problem in time $O(n^k)$ for some fixed positive integer k, where n is the length of the input for the given instance.*

Definition 1.6.5. ***NP** is the class of all decision problems which can be solved in time proportional to a polynomial of the input size by a non-deterministic algorithm.*

The fundamental open question in computational complexity is whether the class P equals the class NP. By definition, the class NP contains all problems in class P. The generally accepted belief is that P$\neq$ NP (see Garey and Johnson (1979)).

Definition 1.6.6. *A problem $\mathcal{P}_1$ is said to be polynomial time reducible to a problem $\mathcal{P}_2$, denoted by $\mathcal{P}_1 \leq_p \mathcal{P}_2$ if the following two conditions hold,*

- *There exists a function f which maps any instance of $\mathcal{P}_1$ to an instance of $\mathcal{P}_2$ in such a way that I_1 is a 'yes' instance of $\mathcal{P}_1$ if and only if $f(I_1)$ is a 'yes' instance of $\mathcal{P}_2$.*

- *For any instance I_1, the instance $f(I_1)$ can be constructed in polynomial time.*

If $\mathcal{P}_1 \leq_p \mathcal{P}_2$, then any algorithm for solving $\mathcal{P}_2$ can be used to solve $\mathcal{P}_1$. Intuitively, problem $\mathcal{P}_1$ is 'no harder' to solve than problem $\mathcal{P}_2$.

Definition 1.6.7. *A problem $\mathcal{P}$ is said to be NP-hard if for every problem $\mathcal{P}' \in$ NP, $\mathcal{P}' \leq_p \mathcal{P}$.*

Definition 1.6.8. *A problem $\mathcal{P}$ is said to be NP-complete if $\mathcal{P} \in$ NP and for every problem $\mathcal{P}' \in$ NP, $\mathcal{P}' \leq_p \mathcal{P}$.*

Since the relation $\leq_p$ is transitive, if a problem $\mathcal{P}$ satisfies the following two conditions, then it is NP-complete.

- $\mathcal{P} \in$ NP.
- There exits an NP-complete problem $\mathcal{P}'$ such that $\mathcal{P}' \leq_p \mathcal{P}$.

Definition 1.6.9. *Given an optimization problem $\mathcal{P}$ and an* ***approximation algorithm*** *$\mathcal{A}$ for $\mathcal{P}$, we say that $\mathcal{A}$ is an r-approximation algorithm for $\mathcal{P}$ if, given any input instance x of $\mathcal{P}$, the performance ratio of the approximate solution $\mathcal{A}(\chi)$ is bounded by r that is: $R(x, \mathcal{A}(\chi)) \leq r$.*

Definition 1.6.10. *A* ***greedy algorithm*** *always makes the choice that looks best at the moment. That is, it makes a locally optimal choice in the hope that this choice will lead to a globally optimal solution*

Definition 1.6.11. *A* ***heuristic*** *algorithm is a procedure that produces a feasible, though not necessarily optimal, solution to every problem instance.*

Depth First Search (DFS) explores edges out of the most recently discovered vertex that still has unexplored edges leaving it. Once all the edges of vertex v have been explored, the search backtracks to explore the edges leaving the vertex from which the vertex v was discovered. This process continues until we have discovered all the vertices that are reachable from the original source vertex.

To solve a given problem, algorithms call themselves ***Recursively*** one or more times to deal with the closely related subproblems.

1.7 ORGANIZATION OF THE THESIS

The proposed thesis will have seven chapters. The relevant fundamentals and introductory concepts are explained in Chapter 1. Chapter 2 introduces a new domination parameter, 2-part degree restricted domination by imposing a restriction on the degree of the vertices in a dominating set. This chapter includes some basic properties of 2-part degree restricted dominating sets, the 2-part degree restricted domination number of some well known graphs and some bounds on $\gamma_{\frac{d}{2}}$. Chapter 3 has a generalization of the concept 2-part degree restricted domination to k-part degree restricted domination for any positive integer k. This chapter presents, k-part degree restricted domination number of some well known graphs. Since there is no explicit formula to obtain $\gamma_{\frac{d}{k}}$ of any given graph, several bounds on $\gamma_{\frac{d}{k}}$ are obtained. Bounds on $\gamma_{\frac{d}{k}}$ of join of two graphs, bounds in terms of independence and covering number are discussed. In Chapter 4, a relation between k-part degree restricted domination and some other domination invariants such as domination, k-domination and efficient domination are discussed. It also includes, an algorithm, which verifies whether the given dominating set is k-part degree restricted dominating set (k-DRD set) or not. In chapter 5, the complexity of k-part degree restricted domination problem is discussed. The problem of finding minimum k-part degree restricted domination number has been proved to be NP-complete for bipartite graphs, chordal graphs, undirected path graphs, chordal bipartite graphs, circle graphs, planar graphs and even when restricted to split graphs. Also, a polynomial time algorithm to find a minimum k-part degree restricted domination number of trees and an exponential time algorithm to find a minimum k-part degree restricted domination number of interval graphs is given. In Chapter 6, the critical aspects of the 2-part degree restricted domination number upon the removal of any vertex or an edge is discussed. Chapter 7 is on the thesis conclusion and the scope for the future work on the concepts introduced in the thesis.

CHAPTER 2

2-PART DEGREE RESTRICTED DOMINATION

2.1 INTRODUCTION

The concept of domination has emerged as one of the most studied areas extensively from theoretical as well as algorithmic point of view. Many domination parameters are formed by combining domination with some graph theoretical properties P. There are certain parameters identified by imposing a further restriction on the dominant set. Haynes et al. (1998) defined the conditional domination number as the smallest cardinality of a dominating set $D \subseteq V$ such that the subgraph $\langle D \rangle$ induced by D satisfies property P. For example, if $\langle D \rangle$ has no edges, then D is *independent dominating* set. If $\langle D \rangle$ has no isolated vertices, then D is *total dominating* set. If $\langle D \rangle$ is connected, then D is *connected dominating* set. Some new dominations are defined by imposing conditions on the dominated set $V - D$, or on V, or on the method by which vertices in $V - D$ are dominated. These include the *multiple domination* in which each vertex in $V - D$ is dominated by at least k vertices in D for a fixed positive integer k. A domination in which each vertex in $V - D$ is within distance k from at least one vertex in D for a fixed positive integer k is called *Distance domination*. A *strong domination* in which each vertex v in $V - D$ is dominated by at least one vertex in D whose degree is greater than or equal to the degree of v. A similar notion of *weak domination* specifies that each vertex v in $V - D$ is dominated by at least one vertex in D whose degree is less than or equal to the degree of v. This type of domination has various applications in the analysis of communication network. Similarly, a new domination parameter by applying some conditions on dominating set D is introduced and called as k-part degree restricted domination.

2.2 MOTIVATION

The concept of network is predominantly used in several applications of computer communication networks. It is also a fact that the dominating set in a communication network acts as a virtual backbone. Since every vertex is communicating with all its neighbors, vertex with more number of neighbors should carry a huge amount of data, which in turn will decrease the efficiency of the network. To balance the load on the dominating vertices (or vertices in the dominating set), one must enforce certain restrictions on the data flow from each vertex. This has been the motivation to introduce a new parameter namely *2-part degree restricted domination*, by imposing a restriction on the degree of the vertices in a dominating set. The vertex u in a *2-part degree restricted dominating* set can dominate at most $\left\lceil \frac{d(u)}{2} \right\rceil$ other vertices (excluding itself) in a given graph, instead of all the vertices in the neighborhood of u. As a further generalization, the concept of k-part degree restricted domination is also introduced and discussed in Chapter 3. The formal definition of the 2-part degree restricted dominating set is stated as follows:

Definition 2.2.1. *A dominating set D of a graph G is a 2-part degree restricted dominating (2-DRD) set, if for all $u \in D$, there exists a set $C_u \subseteq N(u) \cap (V - D)$ such that $|C_u| \leq \left\lceil \frac{d(u)}{2} \right\rceil$ and $\bigcup\limits_{u \in D} C_u = V - D$. The minimum cardinality of a 2-part degree restricted dominating set of a graph G is the 2-part degree restricted domination number of G and is expressed as $\gamma_{\frac{d}{2}}(G)$.*

A 2-DRD set of cardinality $\gamma_{\frac{d}{2}}(G)$ in G is called a $\gamma_{\frac{d}{2}}$-set of G. A set $C \subseteq V$ is said to be dominated by a vertex v in a 2-*DRD* set if $C \subseteq C_v$ and vertex v can dominate at most $\left\lceil \frac{d(v)}{2} \right\rceil$ number of its neighbors. A few examples are given below to illustrate the above definition.

Example 2.2.2. *In Figure 2.1, vertices of degree one and two can dominate only one of its neighbor and vertices of degree three can dominate two of its neighbors. Here, $\{v_2, v_3\}$ is a 2-DRD set with $C_{v_2} = \{v_6, v_4\}$, $C_{v_3} = \{v_1, v_5\}$ and $\bigcup\limits_{u \in D} C_u = C_{v_2} \cup C_{v_3} = \{v_1, v_4, v_5, v_6\} = V - D$ or we can also consider $C_{v_2} = \{v_1, v_6\}$, $C_{v_3} = \{v_4, v_5\}$. Also, $\{v_2, v_4\}$ is a 2-DRD set with $C_{v_2} = \{v_1, v_6\}$ and $C_{v_4} = \{v_3, v_5\}$, $\{v_1, v_3, v_6\}$ is a 2-DRD set with $C_{v_1} = \emptyset$, $C_{v_3} = \{v_4, v_5\}$ and $C_{v_6} = \{v_2\}$. The 2-part degree restricted domination number of graph in Figure 2.1is 2, that is $\gamma_{\frac{d}{2}} = 2$. In Figure 2.2 vertices of degree one can dominate only one of its neighbor and vertices of degree four can dominate two of its neighbors. Hence, $\{v_1, v_2, v_5\}$ $\{v_1, v_3, v_5\}$, $\{v_1, v_4, v_5\}$, $\{v_2, v_3, v_5\}$, $\{v_2, v_4, v_5\}$, $\{v_3, v_4, v_5\}$ are the minimum 2-DRD sets, $\{v_1, v_2, v_3, v_4\}$ is a minimal 2-DRD set of Star T in Figure 2.2 and $\gamma_{\frac{d}{2}}(T) = 2$.*

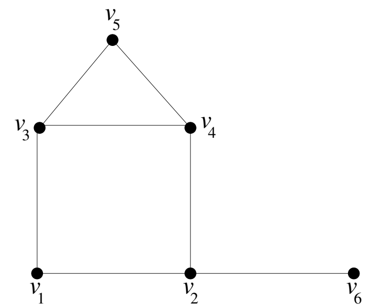


Figure 2.1 An illustration for 2-DRD sets in a graph

2.3 SOME BASIC OBSERVATIONS

As an immediate consequence of the definition of 2-DRD set, we can observe the following:

1. Every graph G has a trivial 2-*DRD* set namely $V(G)$ with $C_u = \emptyset$ for every $u \in V(G)$.

2. For any graph G, every 2-DRD set contains a minimal dominating set.

3. Every 2-*DRD* set is a dominating set but not conversely. For example, consider the graph in Figure 2.1. Here, $\{v_2, v_5\}$ is a dominating set but not a 2-*DRD* set. Since $d(v_2) = 3$, order of the set C_{v_2} can not exceed 2. Similarly $d(v_5) = 2$, order of the set C_{v_5} can not exceed 1. Hence, $|C_{v_5} \cup C_{v_2}| \leq 3 < 4 = |V - D|$. Also $\{v_3, v_6\}$ can not be a 2-*DRD* set of graph though it is a dominating set.

4. If D is a 2-*DRD* set of a graph G, then every superset $D' \supseteq D$ is also a 2-*DRD* set.

5. Suppose G is a graph without isolated vertices and D is a $\gamma_{\frac{d}{2}}$-set of G. Then, $V - D$ need not be a 2-*DRD* set (also dominating set) of G. In Figure 2.2, $D = \{v_1, v_4, v_5\}$ is a $\gamma_{\frac{d}{2}}$-set of T, but $V - D = \{v_2, v_3\}$ is not a dominating set.

6. For any 2-DRD set D, $\sum\limits_{u \in D} \left\lceil \frac{d(u)}{2} \right\rceil \geq |V - D|$.

7. If there exists a dominating set D of graph G such that $\sum\limits_{u \in D} \left\lceil \frac{d(u)}{2} \right\rceil \geq |V - D|$, then D need not be a 2-DRD set. In Figure 2.3, $D = \{v_2, v_3, v_5\}$ is a dominating set such that $\sum\limits_{u \in D} \left\lceil \frac{d(u)}{2} \right\rceil = 6 \geq 5 = |V - D|$, but D is not a 2-DRD set of graph H.

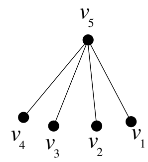


Figure 2.2 A star graph T

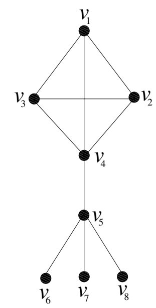


Figure 2.3 An illustration for the property listed in observation 2.3

The 2-part degree restricted domination number of some well known graphs are given below:

1. $\gamma_{\frac{d}{2}}(P_n) = \lceil \frac{n}{2} \rceil$.
2. $\gamma_{\frac{d}{2}}(C_n) = \lceil \frac{n}{2} \rceil$.
3. $\gamma_{\frac{d}{2}}(K_n) = 2$ for all $n \geq 3$.
4. $\gamma_{\frac{d}{2}}(K_{m,1}) = \lceil \frac{m+1}{2} \rceil$.
5. $\gamma_{\frac{d}{2}}(K_{m,2}) = 2$, for all $n \geq 2$.
6. $\gamma_{\frac{d}{2}}(K_{3,3}) = 2$.
7. $\gamma_{\frac{d}{2}}(K_{n,m}) = 3$, for all $n > 3$ and $3 \leq m \leq 5$.
8. $\gamma_{\frac{d}{2}}(K_{n,m}) = 4$, for all $n, m \geq 6$.
9. For wheel graph W_n, $\gamma_{\frac{d}{2}}(W_n) = \begin{cases} 1 + \lceil \frac{n-1}{6} \rceil & if\ n\ is\ odd. \\ 1 + \lceil \frac{n-2}{6} \rceil & if\ n\ is\ even. \end{cases}$
10. If G is a Petersen Graph, then $\gamma_{\frac{d}{2}}(G) = 4$.

2.4 BOUNDS ON 2-PART DEGREE RESTRICTED DOMINATION NUMBER

In this section, we describe some bounds for 2-part degree restricted domination number and some bounds on $\gamma_{\frac{d}{2}}$ of join of two graphs.

Proposition 2.4.1. *For any $\gamma_{\frac{d}{2}}$-set D of graph G, $|V-D| = \sum_{u\in D} \left\lceil \frac{d(u)}{2} \right\rceil$ if and only if $|C_u| = \left\lceil \frac{d(u)}{2} \right\rceil$ for every $u \in D$.*

Proof. Let D be a $\gamma_{\frac{d}{2}}$-set of graph G and $|V-D| = \sum_{u\in D} \left\lceil \frac{d(u)}{2} \right\rceil$. If $|C_v| < \left\lceil \frac{d(v)}{2} \right\rceil$ for some $v \in D$, then $|V-D| = |\bigcup_{u\in D} C_u| < \sum_{u\in D} \left\lceil \frac{d(u)}{2} \right\rceil$, not possible. Conversely, $|V-D| = |\bigcup_{u\in D} C_u| = \sum_{u\in D} \left\lceil \frac{d(u)}{2} \right\rceil$. □

Lemma 2.4.2. *For every 2-DRD set D of a graph G, there exists a partition $\{C_u : u \in D\}$ of $V-D$ such that $C_u \subseteq N(u)\cap(V-D)$ and $|C_u| \leq \left\lceil \frac{d(u)}{2} \right\rceil$.*

Proof. Let D be a 2-*DRD* set of a graph G. Then, for all $u \in D$, there exists a set $C_u \subseteq N(u)\cap(V-D)$ such that $|C_u| \leq \left\lceil \frac{d(u)}{2} \right\rceil$ and $\bigcup_{u\in D} C_u = V-D$. Suppose $C_u \cap C_v \neq \phi$ for some $u, v \in D$, define $C_u^{'} = C_u - (C_u \cap C_v)$ and $C_v^{'} = C_v$. If $C_w^{'} \cap C_x \neq \phi$ or $C_y^{'} \cap C_z^{'} \neq \phi$ for some $w, x, y, z \in D$, then define $C_w^{''} = C_w^{'} - (C_w^{'} \cap C_x)$, $C_x^{''} = C_x$ and $C_y^{''} = C_y^{'} - (C_y^{'} \cap C_z^{'})$, $C_z^{''} = C_z^{'}$. Proceeding like this, we get a partition $\{C_u^* : u \in D\}$ of $V-D$.

□

Proposition 2.4.3. $\gamma_{\frac{d}{2}}(G) = 1$ *if and only if G is either K_1 or K_2.*

Proof. If $\gamma_{\frac{d}{2}}(G) = 1$, then $D = \{u\}$ is a $\gamma_{\frac{d}{2}}$-set of G for some $u \in V(G)$. Then, there exists a set $C_u \subseteq N(u)\cap(V-D)$ such that $|C_u| \leq \left\lceil \frac{d(u)}{2} \right\rceil$. Since $D = \{u\}$, we have $C_u = V-D$. Then, $|C_u| = |V-\{u\}| = n-1 \leq \left\lceil \frac{d(u)}{2} \right\rceil \leq \left\lceil \frac{n-1}{2} \right\rceil$, which implies $n \leq 2$. Hence, G is either K_1 or K_2. Conversely, we can observe that $\gamma_{\frac{d}{2}}(K_1) = \gamma_{\frac{d}{2}}(K_2) = 1$. □

Proposition 2.4.4. *For any graph G, $\gamma(G \circ K_1) = \gamma_{\frac{d}{2}}(G \circ K_1)$.*

Proof. Clearly, $V(G)$ is a minimum dominating set of $G \circ K_1$. Since $d(v) \geq 1$ for any vertex $v \in V(G \circ K_1)$, each vertex in $V(G)$ can dominate its pendant neighbor in $V(G \circ K_1)$. Hence, $V(G)$ is a 2-*DRD* set of $G \circ K_1$. Hence, $|V(G)| = \gamma(G \circ K_1) \leq \gamma_{\frac{d}{2}}(G \circ K_1) \leq |V(G)|$ and $\gamma(G \circ K_1) = \gamma_{\frac{d}{2}}(G \circ K_1)$. □

Proposition 2.4.5. *Let G be a connected graph of order n. Then,*

$$\left\lceil \frac{n}{\lceil \frac{\Delta(G)}{2} \rceil + 1} \right\rceil \leq \gamma_{\frac{d}{2}}(G) \leq n - \left\lceil \frac{\Delta(G)}{2} \right\rceil.$$

Proof. Let G be a graph of order n and D be a $\gamma_{\frac{d}{2}}$-set of G. Since for every $u \in D$ order of C_u can not exceed $\left\lceil \frac{\Delta(G)}{2} \right\rceil$, we have $\left\lceil \frac{n}{\lceil \frac{\Delta(G)}{2} \rceil + 1} \right\rceil \leq \gamma_{\frac{d}{2}}(G)$. Let $v \in V$ such that $d(v) =$

$\Delta(G)$ and $N(v) = \{u_1, u_2, \ldots, u_{\Delta(G)}\}$. Choosing arbitrarily $\left\lceil \frac{\Delta(G)}{2} \right\rceil$ number of vertices from $N(v)$, we define $C_v = \{u_1, u_2, \ldots, u_{\left\lceil \frac{\Delta(G)}{2} \right\rceil}\}$ and for every $w \in V - (C_v \cup \{v\})$, $C_w = \emptyset$. Then, $V - C_v$ is a 2-*DRD* set of G and $\gamma_{\frac{d}{2}}(G) \leq |V - C_v| = n - \left\lceil \frac{\Delta(G)}{2} \right\rceil$. □

Lemma 2.4.6. *If T is a tree having no strong support and degree of each vertex is odd, then T is an infinite tree.*

Proof. Let T be a finite rooted tree, $v \in V(T)$ be a vertex in the last level say m and u be the parent vertex of v. Since degree of each vertex is odd and u lies in $(m-1)^{th}$ level, $d(u) \geq 3$. Note that u has a pendant neighbor that lies in m^{th} level. Since $d(u) \geq 3$, there exists a vertex at a distance two from u and lies in $(m+1)^{th}$ level, a contradiction. Hence, T is an infinite tree. □

Lemma 2.4.7. *For any tree T and a pendant vertex v of T, $\gamma_{\frac{d}{2}}(T - v) \leq \gamma_{\frac{d}{2}}(T)$.*

Proof. Let D be a $\gamma_{\frac{d}{2}}$-set of T and u be the support vertex of v. If both $u, v \in D$, then $C_v = \emptyset$ and $C_u \neq \emptyset$. Then, $D' = (D \cup \{w\}) - \{v\}$ is a 2-DRD set of $T - v$, where $w \in C_u$. If $u \in D$ and $v \notin D$, then $v \in C_u$ and D is a 2-DRD set of $T - v$. If $v \in D$ and $u \notin D$, then $D' = (D \cup \{u\}) - \{v\}$ is a 2-DRD set of $T - v$. Hence, $\gamma_{\frac{d}{2}}(T - v) \leq |D'| \leq |D| = \gamma_{\frac{d}{2}}(T)$. □

Lemma 2.4.8. *For any finite tree T, $\gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$.*

Proof. We prove the result by induction on n. Clearly, the result holds for $n = 1, 2, 3, 4$. Assume that the result holds for all the trees of order less than n. Let T be a tree of order n.

Case 1: n is odd. For each edge $e \in E(T)$, $T - e$ has two components say, T_1 and T_2 such that the order of T_1 is even and the order of T_2 is odd. Then, by the induction, $\gamma_{\frac{d}{2}}(T) \leq \gamma_{\frac{d}{2}}(T_1) + \gamma_{\frac{d}{2}}(T_2) \leq \lceil \frac{|V(T_1)|}{2} \rceil + \left\lceil \frac{|V(T_2)|}{2} \right\rceil \leq \lceil \frac{n}{2} \rceil$.

Case 2: n is even. If T has an edge $e \in E(T)$ such that $T - e$ has two components of even order, then the result holds. Suppose for every edge $e \in E(T)$, $T - e$ has two components of odd order. Then, degree of each vertex in T is odd. By Lemma 2.4.6, there exists a vertex say, w such that at least two pendant vertices say, w_1, w_2 are adjacent to w. Let D be a minimum 2-DRD set of $T - w_2$. Then, any one of the vertex in $\{w, w_1\}$ should be in D. Assume that $w \in D$. Since $d_T(w)$ is odd, $\left\lceil \frac{d_T(w)-1}{2} \right\rceil + 1 = \left\lceil \frac{d_T(w)}{2} \right\rceil$. Now w dominates w_1 in T and D is a 2-DRD set of T. Hence, $\gamma_{\frac{d}{2}}(T) \leq |D| = \gamma_{\frac{d}{2}}(T - w_2) \leq \lceil \frac{n-1}{2} \rceil \leq \lceil \frac{n}{2} \rceil$. □

Theorem 2.4.9. *For any connected graph G, $\gamma_{\frac{d}{2}}(G) \leq \lceil \frac{n}{2} \rceil$.*

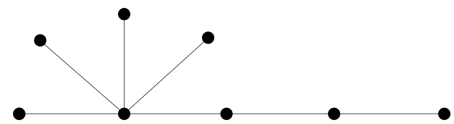


Figure 2.4 Graph G with $\gamma_{\frac{d}{2}}(G) = \lceil \frac{n}{2} \rceil$

Proof. Let T be a spanning tree of G. Then, by Lemma 2.4.8 $\gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$. Note that $d_T(w) \leq d_G(w)$ for every $w \in V$ and hence $\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$. □

Observations:

- Let G be a graph order n, where n is even and D be a minimum 2-DRD set such that $C_u \neq \emptyset$, for every $u \in D$. Then, $\gamma_{\frac{d}{2}}(G) = \lceil \frac{n}{2} \rceil$ if and only if $|C_u| = 1$, for every $u \in D$.

- Let D be a minimum 2-DRD set of a graph G of even order n and $A \subseteq D$ such that each vertex in A dominates at least one vertex in $V - D$. Then, $\gamma_{\frac{d}{2}}(G) = \frac{n}{2}$ if and only if $|V - D| - |A| = |D - A|$.

- Bounds on $\gamma_{\frac{d}{2}}$ given in Theorem 2.4.9 is sharp. For example, the graphs G in Figure 2.4, $\gamma_{\frac{d}{2}}(G) = 4 = \lceil \frac{n}{2} \rceil$.

Lemma 2.4.10. *Let T be a tree, $e \in E(T)$ and T_1, T_2 be the components of $T - e$ such that either $\gamma_{\frac{d}{2}}(T_1) < \left\lceil \frac{|V(T_1)|}{2} \right\rceil$ or $\gamma_{\frac{d}{2}}(T_2) < \left\lceil \frac{|V(T_2)|}{2} \right\rceil$. Then,*

1. *If any one of T_1 , T_2 is of even order, then $\gamma_{\frac{d}{2}}(T) < \lceil \frac{n}{2} \rceil$.*
2. *If n is odd, then $\gamma_{\frac{d}{2}}(T) < \lceil \frac{n}{2} \rceil$.*

Proof. Suppose that there exists an edge $e \in E(T)$ such that either $\gamma_{\frac{d}{2}}(T_1) < \left\lceil \frac{|V(T_1)|}{2} \right\rceil$ or $\gamma_{\frac{d}{2}}(T_2) < \left\lceil \frac{|V(T_1)|}{2} \right\rceil$ and any one of T_1 , T_2 is of even order. Then,

$$\gamma_{\frac{d}{2}}(T) \leq \gamma_{\frac{d}{2}}(T_1) + \gamma_{\frac{d}{2}}(T_2) < \left\lceil \frac{|V(T_1)|}{2} \right\rceil + \left\lceil \frac{|V(T_2)|}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.$$

If n is odd, then one among T_1, T_2 is of even order and result holds by first statement. □

Proposition 2.4.11. *Let T be a tree such that $\gamma_{\frac{d}{2}}(T) = \lceil \frac{n}{2} \rceil$. For an edge $e \in E(T)$, if T_1 and T_2 are the components of $T - e$, of order n_1 and n_2 respectively, then*

1. *$\gamma_{\frac{d}{2}}(T_1) \leq \lceil \frac{n_1}{2} \rceil$ and $\gamma_{\frac{d}{2}}(T_2) \leq \lceil \frac{n_2}{2} \rceil$.*

2. *If n is odd, then* $\gamma_{\frac{d}{2}}(T_1) = \lceil \frac{n_1}{2} \rceil$ *and* $\gamma_{\frac{d}{2}}(T_2) = \lceil \frac{n_2}{2} \rceil$.

Proof. The statement (1) holds trivially. If n is odd and there exists an edge $e \in E(T)$ such that either $\gamma_{\frac{d}{2}}(T_1) < \lceil \frac{n_1}{2} \rceil$ or $\gamma_{\frac{d}{2}}(T_2) < \lceil \frac{n_2}{2} \rceil$. Then, by Lemma 2.4.10 $\gamma_{\frac{d}{2}}(T) < \lceil \frac{n}{2} \rceil$, a contradiction. □

Proposition 2.4.12. *For any connected graph G,*

1. $\gamma_{\frac{d}{2}}(G) + \gamma(G) \leq n$.
2. *If n is even, then* $\gamma_{\frac{d}{2}}(G) + \gamma(G) = n$ *if and only if the components of G are either* C_4 *or* $H \circ K_1$*, for any connected graph H.*

Proof. For any connected graph G, $\gamma_{\frac{d}{2}}(G) \leq \lceil \frac{n}{2} \rceil$ and $\gamma(G) \leq \frac{n}{2}$. Hence, $\gamma_{\frac{d}{2}}(G) + \gamma(G) \leq n$. Suppose n is even. Then, $\gamma_{\frac{d}{2}}(G) + \gamma(G) = n$ if and only if $\gamma_{\frac{d}{2}}(G) = \gamma(G) = \frac{n}{2}$ if and only if the components of G are either C_4 or $H \circ K_1$, for any connected graph H. □

Proposition 2.4.13. *Let G be a graph of odd order. If G has a strong support of odd degree, then* $\gamma_{\frac{d}{2}}(G) < \lceil \frac{n}{2} \rceil$.

Proof. Let $v \in V(G)$ be a strong support of odd degree and $u, w \in V(G)$ be pendant neighbors of v. Now, $\gamma_{\frac{d}{2}}(G-u) \leq \lceil \frac{n-1}{2} \rceil$ and D be a minimum 2-DRD set of $G-u$. Since w is a pendant vertex in $G-u$, either w or v should be in D. If $w \in D$ and $v \notin D$, then $D \cup \{v\} - \{w\}$ is a minimum 2-DRD set of $G-u$. Hence, there exists a minimum 2-DRD set D' of $G-u$ such that $v \in D'$. Since degree of v is even in $G-u$, $\left\lceil \frac{d_G(v)}{2} \right\rceil = \left\lceil \frac{d_{G-u}(v)}{2} \right\rceil + 1$ and v can dominate u in G. Then, D' is a 2-DRD set of G and $\gamma_{\frac{d}{2}}(G) \leq |D'| = \gamma_{\frac{d}{2}}(G-u) \leq \lceil \frac{n-1}{2} \rceil < \lceil \frac{n}{2} \rceil$. □

Remark 2.4.14. *The converse of Proposition 2.4.13 need not be true. For the graph F in the Figure 2.5,* $\gamma_{\frac{d}{2}}(F) = 4 < 5 = \lceil \frac{9}{2} \rceil$*, but there is no vertex of odd degree other than pendant vertices.*

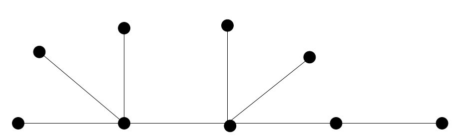


Figure 2.5 Graph F with $\gamma_{\frac{d}{2}}(F) < \lceil \frac{n}{2} \rceil$

Proposition 2.4.15. *For any tree T, if there exists an edge* $e \in E(T)$ *such that at least one component of* $T - e$ *is a bistar* $B_{r,m}$*, where r, m are odd, then* $\gamma_{\frac{d}{2}}(T) < \lceil \frac{n}{2} \rceil$.

Proof. If n is odd, then the result follows immediately from Proposition 2.4.13. Assume that n is even and $e \in E(T)$. Let T_1 and T_2 be the components of $T-e$ such that one among T_1, T_2 is a bistar. Without loss of generality assume that $T_1 = B_{r,m}$, where r, m are odd. Now as n is even, T_2 must be of even order. Further, $\gamma_{\frac{d}{2}}(T_1) = |V(T_1)| - \lceil \frac{r}{2} \rceil - \lceil \frac{m}{2} \rceil < \left\lceil \frac{|V(T_1)|}{2} \right\rceil$. Since $\gamma_{\frac{d}{2}}(T_1) < \left\lceil \frac{|V(T_1)|}{2} \right\rceil$, Lemma 2.4.10 implies that $\gamma_{\frac{d}{2}}(T) < \lceil \frac{n}{2} \rceil$. □

Remark 2.4.16. *The converse of Proposition 2.4.15 need not be true. For example, consider the graph G in the Figure 2.4. Here, $\gamma_{\frac{d}{2}}(G) = 3 < 4 = \lceil \frac{n}{2} \rceil$, but the component of $T-e$ is not a bistar having two vertices of odd degree greater than one, for any edge $e \in E(G)$.*

2.4.1 Nordhaus-Gaddum type results

Proposition 2.4.17. *For any graph G, $\gamma_{\frac{d}{2}}(G) + \gamma_{\frac{d}{2}}(\overline{G}) \leq n + \frac{m}{2}$, where m is the total number of odd components in the graph G and $\overline{G}$.*

Proof. Let $G_1, G_2, \ldots, G_r$ be the components of graph in G, $\overline{G}$ of order $n_1, n_2, \ldots, n_r$ respectively. Then,

$$\gamma_{\frac{d}{2}}(G) + \gamma_{\frac{d}{2}}(\overline{G}) = \gamma_{\frac{d}{2}}(G_1) + \gamma_{\frac{d}{2}}(G_2) + \cdots + \gamma_{\frac{d}{2}}(G_r) \leq \left\lceil \frac{n_1}{2} \right\rceil + \left\lceil \frac{n_2}{2} \right\rceil + \cdots + \left\lceil \frac{n_r}{2} \right\rceil = n + \frac{m}{2},$$

where m is the number of odd components in graph G and $\overline{G}$. □

Corollary 2.4.18. *Let G be a graph such that the components of G and $\overline{G}$ are of even order. Then, $\gamma_{\frac{d}{2}}(G) + \gamma_{\frac{d}{2}}(\overline{G}) = n$ if and only if $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(\overline{G}) = \frac{n}{2}$.*

Theorem 2.4.19. *For any nontrivial tree other than star,*

1. $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq n$.
2. $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) = n$ *if and only if* $T = P_4$ *or* $T = P_5$.

Proof. Let T be a tree such that $T \neq K_{1,n-1}$. Then, T has a vertex which is not adjacent to a vertex of maximum degree and there are at least 2 pendant vertices having no common neighbors. Then, $\overline{T}$ is connected and has at least two vertices of degree $n-2$. By Proposition 2.4.17, $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq n+1$. If $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) = n+1$, then n must be odd. Suppose n is even. Then, $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq \lceil \frac{n}{2} \rceil + \lceil \frac{n}{2} \rceil = n$. Further, as $\overline{T}$ has at least two vertices of degree $n-2$ and has no common neighbors in T, we get $\gamma_{\frac{d}{2}}(\overline{T}) = 2$. By Theorem 2.4.9, $\gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$. Then, $n+1 = \gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq \frac{n+1}{2} + 2$, which implies $n \leq 3$. Hence, T must be a star, a contradiction. Therefore, $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) \leq n$. Suppose that $\gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) = n$. By Theorem 2.4.9, $\gamma_{\frac{d}{2}}(T) \leq \lceil \frac{n}{2} \rceil$ and $\gamma_{\frac{d}{2}}(\overline{T}) \leq \lceil \frac{n}{2} \rceil$, which implies $\gamma_{\frac{d}{2}}(T) = \lceil \frac{n}{2} \rceil$ and $\gamma_{\frac{d}{2}}(\overline{T}) = \lfloor \frac{n}{2} \rfloor$ or $\gamma_{\frac{d}{2}}(T) = \lfloor \frac{n}{2} \rfloor$ and $\gamma_{\frac{d}{2}}(\overline{T}) = \lceil \frac{n}{2} \rceil$. Since $\gamma_{\frac{d}{2}}(\overline{T}) = 2$, $n \leq 5$. If $n = 4$, then tree with 4 vertices other than $K_{1,3}$ is P_4. If $n = 5$, then $n = \gamma_{\frac{d}{2}}(T) + \gamma_{\frac{d}{2}}(\overline{T}) = \gamma_{\frac{d}{2}}(T) + 2 = 5$. Tree with 5 vertices having 2-domination number 3 is P_5. Conversely $\gamma_{\frac{d}{2}}(P_4) = \gamma_{\frac{d}{2}}(\overline{P_4}) = 2$ and $\gamma_{\frac{d}{2}}(P_5) = 3$, $\gamma_{\frac{d}{2}}(\overline{P_5}) = 2$. Hence, the result holds. □

2.4.2 Bounds on $\gamma_{\frac{d}{2}}$ of join of two graphs

In this section, we discuss the bounds on $\gamma_{\frac{d}{2}}$ for join of two graphs. One can observe that, $\gamma_{\frac{d}{2}}$ depends on the degree of the vertices, because as the degree of vertex is more less number of vertices are required to dominate the whole graph. Hence, here we consider the dense graph obtained from the join of two graphs. Throughout this section, it is assumed that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two connected graphs of order n_1 and n_2, respectively, unless otherwise specified.

Proposition 2.4.20. $\gamma_{\frac{d}{2}}(G_1 + G_2) = 1$ *if and only if* $G_1 = G_2 = K_1$.

Proof. Proposition 2.4.3 implies that $\gamma_{\frac{d}{2}}(G_1 + G_2) = 1$ if and only if $G_1 + G_2 = K_2$. Hence, $G_1 = G_2 = K_1$. □

Proposition 2.4.21. *For any two graphs* G_1, G_2 *of order* $2 \leq n_1, n_2$,

$$2 \leq \gamma_{\frac{d}{2}}(G_1 + G_2) \leq 4.$$

Proof. In the graph $G_1 + G_2$, at most $l = \left\lceil \frac{n_2}{\lceil \frac{n_2}{2} \rceil} \right\rceil$ vertices from V_1 will be sufficient to dominate V_2; and the remaining $n_1 - l$ vertices of V_1 will require at most $\left\lceil \frac{n_1 - l}{\lceil \frac{n_1}{2} \rceil} \right\rceil$ vertices from V_2. Then, $\gamma_{\frac{d}{2}}(G_1 + G_2) \leq \left\lceil \frac{n_1 - 2}{\lceil \frac{n_1}{2} \rceil} \right\rceil + 2 \leq 4$. From Proposition 2.4.20 $\gamma_{\frac{d}{2}}(G_1 + G_2) = 1$ if and only if $G_1 = G_2 = K_1$, but $2 \leq n_1, n_2$. Hence, $2 \leq \gamma_{\frac{d}{2}}(G_1 + G_2)$. □

Proposition 2.4.22. *For any two graphs* G_1, G_2 *of order* $n > 1$, $\gamma_{\frac{d}{2}}(G_1 + G_2) = 2$ *if and only if there exists two vertices* $v, u \in V_1 \cup V_2$ *such that* $N(\{u, v\}) = V_1 \cup V_2$ *satisfying one of the following conditions.*

1. *If* $d(u) = n - 1$, *then* $d(v) \geq n - 5$.
2. *If* $d(u) = n - 2$, *then* $d(v) \geq n - 3$.
3. *If* $d(u) = n - 3$, *then* $d(v) \geq n - 3$, *where* $d(u)$, $d(v)$ *are degrees of vertices* u, v *in its corresponding graph.*

Proof. Assume that $\gamma_{\frac{d}{2}}(G_1+G_2)=2$. Then, $D=\{u,v\}$ is a $\gamma_{\frac{d}{2}}$-set of G_1+G_2 for some $u,v\in V_1\cup V_2$, $N(u)\cup N(v)=V_1\cup V_2$ and

$$\left\lceil\frac{d_{G_1+G_2}(v)}{2}\right\rceil+\left\lceil\frac{d_{G_1+G_2}(u)}{2}\right\rceil\geq 2n-2$$

$$\implies\left\lceil\frac{d(v)+n}{2}\right\rceil+\left\lceil\frac{d(u)+n}{2}\right\rceil\geq 2n-2.$$

Case 1: n is even. Then,

$$\left\lceil\frac{d(v)}{2}\right\rceil+\left\lceil\frac{d(u)}{2}\right\rceil\geq n-2.$$

If degree of both the vertices are even, then $d(u)+d(v)\geq 2n-4$ and $d(u),d(v)\geq n-3$. Therefore, if $d(u)=n-2$, then $d(v)=n-2$. If degree of both the vertices are odd, then $d(u)+d(v)\geq 2n-6$ and $d(u),d(v)\geq n-5$. Hence, if $d(u)=n-1$, then $d(v)\geq n-5$, $d(v)\neq n-2$ and $d(v)\neq n-4$. If $d(u)=n-3$, then either $d(v)=n-3$ or $d(v)=n-1$. If $d(u)=n-5$, then $d(v)=n-1$. If degree of one vertex is odd and another is even, then $d(u)+d(v)\geq 2n-5$ and $d(u),d(v)\geq n-4$. Hence, if degree of one vertex is $n-1$, then degree of another vertex is either $n-2$ or $n-4$ and if degree of one vertex is $n-3$, then degree of another vertex is $n-2$.

Case 2: n is odd. If degree of both the vertices are even, then $d(u)+d(v)\geq 2n-6$ and $d(u),d(v)\geq n-5$. If degree of both the vertices are odd, then $d(u)+d(v)\geq 2n-4$ and $d(u),d(v)\geq n-3$. If degree of one vertex is odd and another is even, then $d(u)+d(v)\geq 2n-5$ and $d(u),d(v)\geq n-5$. As discussed in Case 1, the possible values of $d(u)$ and $d(v)$ are listed in Table 2.1.

Table 2.1 All the possible values for $d(u)$ and $d(v)$

n is odd	$d(u)$	$d(v)$
If degree of both the vertices are even, then $d(u)+d(v) \geq 2n-6$	$n-1$	$n-3, n-5$
	$n-3$	$n-3, n-1$
	$n-5$	$n-1$
If degree one vertex is odd and another vertex is even $d(u)+d(v) \geq 2n-5$	$n-1$	$n-2, n-4$
	$n-2$	$n-1, n-3$
	$n-3$	$n-2$
	$n-4$	$n-1$
If degree of both the vertices are odd, then $d(u)+d(v) \geq 2n-4$	$n-2$	$n-2$

Conversely, if $d(u) = n-1$ and $d(v) \geq n-5$, then $d_{G_1+G_2}(u) = n+n-1$ and $d_{G_1+G_2}(v) \geq n+n-5$. Then, u can dominate n vertices in G_1+G_2 and v can dominate at least $n-2$ vertices in G_1+G_2. Also $N(\{u,v\}) = V_1 \cup V_2$. Hence, $\{u,v\}$ is a $\gamma_{\frac{d}{2}}$-set of G_1+G_2. Similarly in the next two cases $\{u,v\}$ is a $\gamma_{\frac{d}{2}}$-set of G_1+G_2 and $\gamma_{\frac{d}{2}}(G_1+G_2) = 2$. $\square$

Proposition 2.4.23. *Let G_1, G_2 be two graphs of order n, n be even. If 2-part degree restricted domination number of any one of the graphs is 2, then $\gamma_{\frac{d}{2}}(G_1+G_2) = 2$.*

Proof. Assume that $\gamma_{\frac{d}{2}}(G_1) = 2$ and $\{v,u\}$ be a $\gamma_{\frac{d}{2}}$-set of G_1. Then, $N(u) \cup N(v) = V_1$ and there exists two sets $C_u \subseteq N_{G_1}(u) - \{v\}$, $C_v \subseteq N_{G_1}(v) - \{u\}$ such that $|C_u| \leq \left\lceil \frac{d_{G_1}(u)}{2} \right\rceil$, $|C_v| \leq \left\lceil \frac{d_{G_1}(v)}{2} \right\rceil$ and $C_v \cup C_u = V_1 - \{u,v\}$. Let $A \subseteq V_2$ of order $\frac{n}{2}$, $B = V_2 - A$, $C_u' = C_u \cup A$ and $C_v' = C_v \cup B$. Since n is even,

$$|C_u'| = |C_u| + |A| \leq \left\lceil \frac{d_{G_1}(u)}{2} \right\rceil + \frac{n}{2} = \left\lceil \frac{d_{G_1}(u)+n}{2} \right\rceil \text{ and } |C_v'| \leq \left\lceil \frac{d_{G_1}(v)+n}{2} \right\rceil.$$

Hence, there exists two sets $C_u' \subseteq N_{G_1+G_2}(u) - \{v\}$, $C_v' \subseteq N_{G_1+G_2}(v) - \{u\}$ such that $|C_u'| \leq \left\lceil \frac{d_{G_1+G_2}(u)}{2} \right\rceil$, $|C_v'| \leq \left\lceil \frac{d_{G_1+G_2}(v)}{2} \right\rceil$ and $C_v' \cup C_u' = V_1 \cup V_2 - \{u,v\}$. Therefore, $\{u,v\}$ is a 2-*DRD* set of G_1+G_2. Since $\gamma_{\frac{d}{2}}(G_1+G_2) = 1$ if and only if $G_1 = G_2 = K_1$, $\gamma_{\frac{d}{2}}(G_1+G_2) = 2$. $\square$

Example 2.4.24 shows that converse of the Proposition 2.4.23 is not true.

Example 2.4.24. *Let $G_1 = G_2 = K_{1,m}$, where $m > 3$ is odd. Then, $\gamma_{\frac{d}{2}}(G_1+G_2) = 2$, but $\gamma_{\frac{d}{2}}(G_1) = \gamma_{\frac{d}{2}}(G_1) = \lceil \frac{m+1}{2} \rceil > 2$.*

Proposition 2.4.25. *Let G_1, G_2 be two graphs of order n, n be odd. If 2-part degree restricted domination number of any one of the graphs is 2 and degree of at least one vertex in a $\gamma_{\frac{d}{2}}$-set is even, then $\gamma_{\frac{d}{2}})G_1]\ G_2=[\ 2$.*

Proof. Let $\{v.u\}$ be a $\gamma_{\frac{d}{2}}$-set of G_1 and $d)u=[\ 2m$, where $m \in N$. Let $A \subseteq V_2$ of order $\lceil \frac{n}{2} \rceil$ and $B [\ V_2 - A$. Define, $C_u^{'} [\ C_u \cup A$ and $C_v^{'} [\ C_v \cup B$. Since $d)u=$is even,

$$|C_u^{'}| [\ |C_u|]\ |A| \leq \left\lceil \frac{d_{G_1})u=}{2} \right\rceil]\ \left\lceil \frac{n}{2} \right\rceil [\ \left\lceil \frac{d_{G_1})u=]\ n}{2} \right\rceil .$$

$$|C_v^{'}| [\ |C_v|]\ |B| \leq \left\lceil \frac{d_{G_1})v=}{2} \right\rceil]\ \left\lfloor \frac{n}{2} \right\rfloor \leq \left\lceil \frac{d_{G_1})u=]\ n}{2} \right\rceil >$$

Therefore, $\{u.v\}$ is a 2-*DRD* set of $G_1]\ G_2$. Since $\gamma_{\frac{d}{2}})G_1]\ G_2=[\ 1$ if and only if $G_1 [\ G_2 [\ K_1$, $\gamma_{\frac{d}{2}})G_1]\ G_2=[\ 2$. □

Proposition 2.4.26. *Let G_1, G_2 be two graphs of order n. If 2-part degree restricted domination number of any one of the graphs is 2, then $\gamma_{\frac{d}{2}})G_1]\ G_2=\leq 3$.*

Proof. If n is even or degree of at least one vertex in $\gamma_{\frac{d}{2}}$-set is even, then by Proposition 2.4.23 and Proposition 2.4.25 results holds.

Let $\{v.u\}$ be a $\gamma_{\frac{d}{2}}$-set of G_1 such that $C_u \cup C_v [\ V_1 - \{u.v\}$. Let n be odd and $A \subseteq V_2$ of order $\lceil \frac{n}{2} \rceil$ and $B [\ V_2 - A$. Define, $C_u^{'} [\ C_u \cup A$ and $C_v^{'} [\ C_v \cup B$. Suppose $|C_u| ,\ \left\lceil \frac{d)u=}{2} \right\rceil$ (or $|C_v| ,\ \left\lceil \frac{d)v=}{2} \right\rceil$). Then,

$$|C_u^{'}| [\ |C_u|]\ |A| \leq \left\lceil \frac{d_{G_1})u=}{2} \right\rceil]\ \left\lceil \frac{n}{2} \right\rceil \leq \left\lceil \frac{d_{G_1})u=]\ n}{2} \right\rceil$$

$$|C_v^{'}| [\ |C_v|]\ |B| \leq \left\lceil \frac{d_{G_1})v=}{2} \right\rceil]\ \left\lfloor \frac{n}{2} \right\rfloor \leq \left\lceil \frac{d_{G_1})u=]\ n}{2} \right\rceil >$$

Hence, $\{u.v\}$ is a 2-DRD set of $G_1]\ G_2$. Assume that n is odd, for any $\gamma_{\frac{d}{2}}$-set $D [\ \{v_1.v_2\}$ of G_1 (or G_2), $d)v_1=$and $d)v_2=$are odd, $C_{v_1} [\ \left\lceil \frac{d_{G_1})v_1=}{2} \right\rceil$ and $C_{v_2} [\ \left\lceil \frac{d_{G_1})v_2=}{2} \right\rceil$ (or $C_{v_1} [\ \left\lceil \frac{d_{G_2})v_1=}{2} \right\rceil$. $C_{v_2} [\ \left\lceil \frac{d_{G_2})v_2=}{2} \right\rceil$). Since $\{v_1.v_2\}$ is a $\gamma_{\frac{d}{2}}$-set of G_1,

$$|C_{v_1}|]\ |C_{v_2}| [\ \left\lceil \frac{d)v=}{2} \right\rceil]\ \left\lceil \frac{d)u=}{2} \right\rceil \geq n-2$$

$$\Leftrightarrow d)v_1=]\ d)v_2=\geq 2n-6>$$

Then, $d)v_1=[\ n-4$ and $d)v_2=[\ n-2$ or $d)v_1=[\ n-2$ and $d)v_2=[\ n-2$. If $d)v_1=[\ n-4$ and $d)v_2=[\ n-2$, then from Proposition 2.4.22 $\{v_1.v_2\}$ is not a 2-DRD set of

$G_1]\ \ G_2$. But $\{v_1. v_2. v_3\}$ is a 2-DRD set of $G_1]\ \ G_2$ for some $v_3 \in)V_1 \cup V_2 =- \{v_1. v_2\}$ and $\gamma_{\frac{d}{2}})G_1]\ \ G_2 = \leq 3$. ☐

Remark 2.4.27. *Let $G_1. G_2$ be two graphs of order n, n be odd such that 2-part degree restricted domination number of G_1 or G_2 is 2, then $\gamma_{\frac{d}{2}})G_1]\ \ G_2 =$ need not be 2. For example let G be a connected graph of order 11, u and v are vertices of degree 9, 7 respectively, $d)w= \leq 7$ for all $w \in V)G =- \{u. v\}$ and $N(u+\cup N(v+[\ \ V)G \div$ Then $\gamma_{\frac{d}{2}})G=[$ 2 but $\gamma_{\frac{d}{2}})G]\ \ P_{11} =[\ \ 3$.*

Let G_1, G_2 be two graphs of order n. If $\gamma_{\frac{d}{2}})G_1 =[\ \ 3$ and $\gamma_{\frac{d}{2}})G_2 = \geq 3$, then 2-part degree restricted domination number of graph $G_1]\ \ G_2$ need not be always 3 (may be less than 3). For example let G_1 be graph of odd order $n \geq 9$, u and v be vertices of degree $n-1$ and $n-5$ respectively, $d)w= \leq n-6$ for every $w \in V)G_1 =- \{u. v\}$ and $G_2 [$ $P_m\ )m < 4 \div$ Clearly, $\gamma_{\frac{d}{2}})G_2 =[\ \ \lceil \frac{m}{2} \rceil \geq 3$. Then, By Proposition 2.4.22 $\gamma_{\frac{d}{2}})G_1]\ \ G_2 =[\ \ 2$. Now

$$\left\lceil \frac{d)v=}{2} \right\rceil]\ \ \left\lceil \frac{d)u=}{2} \right\rceil [\ \ \left\lceil \frac{n-1}{2} \right\rceil]\ \ \left\lceil \frac{n-5}{2} \right\rceil [\ \ n-3>$$

Since in graph G_1 only $u. v$ can dominate maximum number of vertices and $\{u. v\}$ dominate only $n-3$ vertices, $\{u. v\}$ is is not a 2-DRD set of G. Therefore, $3 [\ \ \gamma_{\frac{d}{2}})G_1 =< 2$.

Lemma 2.4.28. *Let G_1, G_2 be two graphs of order n. If 2-part degree restricted domination number of any one of the graphs is 3, then $\gamma_{\frac{d}{2}})G_1]\ \ G_2 = \leq 3$.*

Proof. Let D be a $\gamma_{\frac{d}{2}}$-set of graph G_1 and $|D| [\ \ 3$. Then, each vertex in D can dominate at least $\lceil \frac{n}{2} \rceil - 1$ vertices from V_2. Hence, D is a 2-DRD set of $G_1]\ \ G_2$. ☐

Proposition 2.4.29. *Let G_1, G_2 be two graphs of order n such that $\gamma_{\frac{d}{2}})G_1 =[\ \ 3$ and $\gamma_{\frac{d}{2}})G_2 = \geq 3$. If $\gamma_{\frac{d}{2}})G_1]\ \ G_2 =[\ \ 3$, then following conditions holds.*

1. *$\Delta)G_1 =,\ \ n-3$ or $\Delta)G_2 =,\ \ n-3$.*
2. *If $\Delta)G_1 =[\ \ n-3$ and $\Delta)G_2 =,\ \ n-3$ and G_1 has more than one vertex of degree $n-3$, then $N(u+\cup N(v+[\!\!/\ V)G_1 =$ for any two vertices of degree $n-3$.*

Proof. Assume that $\gamma_{\frac{d}{2}})G_1]\ \ G_2 =[\ \ 3$. Suppose $d)u=[\ \ \Delta)G_1 = \geq n-3$ and $d)v=[$ $\Delta)G_2 = \geq n-3$. Then, $\{u. v\}$ is a 2-*DRD* set of $G_1]\ \ G_2$, a contradiction. Suppose G_1 has more than 2 vertices of degree $n-3$ and $N(u+\cup N(v+[\ \ V)G_1 =$ for some $u. v \in V)G_1 =$ such that $d)u=[\ \ d)v=[\ \ n-3$. Then, $\{u. v\}$ is a 2-*DRD* set of $G_1]\ \ G_2$, a contradiction. ☐

In this chapter, we have studied some basic properties of 2-part degree restricted domination and some bounds on $\gamma_{\frac{d}{2}}$. In the next chapter, we extend the concept of 2-part degree restricted domination to k-part degree restricted domination for any positive integer k. We also study some more bounds on k-part degree restricted domination number in terms of maximum degree, independence and covering number.

CHAPTER 3

k-PART DEGREE RESTRICTED DOMINATION

In this chapter, we study the extended concept of 2-part degree restricted domination, namely k-part degree restricted domination for any positive integer k. The k-part degree restricted domination is a generalizations of the classical domination, where the case $k = 1$ is the classical domination. Here, we discuss some basic properties of k-part degree restricted dominating set, k-part degree restricted domination number of some well known graphs, bounds on k-part degree restricted domination number in terms of maximum degree, independence and covering number.

3.1 SOME BASIC DEFINITIONS AND OBSERVATIONS

Definition 3.1.1. *For a positive integer k, a dominating set D of a graph G is said to be a k-part degree restricted dominating set (k-DRD set) if for all $u \in D$, there exists a set $C_u \subseteq N(u) \cap (V - D)$ such that $|C_u| \leq \left\lceil \frac{d(u)}{k} \right\rceil$ and $\bigcup\limits_{u \in D} C_u = V - D$. The minimum cardinality of a k-DRD set of a graph G is called the k-part degree restricted domination number of G and is denoted by $\gamma_{\frac{d}{k}}(G)$.*

A k-DRD set of graph G of cardinality $\gamma_{\frac{d}{k}}(G)$ is called $\gamma_{\frac{d}{k}}$-set of G. A subset $C \subseteq V(G)$ is said to be dominated by a vertex v in a k-DRD set if $C \subseteq C_v$ and v can dominate at most $\left\lceil \frac{d(v)}{k} \right\rceil$ vertices.

Example 3.1.2. *In Figure 3.1 a 3-part degree restricted domination and a 4-part degree restricted domination are illustrated. If $k = 3$, then the vertices of degree one, two and three can dominate at most one of its neighbors, the vertices of degree four and five can dominate at most two of its neighbors. Here, $D = \{v_2, v_3\}$ is a 3-DRD set with $C_{v_2} = \{v_1, v_6\}$, $C_{v_3} = \{v_4, v_5\}$ and $\bigcup\limits_{u \in D} C_u = C_{v_2} \cup C_{v_3} = \{v_1, v_4, v_5, v_6\} = V - D$. We* *can also consider $C_{v_3} = \{v_1, v_5\}, C_{v_2} = \{v_4, v_6\}$ or $C_{v_3} = \{v_1, v_4\}, C_{v_2} = \{v_5, v_6\}$. Also, $\{v_1, v_4, v_6\}$ is a 3-DRD set with $C_{v_1} = \{v_3\}$, $C_{v_4} = \{v_5\}$ and $C_{v_6} = \{v_2\}$. The 3-part degree restricted domination number of graph in Figure 3.1 is 2. That is, $\gamma_{\frac{d}{2}} = 2$. If $k = 4$, then the vertices of degree one, two, three and four can dominate at most one of its neighbor and the vertices of degree five can dominate at most two of its neighbors. Here, $D = \{v_2, v_3, v_4\}$ is a 4-DRD set with $C_{v_2} = \{v_1, v_6\}$, $C_{v_3} = \{v_5\}$, $C_{v_4} = \emptyset$. Also $\{v_1, v_5, v_6\}$ is a 4-DRD set with $C_{v_1} = \{v_3\}$, $C_{v_5} = \{v_4\}$ and $C_{v_6} = \{v_2\}$. The 4-part degree restricted domination number of graph in Figure 3.1 is 3. That is, $\gamma_{\frac{d}{4}} = 3$.*

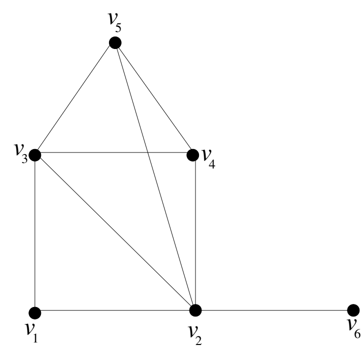


Figure 3.1 An illustration for 3-DRD and 4-DRD sets in a graph

We can observe that, for any positive integer k and $v \in V(G)$, $\left\lceil \frac{d(v)}{k+1} \right\rceil \leq \left\lceil \frac{d(v)}{k} \right\rceil$. Hence, $\gamma(G) \leq \gamma_{\frac{d}{k}}(G) \leq \gamma_{\frac{d}{k+1}}(G)$. For $k \geq \Delta(G)$, $\left\lceil \frac{d(v)}{k} \right\rceil = 1$. Hence, each vertex v in a graph G can dominate at most one of its neighbors and $\gamma_{\frac{d}{\Delta(G)}}(G) = \gamma_{\frac{d}{k}}(G) = \gamma_{\frac{d}{k+1}}(G) = \gamma_{\frac{d}{k+i}}(G)$, for any positive integer i. For $k' \leq k$, we have $\left\lceil \frac{d(v)}{k} \right\rceil \leq \left\lceil \frac{d(v)}{k'} \right\rceil$. Hence, every k-DRD set is a k'-DRD, but a k'-DRD need not be a k-DRD set for $k' < k$. As in the case of 2-part degree restricted domination in Chapter 2, we can also partition the set $V - D$ with the collection of sets $\{C_u : u \in D\}$, for every k-DRD set D of a graph.

k-Part Degree Restricted Domination Number of Some Well Known Graphs

1. $\gamma_{\frac{d}{k}}(P_n) = \lceil \frac{n}{2} \rceil$, for all $k \geq 2$.

2. $\gamma_{\frac{d}{k}}(C_n) = \lceil \frac{n}{2} \rceil$, for all $k \geq 2$.

3. $\gamma_{\frac{d}{k}})K_{1.m}=[\ \ m-\lceil\frac{m}{k}\rceil]\ \ 1.$

4. For wheel graph W_n,

- $\gamma_{\frac{d}{k}})W_n=[\ \ \begin{cases} \lceil\frac{n-)m]\ 1=}{2}\rceil]\ \ 1 & \text{if } n\equiv 1)mod\ k= \text{ and } k\ ,\ \ n-1. \\ \lceil\frac{n-)m]\ 2=}{2}\rceil]\ \ 1 & \text{if } n\not\equiv 1)mod\ k= \text{ and } k\ ,\ \ n-1. \\ \lceil\frac{n}{2}\rceil & if\ k\geq n-1. \end{cases}$

  where $m\ [\ \ \lfloor\frac{n-1}{k}\rfloor$ and $k\geq 3$.

- $\gamma_{\frac{d}{2}})W_n=[\ \ \begin{cases} 1]\ \ \lceil\frac{n-1}{6}\rceil & \text{if n is odd.} \\ 1]\ \ \lceil\frac{n-2}{6}\rceil & \text{if n is even>} \end{cases}$

5. For a prism graph G,

$$\gamma_{\frac{d}{k}})G=[\ \ \begin{cases} \lceil\frac{n}{3}\rceil & \text{if } k\ [\ \ 2. \\ \frac{n}{2} & \text{if } k<2> \end{cases}$$

6. For Petersen Graph G,

$$\gamma_{\frac{d}{k}})G=[\ \ \begin{cases} 4 & \text{if } k\ [\ \ 2. \\ 5 & \text{if } k<2> \end{cases}$$

7. $\gamma_{\frac{d}{k}})K_{n.n}=[\ \ \begin{cases} 2\lfloor\frac{n}{m}\rfloor & if\ n\equiv 0)mod\ m= \\ 2\lfloor\frac{n}{m}\rfloor]\ \ 1 & if\ n\equiv 1)mod\ m= \quad \text{where } m\ [\ \ \lceil\frac{n}{k}\rceil]\ \ 1 \\ 2\lfloor\frac{n}{m}\rfloor]\ \ 2 & Otherwise> \end{cases}$

3.2 BOUNDS ON k-PART DEGREE RESTRICTED DOMINATION NUMBER

In the analysis of subsets of a given type, such as finding the minimum cardinality of different types of dominating sets, or cover, or finding the maximum cardinality of packing, or an independent set, most of these subset problems are NP-complete for arbitrary graphs. Hence, finding some bounds for these numbers is necessary. We explore the NP-completeness of k-part degree restricted problem in Chapter 5. In this section, we discuss some bounds for k-part degree restricted domination number of a graph.

Proposition 3.2.1. *If D is a $\gamma_{\frac{d}{k}}$-set of a graph G such that $C_u\ [\!\!/\ \emptyset$ for every $u\in D$ and $C_u\cap C_v\ [\ \ \emptyset$ for every $u,v\in D$, then $V-D$ is a k-DRD set of G and $\gamma_{\frac{d}{k}})G=\leq\frac{n}{2}$.*

Proof. Let $D = \{v_1, v_2, \ldots, v_m\}$ be a $\gamma_{\frac{d}{k}}$-set of G satisfying the conditions in the hypothesis. For each $v_i \in D$, choose a vertex a_i in C_{v_i} $(a_i \in C_{v_i})$ and let $A = \{a_1, a_2, \ldots, a_m\}$. Clearly, $A \subseteq V - D$. For every $a_i \in A$, define $C_{a_i} = \{v_i\}$ and for every $a_j \in V - (D \cup A)$, $C_{a_j} = \emptyset$. Then, for each $a_i \in A$, $C_{a_i} \subseteq N(a_i) \cap D$, $|C_{a_i}| = 1$ and $\bigcup\limits_{a_j \in V-D} C_{a_j} = (\bigcup\limits_{a_j \in A} C_{a_j}) \cup (\bigcup\limits_{a_j \in (V-(D \cup A))} C_{a_j}) = D = V - (V - D)$. Hence, $V - D$ is a *k-DRD* set and $|D| \leq |V - D|$, which implies $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$. □

Remark 3.2.2. *The bound stated in Theorem 2.4.9 of Chapter 2 does not hold for some graphs, when $k > 2$. For $n \geq 6$, $\gamma_{\frac{d}{3}}(K_{1,m}) = n - \lceil \frac{n-1}{3} \rceil > \lceil \frac{n}{2} \rceil$. Converse of the Proposition 3.2.1 is not true in general. For the graph G in Figure 3.2, $\gamma_{\frac{d}{3}}(G) = 5 < 6 = \lceil \frac{n}{2} \rceil$. Since vertex v_2 has 4 pendant neighbors and $\left\lceil \frac{d(v_2)}{3} \right\rceil = 3$, any $\gamma_{\frac{d}{3}}$-set D of graph G has a vertex $v \in D$ such that $C_v = \emptyset$.*

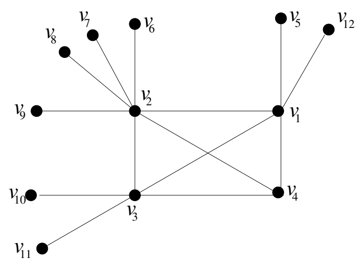


Figure 3.2 An illustration for the Remark 3.2.2

Proposition 3.2.3. *For any $\gamma_{\frac{d}{k}}$-set D of graph G, if $|V - D| = \sum\limits_{u \in D} \left\lceil \frac{d(u)}{k} \right\rceil$, then $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$.*

Proof. The Proposition 2.4.1 in chapter 2 holds for $k > 2$. Hence, if $|V - D| = \sum\limits_{u \in D} \left\lceil \frac{d(u)}{k} \right\rceil$, then $C_u \neq \emptyset$ for every $u \in D$ and Proposition 3.2.1 implies that, $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$. □

Proposition 3.2.4. *Let G be a graph such that every vertex of G is either a pendant vertex or adjacent to at least one pendant vertex. If $A = \{u \in V : d(u) > 1\}$ and k_u is the number of pendant vertices in $N(u)$ for each $u \in A$, then*

$$\gamma_{\frac{d}{k}}(G) = |A| + \sum_{u \in A / k_u \geq \left\lceil \frac{d(u)}{k} \right\rceil} k_u - \left\lceil \frac{d(u)}{k} \right\rceil,$$

where the summation is taken over all the vertices $u \in A$ such that $k_u \geq \left\lceil \frac{d(u)}{k} \right\rceil$.

Proof. For each $u \in A$, we define $C_u = N(u) - A$ if $k_u \leq \left\lceil \frac{d(u)}{k} \right\rceil$ and $C_u \subseteq N(u) - A$ of cardinality $\left\lceil \frac{d(u)}{k} \right\rceil$ if $k_u > \left\lceil \frac{d(u)}{k} \right\rceil$. Then, $D = A \bigcup\limits_{u \in A} (N(u) - C_u)$ is a k-DRD set of G. Since $N(v) \subseteq D$ for every $v \in V - A$, $C_v = \emptyset$ for every $v \in (V - A) \cap D$. Also, the vertices in A dominate its maximum possible vertices in $V - A$. Hence, we get D as a minimum k-DRD set of G. □

Corollary 3.2.5. *Let G be a graph of order n and $G^{'}$ be the graph obtained from G by adding n new vertices such that each newly added vertex is made adjacent to exactly one vertex of G. Then, $\gamma(G^{'}) = \gamma_{\frac{d}{k}}(G^{'}) = n$.*

Proposition 3.2.6. *Let G be a connected graph of order n. Then,*

$$\left\lceil \frac{n}{\lceil \frac{\Delta(G)}{k} \rceil + 1} \right\rceil \leq \gamma_{\frac{d}{k}}(G) \leq n - \left\lceil \frac{\Delta(G)}{k} \right\rceil .$$

Proof. Let G be a graph of order n and D be a $\gamma_{\frac{d}{k}}$-set of G. Since for every $u \in D$ order of C_u can not exceed $\left\lceil \frac{\Delta(G)}{k} \right\rceil$, we have $\left\lceil \frac{n}{\lceil \frac{\Delta(G)}{k} \rceil + 1} \right\rceil \leq \gamma_{\frac{d}{k}}(G)$. Let $v \in V$ such that $d(v) = \Delta(G)$ and $N(v) = \{u_1, u_2, \ldots, u_{\Delta(G)}\}$. Choosing arbitrarily $\left\lceil \frac{\Delta(G)}{k} \right\rceil$ number of vertices from $N(v)$, we define $C_v = \{u_1, u_2, \ldots, u_{\lceil \frac{\Delta(G)}{k} \rceil}\}$ and for every $w \in V - (C_v \cup \{v\})$, $C_w = \emptyset$. Then, $V - C_v$ is a *k-DRD* set of G and $\gamma_{\frac{d}{k}}(G) \leq |V - C_v| = n - \lceil \frac{\Delta(G)}{k} \rceil$. □

Remark 3.2.7. *The upper and lower bounds cited in Proposition 3.2.6 are attained by the graphs $K_{1,n}$ and K_n, respectively.*

Proposition 3.2.8. *Let $k > 1$ and G be any connected graph of order $n \geq 6$. Then, $\gamma_{\frac{d}{k}}(G) = n - \left\lceil \frac{\Delta(G)}{k} \right\rceil$ if and only if $G = K_{1,n-1}$.*

Proof. Let $\gamma_{\frac{d}{k}}(G) = n - \left\lceil \frac{\Delta(G)}{k} \right\rceil$ and $v \in V(G)$ such that $d(v) = \Delta(G)$. We claim that $d(v) = n - 1$. Suppose $d(v) \neq n - 1$. Then, there exists an edge uw such that at least one of u, w is not a neighbor of v. Note that, if $\left\lceil \frac{d(v)}{k} \right\rceil > d(v) - 1$, then $d(v) < 2$. Hence, $d(v) = \Delta(G) = 1$ and $G = K_1$. Since $n \geq 6$ and $d(v) = \Delta(G)$, $\left\lceil \frac{d(v)}{k} \right\rceil \leq d(v) - 1$. Then, we can find a subset S of $N(v) - \{u, w\}$ of cardinality $\left\lceil \frac{\Delta(G)}{k} \right\rceil$ and $D = V - (S \cup \{u\})$ is a k-DRD set of G with $C_v = S$, $C_w = \{u\}$, $C_x = \emptyset$ for all $x \in D - \{v, w\}$. Then, $|D| = |V - (S \cup \{u\})| = n - \left\lceil \frac{\Delta(G)}{k} \right\rceil - 1 < n - \left\lceil \frac{\Delta(G)}{k} \right\rceil$, a contradiction. Hence, $d(v) = n - 1$.

Claim: $G - v = \overline{K}_{n-1}$.

Suppose $G - v \neq \overline{K}_{n-1}$. Then, $G - v$ has at least one edge, say uw. If $\left\lceil \frac{d(v)}{k} \right\rceil > d(v) - 2$, then $n \leq 5$. Since $n \geq 6$, $\left\lceil \frac{d(v)}{k} \right\rceil \leq d(v) - 2$. Then, we can find a subset S of $N(v) - \{u, w\}$

of cardinality $\left\lceil \frac{\Delta(G)}{k} \right\rceil$ and $V - (S \cup \{u\})$ is a k-DRD set of G. Also, $|V - (S \cup \{u\})| = n - \left\lceil \frac{\Delta(G)}{k} \right\rceil - 1 < n - \left\lceil \frac{\Delta(G)}{k} \right\rceil$, a contradiction. Hence, $G = K_{1,n-1}$. □

Proposition 3.2.9. *Let G be a connected graph of order $n \geq 4$. Then, $\gamma_{\frac{d}{k}}(G) = n-1$ if and only if $G = K_{1,n-1}$ and $k \geq n-1$.*

Proof. If $G = K_{1,n-1}$ and $k \geq n-1$, then $\gamma_{\frac{d}{k}}(G) = n-1$. Conversely, assume that G is a connected graph of order $n \geq 4$ and $\gamma_{\frac{d}{k}}(G) = n-1$. Clearly, P_4 is not a subgraph of G. If P_4 is a subgraph of G, then $\gamma_{\frac{d}{k}}(G) \leq n-2$, a contradiction.
Claim 1: $\Delta(G) = n-1$.
Since $n \geq 4$ and G is connected, $\Delta(G) \geq 2$. If $\Delta(G) < n-1$ and u is a vertex of maximum degree in G, then there exists a vertex not adjacent to u but adjacent to some vertices in $N(u)$, which implies P_4 is a subgraph of G, a contradiction.
Claim 2: K_3 is not a subgraph of G.
Assume that K_3 is a subgraph of G. Since $n \geq 4$, there exists a vertex $v \in V$ such that $v \notin V(K_3)$ and adjacent to some vertices in $V(K_3)$. Then, P_4 is a subgraph of G, a contradiction. From Claim 1 and Claim 2 it follows that $G = K_{1,n-1}$. Suppose $k < n-1 = \Delta(G)$. Then, $\left\lceil \frac{\Delta(G)}{k} \right\rceil \geq 2$ and hence $\gamma_{\frac{d}{k}}(G) \leq n-2$, a contradiction to the assumption $\gamma_{\frac{d}{k}}(G) = n-1$. □

3.2.1 Bounds on $\gamma_{\frac{d}{k}}$ of join of two graphs

In this section, we discuss bounds on k-part degree restricted domination number for join of two graphs. For any graph G_1, G_2, we know that $\gamma(G_1 + G_2) \leq 2$, but $\gamma_{\frac{d}{k}}(G_1 + G_2) > max\{\gamma_{\frac{d}{k}}(G_1), \gamma_{\frac{d}{k}}(G_2)\}$, for some graphs. Throughout this section, it is assumed that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two connected graphs of order n_1 and n_2, respectively, unless otherwise specified.

Proposition 3.2.10. *For $k > 1$, $\gamma_{\frac{d}{k}}(G_1 + G_2) = 1$ if and only if $G_1 = G_2 = K_1$.*

Proof. If $\gamma_{\frac{d}{k}}(G_1 + G_2) = 1$, then $D = \{u\}$ is a $\gamma_{\frac{d}{k}}$-set of $G_1 + G_2$ for some $u \in V(G_1 + G_2)$. Let $n_1 + n_2 = n$. Then clearly, $n \geq 2$, $|C_u| = n-1 \leq \left\lceil \frac{d(u)}{k} \right\rceil \leq \left\lceil \frac{n-1}{k} \right\rceil \leq \left\lceil \frac{n-1}{2} \right\rceil$, which implies $n \leq 2$. Hence, $n = 2$ and $G_1 = G_2 = K_1$. Converse is obvious. □

Proposition 3.2.11. *For any two graphs G_1 and G_2 of order n_1 and n_2 respectively, the following results hold.*

1. *If $G_1 \neq K_1$, then $2 \leq \gamma_{\frac{d}{k}}(G_1 + G_2) \leq \gamma_{\frac{d}{k}}(G_1) + \gamma_{\frac{d}{k}}(G_2)$.*

2. *If $k \geq \Delta(G_1 + G_2)$ and $n_1 \leq n_2$, then $n_1 \leq \gamma_{\frac{d}{k}}(G_1 + G_2) \leq n_2$.*

Proof. Since $G_1 [\!/ K_1$ and from proposition 3.2.10, the lower bound in the first statement holds. Let D_1 and D_2 be $\gamma_{\frac{d}{k}}$-sets of G_1 and G_2 respectively. Then, $D_1 \cup D_2$ is a k-DRD set of $G_1]\ \ G_2$ and hence $\gamma_{\frac{d}{k}})G_1]\ \ G_2 = \leq \gamma_{\frac{d}{k}})G_1 = \!\!\!|\ \ \gamma_{\frac{d}{k}})G_2 =\!\!\!\!\cdot$ If $k \geq \Delta)G_1]\ \ G_2 =\!\!\!\!,$ then $|C_v|$ can not exceed one for any vertex v in $V)G_1]\ \ G_2 =\!\!\!\!\cdot$ Hence, $n_1 \leq \lceil \frac{n_1]\ n_2}{2} \rceil \leq \gamma_{\frac{d}{k}})G_1]\ \ G_2 =\!\!\!\!\cdot$ Since $n_1 \leq n_2$, $V)G_2 =$is a k-DRD set of $G_1]\ \ G_2$. Hence, $\gamma_{\frac{d}{k}} G_1]\ \ G_2 = \leq n_2$. □

Proposition 3.2.12. *If G_1 and G_2 are graphs of order $n_1 \geq k$ and $n_2 \geq k$ respectively, then $\gamma_{\frac{d}{k}})G_1]\ \ G_2 = \leq 2k$.*

Proof. In the graph $G_1]\ \ G_2$, at most $l\ [\ \ \left\lceil \frac{n_2}{\lceil \frac{n_2}{k} \rceil} \right\rceil$ vertices from V_1 will be sufficient to dominate V_2; and the remaining $n_1 - l$ vertices of V_1 will require at most $\left\lceil \frac{n_1 - l}{\lceil \frac{n_1}{k} \rceil} \right\rceil$ vertices from V_2. Then, $\gamma_{\frac{d}{k}})G_1]\ \ G_2 = \leq \left\lceil \frac{n_1 - l}{\lceil \frac{n_1}{k} \rceil} \right\rceil\ ]\ \ l \leq 2k$.

□

Proposition 3.2.13. *Let G_1 and G_2 be two graphs of order n_1 and n_2 respectively.*

1. *If $\gamma_{\frac{d}{k}})G_1 = \geq k$ and $n_2 \equiv 0)mod\ k =\!\!\!\!,$ then $\gamma_{\frac{d}{k}})G_1]\ \ G_2 = \leq \gamma_{\frac{d}{k}})G_1 =\!\!\!\!\cdot$*
2. *If $2 \leq k < n_2 \leq n_1$, then $\gamma_{\frac{d}{k}})G_1]\ \ G_2 = < \gamma_{\frac{d}{k}})G_1 =\!\!\!|\ \ k$.*

Proof. Let D be a $\gamma_{\frac{d}{k}}$-set of G_1. Since $n_2 \equiv 0)mod\ k =\!\!\!\!,$ we get $\left\lceil \frac{d)u =\!\!\!|\ n_2}{k} \right\rceil\ [\ \ \left\lceil \frac{d)u=}{k} \right\rceil\ ]\ \ \left\lceil \frac{n_2}{k} \right\rceil$, for any $u \in D$. Hence, each vertex in D can dominate $\frac{n_2}{k}$ vertices from V_2 and $\gamma_{\frac{d}{k}})\frac{n_2}{k} = \geq \frac{kn_2}{k} \geq n_2$. Therefore, D is a k-DRD set of $G_1]\ \ G_2$. Similarly, if $k\ \ n_2$, then each vertex in D can dominate at least $\lceil \frac{n_2}{k} \rceil - 1$ vertices from V_2. Hence, D can dominate $\gamma_{\frac{d}{k}})G_1 =\!\!\!)\lceil \frac{n_2}{k} \rceil - 1 =$vertices from V_2. Since $\lceil \frac{n_2}{k} \rceil \geq 2$ and $n_1 \geq n_2$, we get $n_2 - \gamma_{\frac{d}{k}})\lceil \frac{n_2}{k} \rceil - 1 = \leq n_1 - \gamma_{\frac{d}{k}}$. So we can find a subset D' of $V_1 - D$ of order $\left\lceil \frac{n_2 - \gamma_{\frac{d}{k}})\lceil \frac{n_2}{k} \rceil - 1 =}{\lceil \frac{n_2}{k} \rceil} \right\rceil$, dominate all the remaining vertices in V_2 which are not dominated by D. Therefore, $\gamma_{\frac{d}{k}})G_1]\ \ G_2 = \leq \gamma_{\frac{d}{k}})G_1 =\!\!\!|\ \ \left\lceil \frac{n_2 - \gamma_{\frac{d}{k}})\lceil \frac{n_2}{k} \rceil - 1 =}{\lceil \frac{n_2}{k} \rceil} \right\rceil < \gamma_{\frac{d}{k}})G_1 =\!\!\!|\ \ k$.

□

Remark 3.2.14. *The following examples illustrate that the bounds in Proposition 3.2.12 and Proposition 3.2.13 are sharp.*

1. *Let G_1 and G_2 be two graphs each having perfect matching and $k > \Delta)G_1]\ \ G_2 =\!\!\!\!\cdot$ Then, $\gamma_{\frac{d}{k}})G_1]\ \ G_2 = > max\{\gamma_{\frac{d}{k}})G_1 =\!\!\!\!, \gamma_{\frac{d}{k}})G_2 =\!\!\!|\}$.*

2. *For $k = 3$, $\gamma_{\frac{d}{k}}(K_{12,12}) = \gamma_{\frac{d}{k}}(\overline{K}_{12} + \overline{K}_{12}) = 6 = 2k$ and the bound in Proposition 3.2.12 is attained.*

3. *For $G_1 = C_5$ and $G_2 = C_6$, $\gamma_{\frac{d}{3}}(C_5 + C_6) = 3 = \gamma_{\frac{d}{3}}(C_5)$, which shows that the first equality given in the Proposition 3.2.13 can be attained.*

4. *Let G_1 be a connected graph of order 11 satisfying the following conditions:*

   (a) *$u, v \in V(G_1)$ such that $d(u) = 9$ and $d(v) = 7$.*

   (b) *$d(w) \leq 7$ for all $w \in V(G_1) - \{u, v\}$.*

   (c) *$N(u) \cup N(v) = V(G_1)$.*

   Then, $\gamma_{\frac{d}{2}}(G_1) = 2$ but $\gamma_{\frac{d}{2}}(G_1 + P_{11}) = 3 < 2 + 2 = \gamma_{\frac{d}{2}}(G_1) + k$, which satisfies the second inequality in the Proposition 3.2.12.

3.2.2 Bounds in terms of Independence and Covering Number

In this section, we obtain some bounds on k-part degree restricted domination number $\gamma_{\frac{d}{k}}$ in terms of vertex cover α_0, edge cover α_1, matching number β_1 and vertex independence number β_0. Though we know that, $\gamma(G) \leq \beta_1(G)$ and $\gamma(G) \leq \alpha_0(G)$ for any graph G, $\gamma_{\frac{d}{k}}(G)$, $\beta_1(G)$ and $\gamma_{\frac{d}{k}}(G)$, $\alpha_0(G)$ are incomparable. For any $\gamma_{\frac{d}{k}}$-set D of graph G, if $C_u \neq \emptyset$ for every $u \in D$ or $|V - D| = \sum_{u \in D} \left\lceil \frac{d(u)}{k} \right\rceil$, then $\gamma_{\frac{d}{k}}(G) \leq \beta_1(G)$.

For any given subset $D \subseteq V$ to determine whether it is a k-DRD set or not, first we have to construct C_u, for every $u \in D$. Here, we give a general construction of C_u for every $u \in D$ and we use this construction throughout our discussion.

Let $D = \{v_1, v_2, \ldots v_m\}$ and choose a vertex v_1 from D. If $|N(v_1) \cap (V - D)| \leq \left\lceil \frac{d(v_1)}{k} \right\rceil$, then $C_{v_1} = N(v_1) \cap (V - D)$. Otherwise, choose $\left\lceil \frac{d(v_1)}{k} \right\rceil$ number of vertices from the set $N(v_1) \cap (V - D)$ and name that set as C_{v_1}. For all i, $2 \leq i \leq m$, if $\left|N(v_i) \cap (V - (D \cup (\bigcup_{j=1}^{i-1} C_{v_j})))\right| \leq \left\lceil \frac{d(v_i)}{k} \right\rceil$, then $C_{v_i} = N(v_i) \cap (V - (D \cup (\bigcup_{j=1}^{i-1} C_{v_j})))$. Otherwise, choose $\left\lceil \frac{d(v_i)}{k} \right\rceil$ number of vertices from the set $N(v_i) \cap (V - (D \cup (\bigcup_{j=1}^{i-1} C_{v_j})))$ and name it as C_{v_i}.

Theorem 3.2.15. *For any graph G and $k \geq \Delta(G)$,*

1. *$\gamma_{\frac{d}{k}}(G) \geq \frac{n}{2}$.*

2. *$\gamma_{\frac{d}{k}}(G) + \beta_1(G) = n$.*

3. *$\gamma_{\frac{d}{k}}(G) = \frac{n}{2}$ if and only if G has a perfect matching.*

4. *$\gamma(G) + \gamma_{\frac{d}{k}}(G) = n$ if and only if $\gamma(G) = \beta_1(G)$.*

Proof.

1. Since $k \geq \Delta(G)$, each vertex can dominate at most one vertex other than itself. If every vertex dominate exactly two vertices including itself, then $\gamma_{\frac{d}{k}}(G) = \frac{n}{2}$. Otherwise, $\gamma_{\frac{d}{k}}(G) > \frac{n}{2}$.

2. Let M be a maximum matching of G and U be the set of vertices saturated by M. Since $k \geq \Delta(G)$, each vertex in U can dominate at most one saturated vertex other than itself. Hence, all the neighbors of unsaturated vertices are dominated. Since M is a maximum matching set, only $|M|$ number of vertices can dominate two vertices including itself. Hence, $\gamma_{\frac{d}{k}}(G) = n - 2\beta_1(G) + \beta_1(G) = n - \beta_1(G)$.

3. We know that, $\beta_1(G) = \frac{n}{2}$ if and only if G has a perfect matching and from statement 2, statement 3 is trivial.

4. From statement 2, we have $\gamma(G) + \gamma_{\frac{d}{k}}(G) = n \Leftrightarrow \gamma(G) + n - \beta_1(G) = n \Leftrightarrow \gamma(G) = \beta_1(G)$.

□

Proposition 3.2.16. *For any graph G,*

1. *$\gamma_{\frac{d}{k}}(G) + \beta_1(G) \leq n$.*

2. *If G has a perfect matching, then $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$.*

3. *$\gamma_{\frac{d}{k}}(G) + \gamma(G) \leq n$.*

4. *If G is Hamiltonian, then $\gamma_{\frac{d}{k}}(G) \leq \lceil \frac{n}{2} \rceil$.*

Proof.

1. We know that for any positive integer k, $\gamma_{\frac{d}{k}}(G) \leq \gamma_{\frac{d}{k+1}}(G)$. Therefore, for any $k \leq \Delta(G)$, $\gamma_{\frac{d}{k}}(G) \leq \gamma_{\frac{d}{\Delta(G)}}(G) = n - \beta_1(G)$.

2. The second statement follows trivially from the first statement.

3. Since $\gamma(G) \leq \beta_1(G)$ and from the first inequality, we get $\gamma_{\frac{d}{k}}(G) + \gamma(G) \leq n$.

4. If G is Hamiltonian, then $\beta_1(G) = \lfloor \frac{n}{2} \rfloor$ and from the first inequality $\gamma_{\frac{d}{k}}(G) \leq \lceil \frac{n}{2} \rceil$.

□

Proposition 3.2.17. *If $\gamma(G) + \gamma_{\frac{d}{k}}(G) = n$, then $\gamma(G) = \beta_1(G)$ and $\gamma_{\frac{d}{k}}(G) \geq \frac{n}{2}$.*

Proof. We know that, $\gamma_{\frac{d}{k}}(G) \leq n - \beta_1(G)$. If $\gamma(G) < \beta_1(G)$, then $\gamma_{\frac{d}{k}}(G) < n - \gamma(G)$, which implies $\gamma(G) + \gamma_{\frac{d}{k}}(G) < n$, a contradiction. Also, note that since $\gamma(G) \leq \frac{n}{2}$, we get $\gamma_{\frac{d}{k}}(G) \geq \frac{n}{2}$. □

Remark 3.2.18. *For a graph G of even order, suppose $D \subseteq V$ is both $\gamma(G)$-set and k-DRD set. Then, $\gamma(G) + \gamma_{\frac{d}{k}}(G) = n$ if and only if the components of G are cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

Proposition 3.2.19. *Let G be a graph having an r-factor. If $\left\lceil \frac{\delta(G)}{k} \right\rceil \geq r$, then $\gamma_{\frac{d}{k}}(G) \leq \frac{n}{2}$.*

Proof. Let $G_1, G_2, \ldots, G_m$ be the components of an r-regular spanning subgraph of G. Since $\left\lceil \frac{\delta(G)}{k} \right\rceil \geq r$, union of dominating (1-DRD) set of each G_i's, $1 \leq i \leq m$, will be a k-DRD set of G. Hence,

$$\gamma_{\frac{d}{k}}(G) \leq \sum_{i=1}^{m} \gamma(G_i) \leq \sum_{i=1}^{m} \frac{|V(G_i)|}{2} = \frac{n}{2}.$$

□

Theorem 3.2.20. *For any graph G with $\delta(G) \geq k$, $\gamma(G) \leq \gamma_{\frac{d}{k}}(G) \leq \alpha_0(G)$.*

Proof. Let $D = \{v_1, v_2, \ldots, v_m\}$ be a minimum vertex cover set of G and for each $v_i \in D$, $C_{v_i} \subseteq V - D$ as constructed in the beginning of the Subsection 3.2.2. If $\bigcup_{v_i \in D} C_{v_i} = V - D$, then D is a k-DRD set of G and result holds. Suppose that $\bigcup_{v_i \in D} C_{v_i} \neq V - D$. Then, we can find a vertex $w^* \in (V - D) - (\bigcup_{i=1}^{m} C_{v_i})$. Since D is a vertex cover and $\delta(G) \geq k$, w^* is adjacent to at least k vertices in D. For every $v \in N(w^*) \cap D = B$, $|C_v| = \left\lceil \frac{d(v)}{k} \right\rceil$. Also, for every $u \in N(\bigcup_{x \in B} C_x) \cap D = B_1$, $|C_u| = \left\lceil \frac{d(u)}{k} \right\rceil$. Otherwise, we can find a path $P = w^*, v_1, v_2, v_3$ such that $v_1, v_3 \in D$, $|C_{v_3}| < \left\lceil \frac{d(v_3)}{k} \right\rceil$ and $v_2 \in C_{v_1}$. We redefine, $C_{v_3} = C_{v_3} \cup \{v_2\}, C_{v_1} = (C_{v_1} - \{v_2\}) \cup \{w^*\}$. Then, w^* is dominated by v_1 and D is a k-DRD set. If for every $u \in N(\bigcup_{x \in B_1} C_x) \cap D = B_2$, $|C_u| = \left\lceil \frac{d(u)}{k} \right\rceil$, then continuing the above process, we get the set $C \subseteq D$ with following properties:

($\mathscr{P}_{11}$) $|C_w| = \left\lceil \frac{d(w)}{k} \right\rceil$ for all $w \in C$,

($\mathscr{P}_{12}$) $C_{w_i} \cap C_{w_j} = \emptyset$ for all $w_i, w_j \in C$,

($\mathscr{P}_{13}$) $N(C_w) \cap D \subseteq C$ for all $w \in C$.

Since D is a vertex cover, $\delta(G) \geq k$ and by the above properties, we have $k \sum_{w\in C} |C_w| \leq \sum_{w\in C} d(w)$. If $k \sum_{w\in C} |C_w| = \sum_{w\in C} d(w)$, then the vertices in C are adjacent to only the vertices in $\bigcup_{w\in C} C_w$. But vertices in C are adjacent to w^* and $w^* \notin \bigcup_{w\in C} C_w$. Therefore, $k \sum_{w\in C} |C_w| < \sum_{w\in C} d(w)$, which implies $\sum_{w\in C} |C_w| < \sum_{w\in C} \left\lceil \frac{d(w)}{k} \right\rceil$, a contradiction to Property $\mathscr{P}_{11}$. Hence, w^* should be dominated by some vertices in D, D is a k-DRD set and $\gamma_{\frac{d}{k}}(G) \leq |D| = \alpha_0(G)$. □

Remark 3.2.21. *For any graph G with $\delta(G) < k$, $\gamma_{\frac{d}{k}}(G)$ and $\alpha_0(G)$ are incomparable. For example consider graph G in Figure 3.2, where $\alpha_0(G) = 3 = \gamma_{\frac{d}{2}}(G) < \gamma_{\frac{d}{3}}(G) = 5$. For complete graph K_5 and $k = 5$, $\delta(K_5) = 4 < 5 = k$ and $\gamma_{\frac{d}{5}}(K_5) = 3 < 4 = \alpha_0(K_5)$.*

Proposition 3.2.22. *For any caterpillar T and $k > 2$, $\gamma_{\frac{d}{k}}(T) \geq \alpha_0(T) \geq \beta_1(T)$.*

Proof. Let $A = \{u \in V(T) : d(u) > 2\}$ and S be a minimum vertex cover set of T such that $A \subseteq S$. Then, $\sum_{u\in S} \left\lceil \frac{d(u)}{k} \right\rceil \geq \sum_{v\in S^*} \left\lceil \frac{d(v)}{k} \right\rceil$ for any minimum vertex cover set S^* of T. Since $k > 2$ and $d(u) > 2$ for every $u \in A$, $\left\lceil \frac{d(u)}{k} \right\rceil \leq d(u) - 2$. Also note that vertices in $S - A$ can dominate at most one vertex other than itself. Hence, $|S'| \geq |S| = \alpha_0(T) \geq \beta_1(T)$ for any $\gamma_{\frac{d}{k}}$-set S' of T and $\gamma_{\frac{d}{k}}(T) \geq \alpha_0(T) \geq \beta_1(T)$. □

Theorem 3.2.23. *For any graph G with $\delta(G) > 0$, $\gamma_{\frac{d}{k}}(G) \leq \alpha_1(G)$.*

Proof. Since $\delta(G) > 0$, each vertex can dominate at least one vertex other than itself. By taking one end vertex of each edge in a minimum edge cover, we can construct a k-DRD set of graph G. Hence, $\gamma_{\frac{d}{k}}(G) \leq \alpha_1(G)$. □

Theorem 3.2.24. *For any graph G with $\delta(G) > k$, $\gamma_{\frac{d}{k}}(G) \leq \beta_1(G)$.*

Proof. Let M be a maximum matching set of G and $D = \{v_1, v_2, \ldots v_p\}$ be a dominating set (1-DRD set) of G obtained from the maximum matching M such that $|D| = |M|$. Suppose M is a perfect matching. Then, clearly D is a k-DRD set of G and result holds. Assume M is not a perfect matching and construct C_{v_i} for every $v_i \in D$ as provided in the beginning of Subsection 3.2.2 along with one additional condition. That is, for all i, $1 \leq i \leq p$, if $\left|N(v_i) \cap (V - (D \cup (\bigcup_{j=1}^{i-1} C_{v_j})))\right| > \left\lceil \frac{d(v_i)}{k} \right\rceil$, then choose $\left\lceil \frac{d(v_i)}{k} \right\rceil$ number of vertices along with a vertex u_i such that $v_i u_i \in M$ from the set $N(v_i) \cap (V - (D \cup (\bigcup_{j=1}^{i-1} C_{v_j})))$ and name it as C_{v_i}. If $\bigcup_{v_j\in D} C_{v_j} = V - D$, then D is a k-DRD set. Further, if a vertex $v \in V - (\bigcup_{v_j\in D} C_{v_j} \cup D)$ is adjacent to some vertex $u \in C_w$ with $|C_w| = 1$, then add u to D, w to $V - D$ and construct the set C_u for $u \in D$ as defined above. Since $\delta(G) > k$, u can dominate both w, v. Let A [$V -)\bigcup\limits_{v_j \in D} C_{v_j} \cup D$=. If A [$\emptyset$, then clearly D is a k-DRD set and result holds. Assume that A [/ $\emptyset$ and $w^* \in A$. Since M is a maximum matching and by the above constructions, w^* is not adjacent to any vertices in $V - D$. Hence, w^* is adjacent to at least k vertices in D. Then, as in the proof of the Theorem 3.2.20 either w^* is dominated by some vertex in D or we get a set $C \subseteq D$ satisfying following properties $(\mathscr{P})$:

($\mathscr{P}_{21}$) $|C_w|$ [$\left\lceil \frac{d)w=}{k} \right\rceil$ for all $w \in C$,

($\mathscr{P}_{22}$) $C_{w_i} \cap C_{w_j}$ [$\emptyset$ for all $w_i, w_j \in C$,

($\mathscr{P}_{23}$) $N)C_w$=$\cap D \subseteq C$ for all $w \in C$,

($\mathscr{P}_{24}$) $|C_w| > 1$ for all $w \in C$,

($\mathscr{P}_{25}$) The vertices in $\bigcup\limits_{w \in C} C_w$ has its all neighbor in C.

This leads to a contradiction. Hence, A [$\emptyset$, D is a k-DRD set and $\gamma_{\frac{d}{k}})G$=$\leq |D|$ [$\beta_1)G$=. □

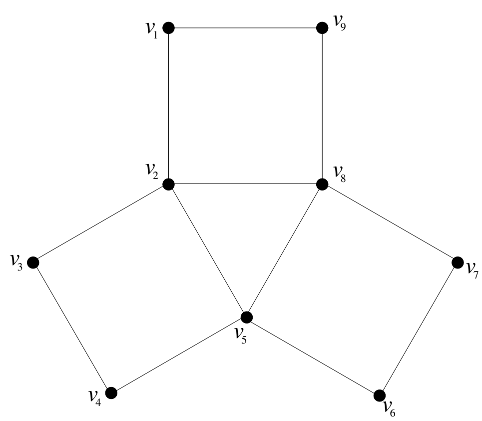


Figure 3.3 Graph H with $\gamma_{\frac{d}{k}})H$=, $\beta_1)H$=for some $k \geq \delta)H$=

Remark 3.2.25. *For any graph G with $\delta)G$=$\leq k$, $\gamma_{\frac{d}{k}})G$=are $\beta_1)G$=are incomparable. For example consider graph G in Figure 3.2 $\beta_1)G$=[3 [$\gamma_{\frac{d}{2}})G$=, $\gamma_{\frac{d}{3}})G$=[5. Also for graph H in Figure 3.3 $\gamma_{\frac{d}{2}})H$=[$\gamma_{\frac{d}{3}})H$=[3 , $\beta_1)H$=[4 , $\gamma_{\frac{d}{4}})H$=[5.*

Corollary 3.2.26. *For any graph G of even order n with $\delta)G$=$< k$, $\gamma_{\frac{d}{k}})G$=[$\beta_1)G$=$\leq n$. If $\gamma_{\frac{d}{k}})G$=[$\beta_1)G$=[n, then G has a perfect matching.*

Theorem 3.2.27. *For any tree T, $\gamma_{\frac{d}{k}})T\!=\leq \beta_0)T\!=\!\!\cdot$*

Proof. Let T be a rooted tree with m levels. Now, label all the vertices in m^{th} level as "0". Label all the vertices in $)m-1\!=^{th}$ level having child in m^{th} level labeled "0" as "1" and label all the remaining vertices in $)m-1\!=^{th}$ level as "0". Similarly, label all the vertices in $)m-2\!=^{th}$ level having child in $)m-1\!=^{th}$ level labeled "0" as "1" and label all the remaining vertices in $)m-2\!=^{th}$ level as "0". Continue the process for all the m levels. Let D be the set of all the vertices labeled "0". Then, D is an independent vertex set. Also, note that all the vertices labeled "1" will be dominated by its child vertices labeled "0". Hence, D is a k-DRD set and $\gamma_{\frac{d}{k}})T\!=\leq |D| \leq \beta_0)T\!=\!\!\cdot$ □

Remark 3.2.28. *For any graph other than tree, the vertex independence number β_0 and $\gamma_{\frac{d}{k}}$ are incomparable. For example the graph G of order $n < 6$ formed by joining two complete graphs by an edge, we get $\beta_0)G\!=,\ \ \gamma_{\frac{d}{2}})G\!=\leq \gamma_{\frac{d}{k}})G\!=\!\!\cdot$ For complete graph K_n, $n < 2$, $1\ [\ \ \ \beta_0)K_n\!=,\ \ \gamma_{\frac{d}{2}})K_n\!=[\ \ \ 2 \leq \gamma_{\frac{d}{k}})K_n\!=\!\!\cdot$*

In this chapter, we have studied the generalized concept of 2-part degree restricted domination. That is, k-part degree restricted domination. We have proposed some bounds on $\gamma_{\frac{d}{k}}$ of join of two graphs and bounds in terms of maximum degree, independence and covering number. In the next chapter, we discuss when a given dominating (1-DRD) set is a k-DRD set for some $k < 1$. We also study the relation between k-part degree restricted domination and some other domination invariants, like k-domination and efficient domination.

CHAPTER 4

RELATION BETWEEN k-DRD SET AND SOME DOMINATION INVARIANTS

There has been a massive amount of work carried out on domination. Several unique and interesting parameters have been adopted, such as independent domination, k-domination, efficient domination by combining domination with another graph theoretical properties. Numerous efforts are made to identify the relationship between domination invariants. In this chapter, some relationship between k-DRD set and dominating set, k-DRD set and k-dominating set as well as a relation between k-DRD set and efficient dominating set of a graph are discussed.

4.1 RELATION BETWEEN DOMINATING SET AND k-DRD SET

Every dominating set need not be a k-DRD set; however, it is true that $\gamma_{\frac{d}{k}}(G) = \gamma(G)$ only for some graphs. But looking at the dominating set it is difficult to determine, whether it is a k-DRD set or not. Clearly, for any dominating set D, if $|V - D| > \sum\limits_{u \in D} \left\lceil \frac{d(u)}{k} \right\rceil$, then D is not a k-DRD set. If $|T_u| > \underset{w \in N(u) \cap D}{\Delta} \left\lceil \frac{d(w)}{k} \right\rceil$ for at least one $u \in V - D$, where $T_u = \{v \in V - D : N(v) \cap D \subseteq N(u) \cap D\} \cup \{u\}$, then also D is not a k-DRD set. For any connected graph G, $1 \leq \gamma(G) \leq \frac{n}{2}$. Similarly, for $k > 1$, $\gamma(G) = \gamma_{\frac{d}{k}}(G) = 1$ if and only if $G = K_1$ or $G = K_2$. Also, for a graph G of even order n, with no isolated vertices and $k > 1$, $\gamma(G) = \gamma_{\frac{d}{k}}(G) = \frac{n}{2}$ if and only if the components of G are cycle C_4 or the corona $H \circ K_1$ for any connected graph H.

Proposition 4.1.1. *For any graph G, if $\gamma(G) < \left\lceil \frac{n}{m+1} \right\rceil$, where $m = \left\lceil \frac{\Delta(G)}{k} \right\rceil$, then $\gamma(G) < \gamma_{\frac{d}{k}}(G)$.*

Proof. Let D be a $\gamma_{\frac{d}{k}}$-set of G. Since for any $u \in D$ order of C_u can not exceed $\left\lceil \frac{\Delta(G)}{k} \right\rceil$, we have $\left\lceil \frac{n}{\lceil \frac{\Delta(G)}{k} \rceil + 1} \right\rceil \leq \gamma_{\frac{d}{k}}(G)$. Hence, result holds. □

Proposition 4.1.2. *Let $k > 1$ and D be an independent γ-set of a tree T such that D has no pendant vertices. Then, D is not a k-DRD set of tree T.*

Proof. Suppose D is a k-DRD set of a tree T satisfying the conditions in the hypothesis. Then, there exists a partition $\{C_u : u \in D\}$ of $V - D$ such that $|C_u| \leq \left\lceil \frac{d(u)}{k} \right\rceil$. Since D is independent and $d(u) > 1$ for every $u \in D$, C_u is a proper subset of $N(u)$ for every $u \in D$. Let $w_1 \in N(u) - C_u$ for some $u \in D$. Since $\bigcup_{u \in D} C_u = V - D$, $w_1 \in C_v$ for some $u \neq v \in D$. Since C_v is a proper subset of $N(v)$, $N(v) - C_v \neq \phi$. Choose a vertex w_2 from $N(v) - C_v$. Furthermore, $w_2 \in C_u$. If $w_2 \in C_u$, then $u - w_1 - v - w_2 - u$ will form a cycle, a contradiction. Continuing the above process, we get a vertex which is not in any of C_u, $u \in D$, a contradiction to the fact that D is a k-DRD set. Hence, D is not a k-DRD set of T. □

The above results clearly tells that, every dominating set is not a k-DRD set. To determine, whether a given dominating set D of a graph G is a k-DRD set or not one has to find C_u for every $u \in D$. Construct C_u for every $u \in D$ as constructed in the beginning of the Subsection 3.2.2. Let A be the collection of all the vertices u in D such that $|C_u| > \lceil \frac{d(u)}{k} \rceil$. Throughout this section, sets S, A are used and defined as follows:

$$S = V - \Big(\bigcup_{u \in D} C_u \cup D\Big),$$

$$A = \left\{ v \in D : |C_v| > \left\lceil \frac{d(v)}{k} \right\rceil \right\},$$

where C_u for $u \in D$ is constructed as defined above. One can observe that, sets S and A changes as C_u changes for u in a given set D. Since for $u \in D$, C_u is not unique, sets S and A are also not unique. If $\bigcup_{u \in D} C_u = V - D$, then D is a k-DRD set.

Lemma 4.1.3. *A γ-set D of a connected graph G is a k-DRD set if and only if, for every subset A of $V - D$, $\sum_{u \in N(A) \cap D} \left\lceil \frac{d(u)}{k} \right\rceil \geq |A|$.*

Proof. Let D be both γ-set and k-DRD set of a graph G and $A \subseteq V - D$. Then, $A \subseteq \bigcup_{u \in N(A) \cap D} C_u$, which implies $|A| \leq |\bigcup_{u \in N(A) \cap D} C_u| \leq \sum_{u \in N(A) \cap D} \left\lceil \frac{d(u)}{k} \right\rceil$. Conversely, assume that, for any subset A of $V - D$, $\sum_{u \in N(A) \cap D} \left\lceil \frac{d(u)}{k} \right\rceil \geq |A|$. For every $u \in D$, construct C_u as defined in the beginning of the Subsection 3.2.2. If $\bigcup_{u \in D} C_u = V - D$, then D is a k-DRD set of G and result holds. Suppose $\bigcup_{u \in D} C_u \neq V - D$. Then, there exists a vertex $w^* \in$

$(V-D)-(\bigcup_{u\in D} C_u)$. Since D is a dominating set, w^* is adjacent to at least one vertex in D. For every $v \in N(w^*)\cap D = B$, $|C_v| = \left\lceil \frac{d(v)}{k} \right\rceil$. Also, for every $u \in N(\bigcup_{x\in B} C_x)\cap D = B_1$, $|C_u| = \left\lceil \frac{d(u)}{k} \right\rceil$. Otherwise, there exists a path $P = w^*, v_1, v_2, v_3$ such that $v_1, v_3 \in D$, $|C_{v_3}| > \left\lceil \frac{d(v_3)}{k} \right\rceil$ and $v_2 \in C_{v_1}$. Redefine, $C_{v_3} = C_{v_3} \cup \{v_2\}, C_{v_1} = (C_{v_1} - \{v_2\}) \cup \{w^*\}$. Then, w^* is dominated by v_1 and D is a k-DRD set. Suppose for every $u \in N(\bigcup_{x\in B_1} C_x)\cap D = B_2$, $|C_u| = \left\lceil \frac{d(u)}{k} \right\rceil$. Then, continuing in this manner, we get the set of vertices $\{w_1, w_2, \ldots, w_l\}$ such that $|C_{w_i}| = \left\lceil \frac{d(w_i)}{k} \right\rceil$ for all i, $1 \le i \le l$ and $(N(\bigcup_{k=1}^{l} C_{w_k}))\cap D \subseteq \{w_1, w_2, \ldots, w_l\}$. Consider $A = \bigcup_{k=1}^{l} C_{w_k} \cup \{w^*\}$. Then, $N(A)\cap D \subseteq \{w_1, w_2, \ldots, w_l\}$ and $\sum_{u\in N(A)\cap D} |C_u| \le \sum_{i=1}^{l} |C_{w_i}| = \sum_{i=1}^{l} \left\lceil \frac{d(w_i)}{k} \right\rceil = |A|-1 > |A|$, a contradiction. Hence, w^* should be dominated by some vertex in D and D is a k-DRD set. □

Proposition 4.1.4. *Let G be a connected graph and $\gamma(G) = 2$. Then, $\gamma_{\frac{d}{k}}(G) = 2$ if and only if $|Pn(u,D)\cap(V-D)| \le \left\lceil \frac{d(u)}{k} \right\rceil$ and $|V-D| \le \sum_{u\in D} \left\lceil \frac{d(u)}{k} \right\rceil$ for any γ-set D of G and $u \in D$, where $Pn(u,D)$ is the private neighborhood of u.*

Proof. Assume that $\gamma(G) = \gamma_{\frac{d}{k}}(G) = 2$ and D is both γ-set and k-DRD set of G. Let $u \in D$ and $A = Pn(u,D)\cap(V-D)$. Since $Pn(u,D)$ is the private neighborhood set of u, $N(A)\cap D = N(Pn(u,D)\cap(V-D))\cap D = \{u\}$. Then, by Lemma 4.1.3 $|Pn(u,D)\cap(V-D)| \le \sum_{u\in N(A)\cap D} \left\lceil \frac{d(u)}{k} \right\rceil = \left\lceil \frac{d(u)}{k} \right\rceil$. Similarly, take $A = V-D$, then by Lemma 4.1.3 $|V-D| \le \sum_{u\in N(A)\cap D = D} \left\lceil \frac{d(u)}{k} \right\rceil$. Conversely, let $D = \{v_1, v_2\}$ be a γ-set of G satisfying above conditions. Let A be any subset of $V-D$,

$$\begin{aligned} A_1 &= \{u \in A : N(u)\cap D = \{v_1\}\}, \\ A_2 &= \{u \in A : N(u)\cap D = \{v_2\}\}, \\ A_3 &= \{u \in A : N(u)\cap D = D\}, \end{aligned}$$

If $A_3 = \emptyset$, then

$$\begin{aligned} |A| = |A_1| + |A_2| &\le |Pn(v_1,D)\cap(V-D)| + |Pn(v_2,D)\cap(V-D)| \\ &\le \left\lceil \frac{d(v_1)}{k} \right\rceil + \left\lceil \frac{d(v_2)}{k} \right\rceil \end{aligned}$$

$$\leq \sum_{u\in N(A)\cap D} \left\lceil \frac{d(u)}{k} \right\rceil,$$

If $A_3 \neq \emptyset$, then

$$|A| \leq |V-D| \leq \sum_{u\in D = N(A)\cap D} \left\lceil \frac{d(u)}{k} \right\rceil,$$

Since for any subset $A \subseteq V-D$, $|A| \leq \sum_{u\in N(A)\cap D} \left\lceil \frac{d(u)}{k} \right\rceil$. By Lemma 4.1.3, D is a k-DRD set of G and $\gamma_{\frac{d}{k}}(G) = 2$.

□

Proposition 4.1.5. *For a graph G having D as a γ-set, there exists a super graph of G having same vertex set V and D as k-DRD set if*

$$\gamma(G) \geq \begin{cases} \left\lceil \frac{n-\gamma(G)}{m} \right\rceil & \text{if } n \equiv 1 (\text{mod } k), \\ \left\lceil \frac{n-\gamma(G)}{m+1} \right\rceil & \text{if } n \not\equiv 1 (\text{mod } k), \end{cases} \quad \text{where } m = \left\lfloor \frac{n-1}{k} \right\rfloor.$$

Proof. Construct a graph G' from G by joining each vertex in D to every other vertex in V. Then, to dominate vertices in $V-D$, at least $\left\lceil \frac{n-\gamma(G)}{\lceil \frac{n-1}{k} \rceil + 1} \right\rceil$ number of vertices in D are required. Then, by the hypothesis D is a k-DRD set of G'. □

Theorem 4.1.6. *A dominating set D of a graph G is a k-DRD set if and only if for every vertex $u \in S$ there exists a path $P_u = u, v_1, v_2, \ldots, v_{2l+1}$ satisfying the following.*

1. *For i, $0 \leq i \leq l$, $v_{2i+1} \in D$.*

2. *For i, $0 > i \leq l$, $v_{2i} \in C_{v_{2i-1}}$.*

3. $|C_{v_{2l+1}}| > \left\lceil \frac{d(v_{2l+1})}{k} \right\rceil$.

4. *If the paths $P_{u_1}, P_{u_2}, \ldots, P_{u_m}$ ends at the same vertex v, then $\left\lceil \frac{d(v)}{k} \right\rceil - |C_v| \geq m$.*

5. *For every $u, w \in S$, $V(P_u) \cap V(P_w) \cap (V-D) = \phi$.*

Proof. Let D be a k-DRD set. Then, for each $u \in D$ there exists $C'_u \subseteq N(u) \cap (V-D)$ such that $|C'_u| \leq \left\lceil \frac{d(u)}{k} \right\rceil$ and $\bigcup_{u\in D} C'_u = V-D$. Construct C_u for every $u \in D$ as defined in Subsection 3.2.2 and find set S. If $S = \emptyset$, then result holds. If $S \neq \emptyset$, then for each vertex $u \in S$, a path P_u satisfying the conditions stated in the theorem is constructed. Consider a vertex u from S. Since D is a k-DRD set, $u \in C'_{v_1}$ for some $v_1 \in D$. If $|C_{v_1}| > \left\lceil \frac{d(v_1)}{k} \right\rceil$, then uv_1 is a path satisfying the above first three conditions. If $|C_{v_1}| = \lceil \frac{d(v_1)}{k} \rceil$, then $C_{v_1} - C'_{v_1} \neq \emptyset$ (Since $u \in C'_{v_1} - C_{v_1}$ and $|C'_{v_1}| \leq \lceil \frac{d(v_1)}{k} \rceil = |C_{v_1}|$, $C_{v_1} - C'_{v_1} \neq \emptyset$). Consider a vertex v_2 from $C_{v_1} - C'_{v_1}$. Since D is a k-DRD set $v_2 \in C'_{v_3}$ for some $v_3 \in D$. If $|C_{v_3}| > \lceil \frac{d(v_3)}{k} \rceil$, then $P_u = u, v_1, v_2, v_3$. If $|C_{v_3}| = \lceil \frac{d(v_3)}{k} \rceil$, then $C_{v_3} - C'_{v_3} \neq \emptyset$, choose a vertex from $C_{v_3} - C'_{v_3} \neq \emptyset$ and continue the process. (Here, we have to choose one vertex from $C_{v_3} - C'_{v_3}$ say v_4, we assume that $v_4 \in C'_{v_5}$ for some $v_5 \in D$ and we continue the process. If $v_4 \in C'_{v_1}$ (or C'_{v_3}), then $C'_{v_1} - C_{v_1}$ (or $C'_{v_3} - C_{v_3}$) has at least 2 vertices. Since $|C_{v_1} - C'_{v_1}| \geq 2$, we can continue the procedure with vertex other than v_2.) Since D is a finite k-DRD set, the above process has to terminate. So, after some finite steps we find a vertex $v_{r-1} \in C_{v_{r-2}} - C'_{v_{r-2}}$, $v_{r-1} \in C'_{v_r}$ such that $|C_{v_r}| > \lceil \frac{d(v_r)}{k} \rceil$. Now, for chosen vertex u from S, we have a path $P_u = u, v_1, v_2, \ldots, v_r$ such that $v_{2i+1} \in D$ for each i, $0 \leq i \leq \frac{r-1}{2}$, $v_{2i} \in C_{v_{2i-1}}$ for each i, $0 > i \leq \frac{r-1}{2}$ ($r \geq 3$), r is odd and $|C_{v_r}| > \lceil \frac{d(v_r)}{k} \rceil$.

Let $P_u = u, u_1, u_2, u_3, \ldots, u_l$, $P_w = w, w_1, w_2, w_3, \ldots, w_q$ be two paths for some $u, w \in S$ such that $u_{2j} = w_{2i}$ and $w_{2i} \in V - D$. Then, by the construction $u_{2j} \in C_{u_{2j-1}}$ and $w_{2i} \in C_{w_{2i-1}}$. Hence, $w_{2i-1} = u_{2i-1}$. Now, $w_{2i-2}, u_{2i-2} \in C'_{w_{2i-1}}$ and $w_{2i-2}, u_{2i-2} \in C_{w_{2i-3}}$. Then, $|C'_{w_{2i-1}} - C_{w_{2i-1}}| \geq 2$. If w_{2i-1} is not an end vertex of path P_w, then $|C_{w_{2i-1}} - C'_{w_{2i-1}}| \geq 2$. Hence, we continue the process as above with vertex other than w_{2i} and we can find one new path P'_w such that $V(P'_u) \cap V(P_w) \cap (V - D) = \emptyset$.

Assume that $w_q = u_l$, $\lceil \frac{d(w_q)}{k} \rceil - |C_{w_q}| = 1$ and there is no other paths satisfying the conditions in the hypothesis for u, w. Let $B_1 = N(u, w) \cap D$, $B'_1 = \bigcup_{u \in B_1} C_u$. For $i \neq 1$, $B_i = N(B'_{i-1}) \cap D$ and $B'_i = \bigcup_{u \in B_i} C_u$. Note that, for u, w there is no path other than P_u, P_w satisfying the above three conditions. Hence, $|C_{w'}| = \lceil \frac{d(w')}{k} \rceil$ for every $w' \in B_i - \{w_q\}$. Since V is finite, there exist $m, n \in N$ such that $B_j = B_{j+1} = B_{j+2}$ for all $j \geq m$ and $B'_l = B'_{l+1} = B'_{l+2}$ for all $l \geq n$. Then, $|B'_n| = \sum_{u \in N(B'_n) \cap D} |C_u|$. Since $u, w \in B'_n$, $\lceil \frac{d(w_q)}{k} \rceil - |C_{w_q}| = 1$ and $|C_{w'}| = \lceil \frac{d(w')}{k} \rceil$ for every $w' \in B_n - \{w_q\}$, $|B'_n| \neq \sum_{u \in N(B'_n) \cap D} \lceil \frac{d(u)}{k} \rceil$, a contradiction to Lemma 4.1.3. Conversely, construct C^*_u for all $u \in D$ which dominates all the vertices of S. First consider a vertex u of S, then there exists a path $u, v_1, v_2, \ldots, v_l$ satisfying the above conditions. Define $C^*_{v_1} = C_{v_1} \cup \{u\} - \{v_2\}$, $C^*_{v_l} = C_{v_l} \cup \{v_{l-1}\}$, $C^*_{v_{2i+1}} = C_{v_{v_{2i+1}}} \cup \{v_{2i}\} - \{v_{2i+2}\}$, *for all* i, $1 \leq i \leq \frac{l-3}{2}$ $(l \geq 3)$. Since $|C_{v_l}| > \lceil \frac{d(v_l)}{k} \rceil$, $|C^*_{v_l}| \leq \lceil \frac{d(v_l)}{k} \rceil$. Also, observe that $|C^*_{v_{2i+1}}| = |C_{v_{2i+1}}| \leq \lceil \frac{d(v_{2i+1})}{k} \rceil$ for all $i, 0 \leq i \leq \frac{l-3}{2}$ and $u \in C^*_{v_1}$ is dominated by D. Since such path exists for all the vertices in S, $\bigcup_{v \in D} C^*_v = V - D$. Hence, D is a k-DRD set. □

Remark 4.1.7. *Let* $t_u = \lceil \frac{d(u)}{k} \rceil - |C_u|$ *for* $u \in A$. *Then, we can observe that if* $\sum_{u \in A} t_u > |S|$, *then* $|V-D| \,[\, \Delta_{u \in D-A} \left\lceil \frac{d=u+}{k} \right\rceil \,]\, \Delta_{v \in A} |C_v| \,]\, |S| \,/\, \Delta_{u \in D} \left\lceil \frac{d=u+}{k} \right\rceil$. *Hence, D is not a k-DRD set. If* $\Delta_{u \in A} t_u \geq |S|$, *then also D need not be a k-DRD set.*

Proposition 4.1.8. *For any dominating set D of a tree T, if* $\langle S \rangle$ *is connected and* $|S| \,/\, |A|$*, then D is not a k-DRD set.*

Proof. If D is a k-DRD set, then by Theorem 4.1.6 there should be a path from each vertex in S to some vertices in A satisfying some conditions. Since $\langle S \rangle$ is connected and T is a tree, there is no path from two different vertices in S which ends at same vertex in A. Since $|S| \,/\, |A|$, there is no path from each vertex in S to a unique vertex in A satisfying the condition in Theorem 4.1.6. Hence, D is not a k-DRD set of T.

□

One can observe that, for a given dominating set D, if there exists path satisfying conditions in Theorem 4.1.6, then D is a k-DRD set. For a given graph G and dominating set D, an algorithm is developed to find paths that satisfy the requirements in Theorem 4.1.6 as follows:

4.1.1 Algorithm to verify whether a given dominating set is a k-DRD set or not

For a given graph G, a dominating set D and for each $u \in D$, initially construct C_u. If $\bigcup_{v \in D} C_v \,[\, V-D$, then D is a k-DRD set. Suppose $=V-D+-\bigcup_{v \in D} C_v \,[\, S \,[\!/\, \phi$. Then, check whether vertices of S can be included in some C_u, $u \in D$. Define, set A as the collection of all the vertices in D such that $|C_u| > \left\lceil \frac{d=u+}{k} \right\rceil$. By Depth first search find the existence of path, from every vertex in S to some vertex in A, which satisfies the conditions in the Theorem 4.1.6. If such path exists for all the vertices in S, then D is a k-DRD set, otherwise D is not a k-DRD set. Throughout the section, the graph labeled by natural numbers are considered.

The key idea in driving Algorithm 4.1 is as follows: First for every vertex i in V find degree $d=i+$, vertex of maximum degree Σ and for every vertex i in D find neighborhood N_i in $V-D$. Add a vertex of minimum degree from N_i to C_i, repeat this step by adding vertex of next minimum degree to C_i until either order of C_i is $\left\lceil \frac{d=i+}{k} \right\rceil$ or N_i becomes empty, update $V-D$ by removing the elements of C_i along with i. Repeat this procedure for each vertex in D, which gives a set C_i for each $i \in D$. If $\bigcup_{i \in D} C_i \,[\, V-D$, then D is a k-DRD set. Otherwise, check the existence of path from each vertex in S to some vertices in A as in Theorem 4.1.6. Depth First Search with stack function P is used to find these paths. Note that Top[0 means $P \,[\, \phi$ and P(Top)[i means P$\cup\{i\}$. Since a vertex of degree one or its neighborhood vertex should be in k-DRD set, for any $i \in D$ while adding vertices to C_i first preference is given to a vertex of minimum degree in N_i.

Theorem 4.1.9. *The Algorithm 4.1 runs in $O(n^3)$ time.*

Proof. For a given graph G and its dominating set D, calculating the degree of each vertex in V it takes $O(n^2)$ time. Similarly to determine the neighborhood of each vertex in $V - D$, takes $O(n^2)$ time. Since cardinality of neighborhood of any vertex is at most $n - 1$, constructing C_v for each vertex $v \in D$ takes $O(n^3)$ time. We find path using DFS which takes $O(n^2)$ time if exists. In total to find such paths for each vertex in S it takes $O(n^3)$ running time. Hence, complexity of the algorithm is $O(n^3)$. □

Algorithm 4.1: Test for dominating set to be a k-DRD set

Input: A simple graph $G \leftarrow (V,E)$, positive integer k, γ-set D, maximum degree Δ.
Output: D is a k-DRD set or not.

begin
  $D' \leftarrow V - D$,
  for $i \in V$ **do**
    $d(i) \leftarrow 0$
    for *each* $j \in V$ **do**
      $d(i) \leftarrow d(i) + a_{ij}$
    end
  end
  for $i \in D$ **do**
    $N_i \leftarrow \phi$
    for *each* $j \in D'$ **do**
      if $a_{ij} \leftarrow 1$ **then**
        $N_i \leftarrow N_i \cup \{j\}$
      end
    end
    $C_i \leftarrow \phi$
    while $|C_i| > \left\lceil \frac{d(i)}{k} \right\rceil$ *and* $N_i \neq \phi$ **do**
      $d'_\Delta \leftarrow \Delta$
      for *each* $j \in N_i$ **do**
        if $d(j) \leq d'_\Delta$ **then**
          $d'_\Delta \leftarrow d(j)$, $d_\Delta \leftarrow j$
        end
      end
      $C_i \leftarrow C_i \cup \{d_\Delta\}$, $N_i \leftarrow N_i - \{d_\Delta\}$
    end
    $D' \leftarrow D' - C_i$
  end
  if $\bigcup_{i \in D} C_i \leftarrow D'$ **then**
    D is k-DRD set
  end
  else
    $S \leftarrow D' - \bigcup_{i \in D} C_i$, $A \leftarrow \{j \in D : |C_j| > \left\lceil \frac{d(j)}{k} \right\rceil\}$
    if $A \leftarrow \phi$ **then**
      D is not a k-DRD set
    end
    else
      for *all* $i \in S$ **do**
        $P \leftarrow$ *call* $Path(i)$
        $P \leftarrow v_o, v_1, v_2, \ldots, v_k$, $\frac{k-3}{2} \leftarrow m$
        for $l \leftarrow 1, 2, \ldots, m$ **do**
          $C_{v_{2l+1}} \leftarrow (C_{v_{2l+1}} \cup \{v_{2l}\})$-$\{v_{2l+2}\}$
        end
        $C_{v_1} \leftarrow (C_{v_1} \cup \{v_o\}) - \{v_2\}$ $C_{v_k} \leftarrow C_{v_k} \cup \{v_{k-1}\}$
      end
      D-is a k-DRD set
    end
  end
end

Table 4.1 Algorithm to verify whether a given dominating set is a k-DRD set or not

Algorithm 4.2: Path(i)

```
begin
    for all g ∈ V do
        Visited[g]=0
    end
    Top [  0, Visited[i]=1, Push=i+
    while P [/ φ do
        j=P(Top)
        if j ∈ D then
            N'_j= {v ∈ C_j : Visited(v) [  0}
        end
        else
            N'_j= {v ∈ D : a_jv [  1<Visited(v) [  0}
        end
        if N'_j [/ φ then
            choose a vertex l from N'_j, Push(l),Visited(l) [  1
            if l ∈ A then
                return P
            end
        end
        else
            pop()
        end
    end
    if P [  φ then
        D is not a k-DRD set
    end
end
```

Table 4.2 Algorithm to find all possible path satisfying the conditions in Theorem 4.1.6

Algorithm 4.3: Pop

```
begin
    Top=Top+1
    P(Top)=i
end
```

Table 4.3 Pop operation

Algorithm 4.4: Push(i)

```
begin
    P(Top)=Null
    Top=Top-1
end
```

Table 4.4 Push operation

4.2 RELATION BETWEEN *k*-DOMINATING SET AND *k*-DRD SET

There are some similarity between names *k*-dominating set and *k*-part degree restricted dominating set, so a study is initiated to find relationship between them. In this section, a relation between *k*-domination number $\gamma_k(G)$ and *k*-part degree restricted domination number $\gamma_{\frac{d}{k}}(G)$ of a graph G is discussed. Also proved that, $\gamma_{\frac{d}{k}}(G) \leq \gamma_k(G)$ for graph G and characterized the trees T for which $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$.

Definition 4.2.1. *Fink and Jacobson (1985) For a positive integer k, a dominating set D of a graph G is called a k-dominating set, if every vertex of* $V - D$ *is adjacent to at least k vertices in D. The k-domination number of G is the minimum cardinality of a k-dominating set in G and is denoted by* $\gamma_k(G)$.

Theorem 4.2.2. *In any graph G, every k-dominating set is a k-DRD set.*

Proof. Without loss of generality, assume that G is connected (otherwise, we can apply the following argument for each of the components of G). Let D be a k-dominating set of G. Then, each vertex in $V - D$ is adjacent to at least k vertices in D. Construct C_u for every $u \in D$ and the proof follows by the similar argument used in the proof of Theorem 3.2.20 in Chapter 3. □

Corollary 4.2.3. *For any graph G,* $\gamma_{\frac{d}{k}}(G) \leq \gamma_k(G)$.

Proof. Let D be a minimum k-dominating set of graph G. Then, by Theorem 4.1.6, D is a k-DRD set of G and $\gamma_{\frac{d}{k}}(G) \leq |D| = \gamma_k(G)$.

□

Corollary 4.2.4. *For any graph G with* $\delta(G) \geq k$, $\gamma(G) \leq \gamma_{\frac{d}{k}}(G) \leq \gamma_k(G) \leq \alpha_0(G)$.

Proof. Since $\delta(G) \geq k$, every vertex cover set is a k-dominating set. Also note that every k-dominating set is a k-DRD set and every k-DRD set is a dominating set. Hence, above inequality holds.

□

Remark 4.2.5. *For $k = 2$, the bound stated in Corollary 4.2.3 can be attained by the graph P_{2n+1}, C_n and K_n, $n \geq 2$. Also for any graph G and $k \geq \Delta(G)$, $\gamma_{\frac{d}{k}}(G) > \gamma_k(G)$ and $\gamma_{\frac{d}{k}}(G) + \beta_1(G) = \gamma_k(G)$.*

Proposition 4.2.6. *For any graph G with $\delta(G) \geq k$,*

$$\frac{\gamma_{\frac{d}{k}}(G) + \gamma_k(G)}{2} \leq n - \beta_0(G).$$

Proof. For any graph G and $\delta(G) \geq k$, $\gamma_{\frac{d}{k}}(G) \leq \alpha_0(G)$ and $\gamma_k(G) \leq \alpha_0(G)$. Since $\alpha_0(G) + \beta_0(G) = n$,

$$\frac{\gamma_{\frac{d}{k}}(G) + \gamma_k(G)}{2} \leq n - \beta_0(G).$$

□

Lemma 4.2.7. *For any graph G, $\gamma_{\frac{d}{k}}(G) = \gamma_k(G)$ if and only if G has a $\gamma_{\frac{d}{k}}$-set which is a k-dominating set.*

Proof. Assume that $\gamma_{\frac{d}{k}}(G) = \gamma_k(G)$ and D is a minimum k-dominating set of G. Then, by Theorem 4.2.2, D is a k-DRD set of G. Since $\gamma_{\frac{d}{k}}(G) = \gamma_k(G)$, D is a $\gamma_{\frac{d}{k}}$-set which is a k-dominating set. Conversely, suppose G has a $\gamma_{\frac{d}{k}}$-set D, which is a k-dominating set. Then, $\gamma_k(G) \leq |D| = \gamma_{\frac{d}{k}}(G)$. From Corollary 4.2.3, $\gamma_{\frac{d}{k}}(G) \leq \gamma_k(G)$. Hence, $\gamma_{\frac{d}{k}}(G) = \gamma_k(G)$. □

Lemma 4.2.8. *For any tree $T \neq K_2$ and $k \geq 1$, $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$ if and only if there exists a set $D \subseteq V(T)$ satisfying the following properties ($\mathscr{P}$):*

($\mathscr{P}_1$) *All the pendant vertices are in D.*

($\mathscr{P}_2$) *$d(u) = k$ for all $u \in V - D$.*

($\mathscr{P}_3$) *If $uv \in E(T)$, then either $u \in D$ and $v \in V - D$ or $u \in V - D$ and $v \in D$.*

Proof. Assume that T is a rooted tree such that $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. Then, there exists a $\gamma_{\frac{d}{k}}$-set D which is a k-dominating set. Since D is a k-dominating set, property $\mathscr{P}_1$ holds trivially.

Claim: $|C_u| \leq 1$ for all $u \in D$ and if $|C_u| = 1$, then C_u contain the parent vertex of u. Let $u \in D$ be a vertex in the i^{th} level of rooted tree T such that $|C_u| = 2$ and $|C_v| \leq 1$ for all the vertices v in the succeeding level. (If $|C_u| \geq 2$, then apply the same following argument for each of the child neighbor of u in C_u.) Then, at least one vertex in $C_u \subseteq V - D$ say u_1 should be a child of u and $d(u_1) \geq 1$. (Since $u_1 \in V - D$ and D is k-dominating set.) Since D is a k-dominating set, at least one child of u_1 say u_2 should be in D. If $|C_{u_2}| = 0$ (or $d(u_2) = 1$), then u_2 can dominate u_1 and $|C_u| = 1$. If not, then u_2 has at least one child say $u_3 \in C_{u_2}$ in $V - D$. Since D is a k-dominating set, at least one child of u_3 say u_4 should be in D. If $|C_{u_4}| = 0$ (or $d(u_4) = 1$), then u_4 can dominate u_3, u_2 can dominate u_1 and $|C_u| = 1$. If not, then continuing this process a path $P = u, u_1, u_2, \ldots, u_l$ such that $u_i \in D$ if i is even, $u_i \in V - D$ if i is odd and $d(u_l) = 1$ is obtained. Then, by similar rearrangements modify C_u such that $|C_u| = 1$ and C_u contains the parent vertex of u. Now, D is a minimum k-DRD set, which is a k-dominating set such that $|C_u| \leq 1$ and if $|C_u| = 1$, then C_u contains the parent vertex of u.

Since D is a k-dominating set, $d(u) \geq k$ for all $u \in V - D$. Let $d(u) = k + 1$ and N be the set of k neighbor of u in D. By above claim there exists two vertices $v, w \in N$ such that $C_v = \{u\}$ and $C_w = \emptyset$. Since $d(u) = k + 1$ and $\left\lceil \frac{d(u)}{k} \right\rceil = 2$, u can dominate two of its neighbors. Hence, $D - \{v, w\} \cup \{u\}$ is a k-DRD set of tree T with $C_u = \{v, w\}$, a contradiction to the fact that D is a minimum k-DRD set. Hence, property $\mathscr{P}_2$ holds.

If $uv \in E(T)$, then by $\mathscr{P}_2$ both u, v are not in $V - D$. Assume that $u, v \in D$ such that u lies in l^{th} level and v lies in the $l + 1^{th}$ level. Then, $C_v = \emptyset$ and $|C_u| = 1$. Let $C_u = \{u_1\} \subseteq V - D$. Since $d(u_1) = k$ and D is a k-dominating set, all the neighbors of u_1 is in D. If u_1 has at least one child neighbor say $u_2 \neq u$ in D, then u_2 can dominate u_1 and v can dominate u, a contradiction to the fact that D is a minimum k-DRD set. Assume that u_1 has no child other than u in D. Then, $d(u_1) = k = 2$ and parent vertex of u_1 say u_3 is in D. If $|C_{u_3}| = 0$, then it is a contradiction to the fact that D is a $\gamma_{\frac{d}{k}}$-set. If $C_{u_3} = \{u_4\}$, then parent vertex of u_4 is in D, continuing like this we get a path $P = v, u, u_1, u_3, u_4, \ldots, u_r$ from v to root vertex u_r such that $u_i \in D$ if i is odd, $u_i \in V - D$ if i is even for $i \neq 2$. Suppose $u_r \in D$. Then, $C_{u_r} = \emptyset$ and v can dominate u, (by some rearrangement in $V - D$) a contradiction. If $u_r \in V - D$, then at least two child vertices of u_r should be in D. Then, v can dominate u (by some rearrangement in $V - D$), a contradiction. Hence, property $\mathscr{P}_3$ holds.

Conversely, assume that T is a rooted tree having m levels and $D \subseteq V$ satisfying all the above properties. Then, by properties $\mathscr{P}_2$ and $\mathscr{P}_3$, D is a k-dominating set and hence k-DRD set of T. Also, Properties $\mathscr{P}_1$ and $\mathscr{P}_3$ implies that, vertices in $(m-i)^{th}$ level lie in $V - D$ if i is odd and vertices in $(m-i)^{th}$ level lie in D if i is even for all i, $1 \leq i > m$. Let D^* be a minimum k-DRD set of tree T such that $\bigcup_{u \in D^*} C'_u = V - D^*$. Construct a minimum k-DRD set D' of T from D^* such that $V - D \subseteq D'$. Now all the vertices in m^{th} level are in D and all the vertices in $(m-1)^{th}$ level are in $V - D$. If there is a vertex $v \in V - D^*$ lies in $(m-1)^{th}$ level, then pendant neighbor (Since $v \in V - D$, $d(v) \neq 1$) of v say u should be in D^* with $C'_u = \emptyset$ (or $C'_u = \{v\}$). Define, $D_1 = D^* \cup \{v\} - \{u\}$ and $C_v = \{u\}$. If there is a vertex $v \in D_1$ lies in $(m-1)^{th}$ level, such that $C_v = \{v'\}$ and v' is the parent vertex of v, then also pendant neighbor (Since $v \in V - D$, $d(v) \neq 1$) of v say

w should be in D_1 with $C'_w = \emptyset$. (Since $v \in V - D$, $d(v) = k$ and $|C_v|$ can not exceed 1.) Define, $D_2 = D_1 \cup \{v'\} - \{w\}$, $C_{v'} = \emptyset$ and $C_v = \{w\}$. Then, D_2 is a minimum k-DRD set of T such that all the vertices in $(m-1)^{th}$ level is in D_2 and dominates only its pendant neighbor. Since vertices lie in $(m-3)^{th}$ level are in $V - D$, the $(m-3)^{th}$ level vertices are not pendant vertices. If there is a vertex $w \in V - D_2$ that lies in $(m-3)^{th}$ level, then child neighbor of w say w' is in D_2 with $C'_{w'} = \emptyset$ (or $C'_{w'} = \{w\}$) (Since w' has all neighbors except w in $m-1^{th}$ level and all the vertices in $m-1^{th}$ level are in D_2 and only dominating its child vertices). Define, $D_3 = D_2 \cup \{w\} - \{w'\}$, $C_w = \{w'\}$. If there is a vertex $u \in D_3$ that lies in $(m-3)^{th}$ level, such that $C_u = \{u'\}$ and u' is the parent vertex of u, then child of u say w^* should be in D_3 with $C'_{w^*} = \emptyset$ (Since $u \in V - D$, $d(u) = k$ and $|C_u|$ can not exceed 1. Also w^* has all neighbors except u in $m-1^{th}$ level and all the vertices in $m-1^{th}$ level are in D_3 and only dominating its child vertices). Proceeding in this manner, a minimum k-DRD set $D_r = D'$ such that all the vertices in $(m-i)^{th}$ level lie in D' if i is odd and $V - D \subseteq D'$ is obtained. Then, $V - D' \subseteq D$ and $D' = (V - D) \cup (D \cap D')$. Since all the neighbors of D lie in $V - D$, $C'_w = \emptyset$ for all $w \in D \cap D' \subseteq D'$. Since $d(u) = k$ for all $u \in (V - D)$, $|C'_u| = 1$ for all $u \in (V - D) \subseteq D'$. Hence, vertices in $V - D'$ should be dominated by vertices in $V - D$ in D', which implies $|V - D'| \leq |V - D|$. Since D' is a $\gamma_{\frac{d}{k}}$-set of T, we get $|D'| = |D|$. Since D is a minimum k-DRD set and a k-dominating set of T, D is a minimum k-dominating set of T and $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. □

For a positive integer k, ψ_k is the collection of all trees T such that for any $u \in V(T)$ either all the pendant vertices are at odd distance from u or all the pendant vertices are at even distance from u. If a vertex u is at odd distance from a pendant vertex, then $d(u) = k$.

Theorem 4.2.9. *For any tree $T \neq K_2$ and $k \geq 1$, $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$ if and only if $T \in \psi_k$.*

Proof. Let T be a rooted tree and $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. Then, by Lemma 4.2.8 there exists a subset $D \subseteq V(T)$ satisfying properties $\mathscr{P}$. Suppose there exists a vertex $v \in V(T)$ such that v is at odd distance from a pendant vertex v_1 and v is at even distance from a pendant vertex v_2. Then, the first condition $\mathscr{P}_1$ of Lemma 4.2.8 implies that, $v_1, v_2 \in D$. By the third condition $\mathscr{P}_3$ parent vertex of v_1 say v_3 lies in $V - D$ and parent vertex of v_3 lies in D. Since v is at odd distance from v_1, we have $v \in V - D$. Now, $v_2 \in D$ and v is at even distance from v_2. Then, by the similar argument as above $v \in D$, a contradiction. Note that all the vertices at odd distance from a pendant vertex lie in $V - D$. Then, by Property $\mathscr{P}_2$ degree of all the vertices are at odd distance from a pendant vertex is k. Hence, $T \in \psi_k$. Conversely, assume that $T \in \psi_k$. Let $D \subseteq V(T)$ be the collection of all the pendant vertices in $V(T)$ and all the vertices at even distance from pendant vertices.

Since $T \in \psi_k$, we have $V - D$ is the collection of all the vertices at odd distance from pendant vertices and $d(u) = k$ for all $u \in V - D$. Then, $D \subseteq V$ satisfying the first and second conditions in Lemma 4.2.8. Let $uv \in E(T)$. If any one of u, v is a pendant vertex, then third condition in Lemma 4.2.8 holds. Suppose both u, v are not pendant vertices. If $u \in D$, then u is at even distance from a pendant vertex say v_1 and v is at odd distance from the pendant vertex v_1. Hence, $v \in V - D$. If $u \in V - D$, then u is at odd distance from a pendant vertex say v_2 and v is at even distance from the pendant vertex v_2. Hence, $v \in D$. Therefore, there exists a set $D \subseteq V(T)$ satisfying all the three conditions stated in Lemma 4.2.8 and hence $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. □

Corollary 4.2.10. *For any caterpillar T with diametral path $P = v_1, v_2, \ldots, v_m$, where v_1, v_m are pendant vertices and $k > 1$, $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$ if and only if T satisfies following properties.*

1. *m is odd.*
2. *$d(v_{2l-1}) = 2$ for all $1 \leq l \leq \frac{m-3}{2}$.*
3. *$d(v_{2l}) = k$ for all $1 \leq l \leq \frac{m-1}{2}$.*

Proof. Assume that $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. Since both v_1, v_m are pendant vertices and $k > 1$, Theorem 4.2.9 implies that the vertex v_1 at even distance from v_m and m is odd. Note that, vertex v_{2l} is at odd distance from pendant vertex v_1 for $1 \leq l \leq \frac{m-1}{2}$. Hence, from Theorem 4.2.9, $d(v_{2l}) = k$ for $1 \leq l \leq \frac{m-1}{2}$. Since vertex v_{2l-1} is at even distance from pendant vertex v_1 for $1 \leq l \leq \frac{m-3}{2}$, Theorem 4.2.9 implies that all the pendant vertices are at even distance from v_{2l-1}. Hence, vertex v_{2l-1} has no pendant neighbors and $d(v_{2l-1}) = 2$ for all $1 \leq l \leq \frac{m-3}{2}$. Conversely, suppose caterpillar T satisfies all the three conditions in the hypothesis. Then, $T \in \psi_k$ and $\gamma_{\frac{d}{k}}(T) = \gamma_k(T)$. □

Corollary 4.2.11. *For any caterpillar T of order n, $\gamma_{\frac{d}{2}}(T) = \gamma_2(T)$ if and only if $T = P_n$, n is odd.*

Proof. Conversely, for path P_n of odd order n, $\gamma_{\frac{d}{2}}(T) = \lceil \frac{n}{2} \rceil = \gamma_2(T)$. Let T be a caterpillar with diametral path $P = v_1, v_2, \ldots, v_m$, where v_1, v_m be pendant vertices and $\gamma_{\frac{d}{2}}(T) = \gamma_2(T)$. From Corollary 4.2.10 $d(v_{2l-1}) = 2$ for all $1 \leq l \leq \frac{m-3}{2}$ and $d(v_{2l}) = 2$ for all $1 \leq l \leq \frac{m-1}{2}$. Hence, $T = P_n$ and n is odd. □

4.3 RELATION BETWEEN AN EFFICIENT DOMINATING SET AND k-DRD SET

A subset $D \subseteq V$ is a dominating set of G, if $N(D) = V$, or for each $u \in V$, $N(u) \cap D \neq \emptyset$. The efficient domination is an effort to dominate every vertex exactly once. The priority moves from the order of the set to the amount of domination being done. If D is an efficient dominating set, then for every pair of vertices $u, v \in D$, $d(u,v) \geq 3$. This simply means that D is a packing. If G has an efficient dominating set, then the cardinality of any efficient dominating set is the domination number $\gamma(G)$. All efficient dominating sets of G have the same cardinality. Therefore, $\gamma(G) \leq \gamma_{\frac{d}{k}}(G)$.

Definition 4.3.1. *Bange et al. (1988) A dominating set D of a graph G is called an efficient dominating set, if for every vertex $v \in V$, $|N(v) \cap D| = 1$.*

Proposition 4.3.2. *For $k \geq 1$, an efficient dominating set D of a graph G is a k-DRD set if and only if $G = H \circ K_1$, where G is a corona of any connected graph H and K_1.*

Proof. Assume that an efficient dominating set D of a graph G is a k-DRD set. Then, there exists a partition $\{C_u : u \in D\}$ of $V - D$ such that $|C_u| \leq \left\lceil \frac{d(u)}{k} \right\rceil$, for every $u \in D$. Since D is independent, either C_u is a proper subset of $N(u) \cap (V - D)$ or $|C_u| = 1$ for $u \in D$. Also $|N(v) \cap D| = 1$ for every $v \in V - D$. Therefore, $|N(u) \cap (V - D)| = 1$ for every $u \in D$ and $G = H \circ K_1$. Conversely, if G is a corona of any connected graph H and K_1, then clearly collection of all the vertices of degree one in G forms an efficient dominating set, which is again a k-DRD set. $\square$

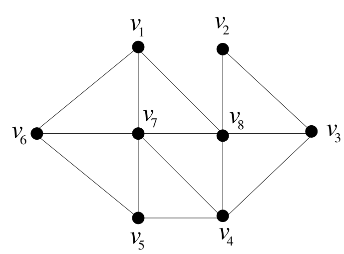


Figure 4.1 Example in reference to Remark 4.3.3

Remark 4.3.3. *Here, we discussed when an efficient dominating set D is a k-DRD set for $k \geq 1$. There are many graphs having same efficient domination number and $\gamma_{\frac{d}{k}}$. For example, consider the graph G in Figure 4.1, where $\gamma(G) = \gamma_{\frac{d}{2}}(G)$, $D_1 = \{v_3, v_6\}$ is an efficient dominating set and $D_2 = \{v_7, v_8\}$ is a 2-DRD set. But none of the efficient dominating sets of graph G in Figure 4.1 are 2-DRD set. At present the characterization of the graph G for which $\gamma(G) = \gamma_{\frac{d}{k}}(G)$ is not known.*

Considering all three types of domination discussed above, that is efficient domination, k-domination and k-part degree restricted domination. We can observe the following:

- *Every k-dominating, efficient dominating and k-DRD sets are dominating set.*
- *A minimum dominating set D of a connected graph G is a k-DRD set if and only if* $\sum_{u \in N(A) \cap D} \left\lceil \frac{d(u)}{k} \right\rceil \geq |A|$ *for every subset A of* $V - D$.
- *Every k-dominating set is a k-DRD set.*
- *Every k-DRD set need not be a k-dominating set.*
- *For* $k \neq 1$*, an efficient dominating set D of a graph G is a k-DRD set if and only if* $G \cong H \circ K_1$*, where G is a corona of any connected graph H and* K_1.
- *A k-dominating set is not an efficient dominating set for* $k \neq 1$.

In this chapter, relationship between k-DRD set and dominating set, k-DRD set and k-dominating set and also relation between k-DRD set and efficient dominating set of a graph are discussed. In the next chapter, the difficulty in computing the k-part degree restricted domination number of an arbitrary graph. That is, the complexity of k-part degree restricted domination problem is studied and algorithm to compute the k-part degree restricted domination number of some graph classes are provided.

CHAPTER 5

k-PART DEGREE RESTRICTED DOMINATION COMPLEXITY AND ALGORITHMS

In Chapter 3 we discussed bounds on $\gamma_{\frac{d}{k}}(G)$ of a graph. We are keen to know the value of $\gamma_{\frac{d}{k}}(G)$ of an arbitrary graph G, so we are searching for an algorithm to measure $\gamma_{\frac{d}{k}}(G)$ that is faster than the brute-force algorithm. We have no algorithm whose complexity is better than exponential time to find $\gamma(G)$ of any graph G. It is universally accepted that the problem of determining the domination number of an arbitrary graph is difficult. This problem has been proved NP-complete and requires exponential time. This study also continued to find algorithms to calculate $\gamma(G)$ of some classes of graphs.

In this chapter, we discuss the complexity of k-part degree restricted domination problem. We prove that the k-part degree restricted domination problem is NP-complete for bipartite graphs, chordal graphs and for split graphs. Also, we propose an exponential time algorithm to find 2-part degree restricted domination number of an interval graph and a polynomial time algorithm to find k-part degree restricted domination number of a tree.

5.1 NP-COMPLETENESS OF *k*-PART DEGREE RESTRICTED DOMINATION PROBLEM

In this section, we prove the NP-completeness of k-part degree restricted domination problem by a polynomial time reduction from the domination problem, which is proved to be NP-complete by Garey and Johnson (1979). The decision version of domination problem is as follows:

Dominating set problem (DOM)

Instance: A graph $G = (V, E)$ and a positive integer t.

Question: Is $\gamma(G) \leq t$?

The decision version of k-part degree restricted domination problem is as follows:

***k*-part Degree Restricted Domination Problem (*k*-PDRDOM)**

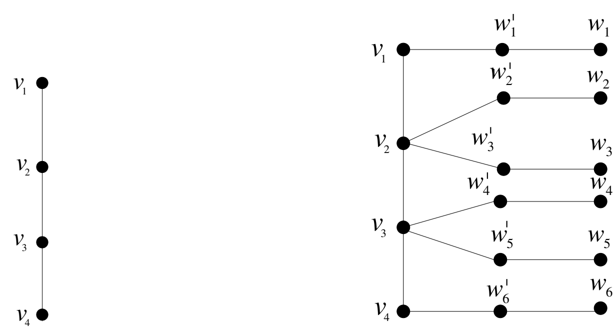
Instance: A graph $G = (V, E)$ and a positive integer t.

Question: Is $\gamma_{\frac{d}{k}}(G) \leq t$?

Theorem 5.1.1. *k-PDRDOM is NP-complete.*

Proof. Clearly, k-PDRDOM is a member of NP, since we can check in polynomial time whether a given set of vertices is a k-DRD set of G or not. Let $G = (V, E, t)$ be the instance of DOM, where $V = \{v_1, v_2, \ldots, v_r\}$ be the vertex set, $E = \{e_1, e_2, \ldots, e_m\}$ be the edge set and t be any positive integer. An instance $G^* = (V^*, E^*, t^*)$ of k-PDRDOM is constructed as follows:

For each vertex u in G, join $d_G(u)(k-1)$ number of new vertices by an edge and subdivide each newly added edge. Now G^* has $|V(G)| + 2\sum_{u \in V} d_G(u)(k-1)$ number of vertices and $|E(G)| + 2\sum_{u \in V} d_G(u)(k-1)$ number of edges. Let $W = \{w_1, w_2, \ldots w_r\}$ be the set of all newly added pendant vertices to G and w_i' be the support vertex of w_i for all i, $1 \leq i \leq r$. Now $V^* = V \cup W \cup W'$, where $W' = \{w_1', w_2', \ldots w_r'\}$.



(a) The graph G (b) The graph G^*

Figure 5.1 The construction of the graph G^* from the graph G, for $k = 2$

Claim: $G = (V.E)$ has a dominating set of cardinality at most t if and only if $G^* = (V^*.E^*)$ has a k-DRD set of cardinality at most $t + 2|E|(k-1)$.

If $k = 1$, then 1-DRD set is a dominating set. Hence, we assume $k > 1$. Let D be a dominating set of G of cardinality at most t. Since $d_{G^*}(u) = d_G(u) + d_G(u)(k-1)$ for every $u \in V^* - (W \cup W')$, vertices in D can dominate all the vertices in $V^* - (W \cup W')$ (as per the definition of k-DRD set) and each newly added pendant vertex can dominate its support vertex. Hence, $D \cup W$ is a k-DRD set of G^*, where $|W| = \sum_{u \in V} d_G(u)(k-1) = 2|E|(k-1)$ and $|D \cup W| \leq t + 2|E|(k-1)$.

Conversely, let D^* be a k-DRD set of G^* of cardinality at most $t + 2|E|(k-1)$. Let w_i be a pendant vertex and w'_i be the support vertex of w_i for any i, $1 \leq i \leq r$ in graph G^*. Then, D^* contains at least one the vertex in $\{w_i.w'_i\}$ for any i, $1 \leq i \leq r$. Let u_i be the neighbor of w'_i other than w_i. Since $d_{G^*}(w'_i) = 2$, $|C_{w'_i}|$ can not exceed 1 and hence, w'_i can not dominate both of its neighbors. If both $w_i.w'_i$ belongs to D^* and $C_{w'_i} = \{u_i\}$, then $D' = D^* \cup \{u_i\} - \{w'_i\}$ is a k-DRD set of G^* of cardinality at most $t + 2|E|(k-1)$. If $C_{w'_i} = \emptyset$, then $D' = D^* - \{w'_i\}$ is a k-DRD set of G^* of cardinality at most $t + 2|E|(k-1)$. If any one of $w_i.w'_i$ is in D^*, then $D' = D^*$ is a k-DRD set of G^* of cardinality at most $t + 2|E|(k-1)$. Now $D' \cap V$ is a dominating set of G and D' contains either the pendant vertex w_i or its support vertex w'_i, but not both for any i, $1 \leq i \leq r$. Hence, $|D' \cap V| \leq t$. □

The following lemma is easy to verify. For definitions of chordal bipartite graph, circle graph, undirected path graph and planar graph, we refer Brandstädt et al. (1999).

Lemma 5.1.2. *Let G^* be the graph constructed from a graph G as shown in Theorem 5.1.1. Then,*

1. *If G is bipartite, then G^* is also bipartite.*
2. *If G is chordal, then G^* is also chordal.*
3. *If G is chordal bipartite, then G^* is also chordal bipartite.*
4. *If G is circle, then G^* is also circle.*
5. *If G is undirected path graph, then G^* is also undirected path graph.*
6. *If G is planar, then G^* is also planar.*

Since the domination problem is NP-complete for bipartite graphs Bertossi (1984), undirected path graphs Booth and Johnson (1982), chordal bipartite graphs Müller and Brandstädt (1987), circle graphs Keil (1993), and planar graphs Garey and Johnson (1979), the k-part degree restricted domination problem is NP-complete for all the above mentioned graphs. Therefore, we have the following theorem.

Theorem 5.1.3. *The k-part degree restricted domination problem is NP-complete for bipartite graphs, chordal graphs, undirected path graphs, chordal bipartite graphs, circle graphs, and planar graphs.*

We have proved that k-part degree restricted domination problem is NP-complete for chordal graphs. Now we show that the k-part degree restricted domination problem is NP-complete for split graphs, a subclass of chordal graphs. Our reduction is from a well-known NP-complete problem, vertex cover problem for general graphs. The vertex cover problem is to find a minimum vertex cover of graph G.

The decision version of vertex cover problem is as follows:

Vertex cover problem (VCP)

Instance: A graph $G = (V,E)$ and a positive integer c.

Question: Does G has a vertex cover of cardinality $\leq c$?

The decision version of k-part degree restricted domination problem is as follows:

***k*-part Degree Restricted Domination Problem (*k*-PDRDOM)**

Instance: A graph $G = (V,E)$ and a positive integer c.

Question: Does G has a k-part degree restricted dominating set of cardinality $\leq c$?

Theorem 5.1.4. *k-PDRDOM is NP-complete for split graphs.*

Proof. Clearly, the k-PDRDOM is a member of NP, since we can check whether a given set of vertices is k-DRD set of G or not in polynomial time. Let $G = (V,E,c)$ be the instance of VCP. We construct the graph $G^* = (V^*,E^*,c^*)$ with vertex set V^* and edge set E^*, where

$$V^* = V \cup V_E \cup U \cup W, \;\; E^* = E_1 \cup E_2 \cup E_3$$

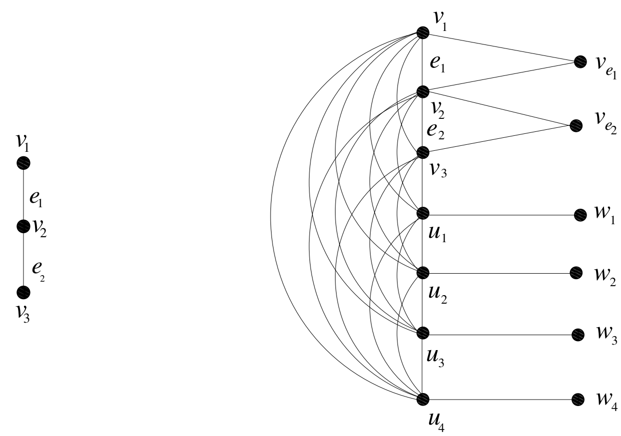
such that

$$\begin{aligned}
V_E &= \{v_e : e \in E\} \\
U &= \{u_1, u_2, \ldots, u_{(n-1)k}\}, (|V(G)| = n) \\
W &= \{w_1, w_2, \ldots, w_{(n-1)k}\} \\
E_1 &= \{vv_e : v \in V, v_e \in V_E, v \text{ is an end point of edge } e\} \\
E_2 &= \{uv : u, v \in V \cup U \textit{ and } u \neq v\} \\
E_3 &= \{w_i u_i : w_i \in W, u_i \in U\}
\end{aligned}$$

Clearly G^* is a split graph and can be constructed in polynomial time.

Claim: $G = (V,E)$ has a vertex cover of cardinality at most c if and only if G^* has a k-DRD set of cardinality at most $c\,]\;(n-1)k$.

Assume that G has a vertex cover C of cardinality at most c. Since $d_{G^*}(v) = d_G(v)\,]\;(n-1)(k]\;1)$ ($That\ is$, $\left\lceil \frac{d_{G^*}(v)}{k} \right\rceil \geq n-1$) for all $v \in V$, vertices in C can dominate all the vertices in V_E as per the definition of *k-DRD* set. Also note that $d_{G^*}(u) = k(n-1)\,]\;\; n$



(a) The graph G (b) The graph G^*

Figure 5.2 The construction of the graph G^* from the graph G, for $k = 2$

for all $u \in U$ and $\left\lceil \frac{d_{G^*}(u)}{k} \right\rceil \geq n-1$, hence U can dominate all the vertices in $(V-C) \cup W$. Therefore, $C \cup U$ is a *k-DRD* set of G^* of cardinality at most $c + (n-1)k$. Assume that D is a *k-DRD* set of G^* of cardinality at most $c + (n-1)k$. Since $d(w_i) = 1$, either the vertex w_i or u_i is in D for every i, $1 \leq i \leq (n-1)k$. If $w_i \in D$ and $u_i \notin D$, then $D \cup \{u_i\} - \{w_i\}$ is also a *k-DRD* set of G^* of cardinality $c + (n-1)k$. Thus, we may assume that D contains all the vertices in U and U can dominate all the vertices in $V - (D \cup V_E)$. Since $d_{G^*}(v) = d_G(v) + (n-1)(k+1)$ for all $v \in V$, vertices in $V \cap D$ can dominate all its neighbors in V_E. Suppose D contains some $v_e \in V_E$. Let v be an end point of edge e. Then, $D^* = (D - v_e) \cup \{v\}$ is a *k-DRD* set of G^* and $D^* \cap V_E = \emptyset$. Since D is a *k-DRD* set of G^*, either v_e or any one of the neighbors of v_e should be in D. Hence, $D^* \cap V$ is a vertex cover of G of cardinality at most c. □

5.2 MINIMAL *k*-DRD SET

In this section, we propose an algorithm which takes adjacency matrix of a simple connected graph $G = (V, E)$ as an input and results in to a minimal k-DRD set.

Theorem 5.2.1. *A k-DRD set D of a graph G is minimal if and only if for each $v \in D$, there exists at least one $u \in C_v \cup \{v\}$ for which there is no path $P_u = u, v_1, v_2, \ldots v_{2l+1}$ satisfying following conditions:*

1. *For each i, $0 \leq i \leq l$, $v_{2i+1} \in D$.*

2. *For each i, $1 \leq i \leq l$, $v_{2i} \in C_{v_{2i-1}}$.*

3. $|C_{v_{2l]\ 1}}| < \left\lceil \frac{d(v_{2l]\ 1})}{k} \right\rceil$.

4. *If all the paths* $P_{u_1}, P_{u_2}.., P_{u_m}$ *end at the same vertex say* w*, then* $\left\lceil \frac{d(w)}{k} \right\rceil - |C_w| \geq m$.

5. *For every* $u, w \in C_v \cup \{v\}$ *we can find paths* P'_u*,* P'_w *from* P_u *and* P_w*, respectively such that* $V(P'_u) \cap V(P'_w) \cap (V - D) = \emptyset$.

Proof. Assume that $D' = D - \{v\}$ is a k-DRD set of G. Then, for each $w \in D'$ there exists a set $C'_w \subseteq N(w) \cap (V - D')$ such that $|C'_w| \leq \left\lceil \frac{d(w)}{k} \right\rceil$ and $\bigcup_{w \in D'} C'_w = V - D'$. Now for each $u \in C_v \cup \{v\}$ we construct a path P_u, which satisfies the above conditions. Consider a vertex u from $C_v \cup \{v\}$. Since D' is k-DRD set, $u \in C'_{v_1}$ for some $v_1 \in D' \subseteq D$. If $|C_{v_1}| < \left\lceil \frac{d(v_1)}{k} \right\rceil$, then uv_1 is a path satisfying the above first three conditions. If $|C_{v_1}| = \left\lceil \frac{d(v_1)}{k} \right\rceil$, then $C_{v_1} - C'_{v_1} \neq \emptyset$ (Since $u \in C'_{v_1} - C_{v_1}$ and $|C'_{v_1}| \leq \left\lceil \frac{d(v_1)}{k} \right\rceil = |C_{v_1}|$, $C_{v_1} - C'_{v_1} \neq \emptyset$). Consider a vertex v_2 from $C_{v_1} - C'_{v_1}$. Since D' is a k-DRD set $v_2 \in C'_{v_3}$ for some $v_3 \in D' \subseteq D$. If $|C_{v_3}| < \left\lceil \frac{d(v_3)}{k} \right\rceil$, then $P_u = u, v_1, v_2, v_3$. If $|C_{v_3}| = \left\lceil \frac{d(v_3)}{k} \right\rceil$, then $C_{v_3} - C'_{v_3} \neq \emptyset$, choose a vertex from $C_{v_3} - C'_{v_3} \neq \emptyset$ and continuing the process (Here, we have to choose one vertex from $C_{v_3} - C'_{v_3}$ say v_4, we assume that $v_4 \in C'_{v_5}$ for some $v_5 \in D'$ and we continue the process. If $v_4 \in C'_{v_1}$, then $C'_{v_1} - C_{v_1}$ has at least 2 vertices. Since $|C_{v_1} - C'_{v_1}| \geq 2$, we can continue the procedure with vertex other than v_2). Since D is a finite k-DRD set, the above process has to terminate. So after some finite steps we find a vertex $v_{s-1} \in C_{v_{s-2}} - C'_{v_{s-2}}$, $v_{s-1} \in C'_{v_s}$ such that $|C_{v_s}| < \left\lceil \frac{d(v_s)}{k} \right\rceil$. Now for chosen vertex u from $C_v \cup \{v\}$, we have a path $P_u = u, v_1, v_2 ..., v_s$ such that $v_{2i]\ 1} \in D$ for each i, $0 \leq i \leq \frac{s-1}{2}$, $v_{2i} \in C_{v_{2i-1}}$ for each i, $0 < i \leq \frac{s-1}{2}$ ($s \geq 3$), s is odd and $|C_{v_s}| < \left\lceil \frac{d(v_s)}{k} \right\rceil$.

Let $P_u = u, u_1, u_2, u_3, \ldots, u_l$, $P_w = w, w_1, w_2, w_3, \ldots, w_q$ be two paths for some $u, w \in C_v \cup \{v\}$ such that $u_{2j} = w_{2i}$ and $w_{2i} \in V - D$. Then, by the construction $u_{2j} \in C_{u_{2j-1}}$ and $w_{2i} \in C_{w_{2i-1}}$. Hence, $w_{2i-1} = u_{2i-1}$. Now $w_{2i-2}, u_{2i-2} \in C'_{w_{2i-1}}$ and $w_{2i-2}, u_{2i-2} \in C_{w_{2i-3}}$. Then, $|C'_{w_{2i-1}} - C_{w_{2i-1}}| \geq 2$. If w_{2i-1} is not an end vertex of path P_w, then $|C_{w_{2i-1}} - C'_{w_{2i-1}}| \geq 2$. Hence, we continue the process as explained above with vertex other than w_{2i} and we can find one new path P'_w such that $V(P'_u) \cap V(P_w) \cap (V - D) = \emptyset$.

Assume that $w_q = u_l$, $\left\lceil \frac{d(w_q)}{k} \right\rceil - |C_{w_q}| = 1$ and there is no other such paths for u, w. Let $B_1 = N(u, w) \cap D$, $B'_1 = \bigcup_{u \in B_1} C_u$. For $i > 1$, $B_i = N(B'_{i-1}) \cap D$ and $B'_i = \bigcup_{u \in B_i} C_u$. Note that, for u, w there is no path other than P_u and P_w satisfying the above three conditions . Hence, $|C_{w'}| = \left\lceil \frac{d(w')}{k} \right\rceil$ for every $w' \in B_i - \{w_q\}$. Since V is finite, there exist $m, n \in N$ such that $B_j = B_{j]\ 1} = B_{j]\ 2}$ for all $j \geq m$ and $B'_l = B'_{l]\ 1} = B'_{l]\ 2}$ for all $l \geq n$. Then, $|B'_n| = \sum_{u \in N(B'_n) \cap D} |C_u|$. Since $u, w \in B'_n$, $\left\lceil \frac{d(w_q)}{k} \right\rceil - |C_{w_q}| = 1$ and $|C_{w'}| = \left\lceil \frac{d(w')}{k} \right\rceil$ for every $w' \in B_n - \{w_q\}$, $|B'_n| > \sum_{u \in N(B'_n) \cap D} \left\lceil \frac{d(u)}{k} \right\rceil$, we arrive at a contradiction.

Conversely, assume that for each vertex $u \in C_v \cup \{v\}$, there exists a path $P_u = u, v_1, v_2, \ldots, v_l$ satisfying the above conditions. Define $C'_{v_1} = (C_{v_1} \cup \{u\}) - \{v_2\}$, $C'_{v_l} = C_{v_l} \cup \{v_{l-1}\}$ and $C'_{v_{2i-1}} = C_{v_{2i-1}} \cup \{v_{2i}\} - \{v_{2i-2}\}$ for all i, $1 \leq i \leq \frac{l-3}{2}(l \geq 3)$. Since $|C_{v_l}| < \left\lceil \frac{d(v_l)}{k} \right\rceil$, $|C'_{v_l}| \leq \left\lceil \frac{d(v_l)}{k} \right\rceil$. Also, $|C'_{v_{2i-1}}| = |C_{v_{2i-1}}| \leq \left\lceil \frac{d(v_{2i-1})}{k} \right\rceil$ for all i, $0 \leq i \leq \frac{l-3}{2}$ and u is dominated by v_1. Since such path exists for all the vertices in $C_v \cup \{v\}$ and by the fourth and fifth condition in the hypothesis, $D' = D - \{v\}$ is a k-DRD set of G. Hence, D is not a minimal k-DRD set of G. □

Algorithm to Find a Minimal k-DRD set of a Graph

In this section, we present an algorithm which takes adjacency matrix of a simple connected graph $G = (V, E)$ as an input and returns a minimal k-DRD set. Here, first we find a k-DRD set by taking degree as a major parameter, then we look for its subset which is again a k-DRD set. The basic idea of the algorithm is as follows:

First we find the degree of each vertex i in G and neighborhood N_i. Next, we choose a vertex i of maximum $|N_i|$ in V and we add a vertex of minimum degree from N_i to C_i. We repeat this step by adding a vertex of next minimum degree to C_i until the order of C_i is $\left\lceil \frac{d(i)}{k} \right\rceil$ or N_i becomes empty and update V by removing the elements of C_i along with i. Repeat the procedure until V becomes empty, which results into a k-DRD set D and C_i corresponding to each i in D. We define, $A = \{j \in D : |C_j| < \left\lceil \frac{d(j)}{k} \right\rceil\}$. If $|C_i| = \left\lceil \frac{d(i)}{k} \right\rceil$ for every $i \in D$, then D is a minimal k-DRD set. If A is non empty, then we proceed with Algorithm 5.2. That is, Test-Minimal. In Test-Minimal we check whether D has any subset which is again a k-DRD set. We can observe that, either a pendant vertex or the vertex adjacent to a pendant vertex should be in k-DRD set, therefore for any $i \in D$ while adding vertices to C_i, we give first preference to a vertex of minimum degree in N_i. The procedure of the algorithm Test-Minimal is as follows:

By the Theorem 5.2.1, if there exists a path P_u for each $u \in C_v \cup \{v\}$, where $v \in D$ with some conditions, then $D - \{v\}$ is a k-DRD set. In this algorithm we find all possible paths satisfying the conditions mentioned in Theorem 5.2.1 using Depth First Search (DFS) technique. First we choose a vertex i from D and j from C_i. We check for path satisfying the conditions in Theorem 5.2.1 from j to some vertex in A. Here, we use DFS technique with stack function to find such path. If there exists such path, then we shift j to some set C_l, $l \in D - \{i\}$ and we redefine C_l for all $l \in V(P_j) \cap D$. If such path exists for all the vertices in C_j and at least one vertex in $N_i \cap D$, then $D - \{i\}$ is a k-DRD set. We update D by removing i from D and we continue the same procedure for all the vertices in $D - \{i\}$ with updated C_l, $l \in D$.

Algorithm 5.1: Finding k-DRD set of a graph

Input: Adjacency matrix $[a_{ij}]_{n\times n}$ of a graph $G = (V,E)$, positive integer k
Output: k-part degree restricted dominating set D
begin
    $D = \phi, \Delta = 0$
    for *each* $i \in V$ **do**
        $d(i) = 0$
        for $j \in V$ **do**
            $d(i) = d(i) + a_{ij}$
        end
        if $d(i) > \Delta$ **then**
            $\Delta = d(i)$
        end
    end
    while $V \neq \phi$ **do**
        $d' = 0$
        for *each* $i \in V$ **do**
            $N_i = \phi$
            for *each* $j \in V$ **do**
                if $a_{ij} = 1$ **then**
                    $N_i = N_i \cup \{j\}$
                end
            end
            if $d' \leq |N_i|$ **then**
                $d' = |N_i|$
                $a = i$
            end
        end
        $D = D \cup \{a\}$
        $C_a = \phi$
        while $|C_a| < \left\lceil \frac{d(a)}{k} \right\rceil$ *and* $N_a \neq \phi$ **do**
            $d'_{\Delta} = \Delta$
            for *each* $j \in N_a$ **do**
                if $d(j) \leq d'_{\Delta}$ **then**
                    $d'_{\Delta} = d(j)$
                    $d_{\Delta} = j$
                end
            end
            $C_a = C_a \cup \{d_{\Delta}\}$
            $N_a = N_a - \{d_{\Delta}\}$
        end
        $A_a = C_a \cup \{a\}, V = V - A_a$
    end
    return D Call Test minimal
end

Table 5.1 Algorithm to find k-DRD set of a graph

Algorithm 5.2: Test Minimal

Input: k-DRD set D, $A = \left\{ j \in D : |C_j| < \left\lceil \frac{d(j)}{k} \right\rceil \right\}$
Output: Minimal k-DRD set
begin
    if $A = \phi$ **then**
        D is a minimal k-DRD set
    end
    else
        Procedure:
        for *all* $j \in D$ **do**
            $C'_j = C_j$
            for *all* $i \in C = C'_j$ **do**
                $P = Path(i)$
                if $P = \phi$ **then**
                    go to **Procedure**
                end
                else
                    $P = \{v_0, v_1, v_2, \ldots, v_k\} \neq \phi$, where vertex $i = v_0$
                    for $l = 0, 1, 2, \ldots, \frac{k-3}{2}$ **do**
                        $C'_{v_{2l]\ 1}} = (C'_{v_{2l]\ 1}} \cup \{v_{2l}\}) - \{v_{2l]\ 2}\}$
                    end
                    $C'_{v_k} = C'_{v_k} \cup \{v_{k-1}\}, C = C - \{i\}$
                    if $|C'_{v_k}| = \left\lceil \frac{d(v_k)}{k} \right\rceil$ **then**
                        $A' = A - \{v_k\}$
                    end
                end
            end
            for *all* $i \in D \cap N_j$ **do**
                $P = Path(i)$
                if $P = \{i, v_1, v_2, \ldots, v_k\} \neq \phi$ **then**
                    for $l = 1, 2, \ldots, \frac{k-2}{2}$ **do**
                        $C'_{v_{2l}} = (C'_{v_{2l}} \cup \{v_{2l-1}\}) - \{v_{2l]\ 1}\}$
                    end
                    $C'_i = (C'_i \cup \{j\}) - \{v_1\}$
                    $C'_{v_k} = C'_{v_k} \cup \{v_{k-1}\}$
                    if $|C'_{v_k}| = \left\lceil \frac{d(v_k)}{k} \right\rceil$ **then**
                        $A' = A' - \{v_k\}$
                    end
                    $D = D - \{j\}, A = A'$
                    for *all* $i \in D$ **do**
                        $C_j = C'_j$
                    end
                    go to **Procedure**
                end
            end
        end
    end
    return D
end

Table 5.2 Algorithm to check if the given set D has a k-DRD set as a its proper subset

Algorithm 5.3: Path(i)

```
begin
    for all g ∈ V do
        Visited[g]=0
    end
    Top = 0, Visited[i]=1, Push(i)
    while P ≠ φ do
        j=P(Top)
        if j ∈ D then
            N'_j= {v ∈ C_j : Visited[v] = 0}
            else
                N'_j= {v ∈ (D − {j}) ∩ N_j : Visited[v] = 0}
            end
        end
        if N_j ≠ φ then
            choose a vertex l from N'_j, Push(l),Visited[l] = 1
            if l ∈ A then
                return P
            end
        end
        else
            pop()
        end
    end
end
```

Table 5.3 Algorithm to find all possible path satisfying the conditions in Theorem 5.2.1

Algorithm 5.4: Push(i)
begin Top=Top+1 P(Top)=i **end**

Table 5.4 Push operation

Algorithm 5.5: Pop
begin P(Top)=Null Top=Top-1 **end**

Table 5.5 Pop operation

Theorem 5.2.2. *Resultant set D of Algorithm 5.1 is a k-DRD set.*

Proof. Let $D = \{1, 2, ...p\}$. By the construction $|C_i| \leq \left\lceil \frac{d(i)}{k} \right\rceil$ and $C_i \subseteq N_i \cap (V - D)$, for all i, $1 \leq i \leq p$. Also note that

$$V = \bigcup_{i=1}^{p} A_i = \bigcup_{i=1}^{p} C_i \cup D \Rightarrow V - D = \bigcup_{i=1}^{p} C_i.$$

Hence, D is a k-DRD set. □

Theorem 5.2.3. *Resultant set D of Algorithm 5.2 is a minimal k-DRD set.*

Proof. Let A and D be the outputs obtained by Algorithm 5.1 and Algorithm 5.2, respectively and $v \in D$. Initially, we choose a vertex u from $C_v \cup \{v\}$ and using DFS technique we find a path P_u from u to u_l, where $u_l \in A$. Since $u_l \in A$, $|C_{u_l}| < \left\lceil \frac{d(u_l)}{k} \right\rceil$. Next, for every $w \in V(P_u) \cap D$, we find C'_w. If such path exists for every vertex in $C_v \cup \{v\}$, then we relabel C'_w as C_w for every $w \in V(P_u) \cap D$. If $|C'_{u_l}| = \left\lceil \frac{d(u_l)}{k} \right\rceil$, then we update A by removing u_l from A. Hence, $\left\lceil \frac{d(u_l)}{k} \right\rceil - |C_{u_l}| \geq m$ for the paths $P_{v_1}, P_{v_2} .., P_{v_m}$, which end at the same vertex u_l. In Algorithm 5.2, for all the vertices in $\bigcup_{v \in D} C_v$, we check the existence of path having the property as defined in Theorem 5.2.1. Hence, from Theorem 5.2.1 output set D of Algorithm 5.2 is a minimal k-DRD set. □

Theorem 5.2.4. *Algorithm 5.2 used to compute minimal k-DRD set of a given graph runs in $O(n^4)$ time.*

Proof. For a given graph G, computing degree of all the vertices using adjacency matrix takes $O(n^2)$ time. Finding neighborhood of all the vertices takes $O(n^2)$ running time, and construction of C_v, whose cardinality is at most degree of vertex v, takes $O(n^2)$ time. Finding neighborhood of all the vertices in updated V in the Algorithm 5.1 takes $O(n^2)$ time. Finding the vertex a having maximum neighbor in V takes $O(n)$ time. The construction of C_a takes $O(n^2)$ time. In worst case first while loop in Algorithm 5.1 repeats n times and each time V gets updated. Hence, running time of Algorithm 5.1 is $O(n^3)$. We use the resultant set D of Algorithm 5.1 in Algorithm 5.2 Test Minimal. Now, to find the path which satisfies the conditions in Theorem 5.2.1 from all the vertices in $C_v \cup (N_v \cap D)$, $v \in D$ to some vertex in A by DFS technique in Algorithm 5.2 takes $O(n^3)$ time. We repeat this procedure to all the vertices in D so will take $O(n^4)$ time. Hence, complexity of the Algorithm 5.2 is $O(n^4)$. □

5.3 ALGORITHM TO FIND A MINIMUM k-DRD SET OF A TREE

In this section, we discuss an algorithm to find a minimum k-DRD set of a tree. Here, we use recursive labeling of a tree. The definition of a recursive tree was presented by Meir and W.Moon (1974). A tree T having M vertices labeled $1,2,\dots,M$ is recursive if either $M = 1$ or $M > 1$ and T was iteratively constructed by joining the vertex with label i to one of the $i-1$ previous vertices, for every i, $2 \le i \le M$. From the definition recursive tree one can observe that recursive labeling of a tree T with M vertices is any assignment of the labels $1,2,\dots,M$ to the vertices of T which has the property that every vertex, except the vertex labeled 1 is adjacent to exactly one vertex with a smaller label. In the following algorithm, We choose vertex labeled "$1''$ as the root vertex and label the tree recursively with one extra condition, that is the vertices labeled in the m^{th} level should be greater than all the vertices labeled in $(m-1)^{th}$ level.

Analysis of the Algorithm

We consider a tree $T =(V,E)$ having n vertices and labeled $1,2,\dots,n$ recursively as defined above. We find degree and neighborhood of each vertex in V. We label all the vertices of T as "Bound". Initially, we choose the vertex n from V, whose label is "Bound" and we relabel the parent of n as "Required". Next, we choose the vertex labeled as $n-1$. If label of $n-1$ is "Bound", then we relabel the parent of $n-1$ as "Required". If label of $n-1$ is "Required", then we add the vertex $n-1$ to D, where D is the minimum k-DRD set of T which is initially an empty set. We construct C_{n-1} as follows:

Algorithm 5.6: $\gamma_{\frac{d}{k}}$-set of a tree

```
Input: Adjacency matrix [a_ij]_{n×n} of tree T = (V,E)
Output: Minimum k-DRD set.
begin
    V = {1,2,3,...,n}, D = φ
    for i ∈ V do
        d(i) = 0
        for each j ∈ V do
            d(i) = d(i) + a_ij
        end
    end
    for i=1 to n do
        Label[i] = Bound
    end
    for i = n; i > 1; i−− do
        if Label[i] = Bound then
            Label[Parent[i]] = Required
        end
        if Label[i] = Required then
            D = D ∪ {i}, N_i = φ
            for each j ∈ V do
                if a_ij = 1 then
                    N_i = N_i ∪ {j}
                end
            end
            C_i = φ, N_i = N_i − {Parent[i]}
            for each j ∈ N_i do
                while |C_i| < ⌈d(i)/k⌉ and N_i ≠ φ do
                    if Label[j] = Bound then
                        C_i = C_i ∪ {j}
                    end
                    N_i = N_i − {j}
                end
                V = V − (C_i ∪ {i})
                if |C_i| < ⌈d(i)/k⌉ then
                    Label[Parent[i]] = Free
                end
            end
        end
    end
    D = D ∪ {l ∈ V : Label[l] = Bound}
    return D
end
```

Table 5.6 Algorithm to find minimum k-DRD set of a tree

Let N'_{n-1} be the collection of all the child neighbors of $n-1$ labeled as “Bound”. If $|N'_{n-1}| \geq \left\lceil \frac{d(n-1)}{k} \right\rceil$, then C_{n-1} is a subset of N'_{n-1} of cardinality $\left\lceil \frac{d(n-1)}{k} \right\rceil$. If $|N'_{n-1}| < \left\lceil \frac{d(n-1)}{k} \right\rceil$, then $C_{n-1} = N'_{n-1}$ and we relabel parent vertex of $n-1$ as “Free ". We repeat the above procedure for all the vertices $n-2, n-3, \ldots 2$. That is, in the decreasing order of their labeling. In the final step, we add all the vertices which are not dominated by any of the vertices in D and labeled as “Bound" to D, which results into a minimum k-DRD set of given tree T.

Let D be a minimum k-DRD set of a rooted tree T obtained from the Algorithm 5.6, $u \in V(T)$ and $T_1, T_2, \ldots, T_{d(u)-1}$ be the components of $T-u$ containing child neighbors of vertex u. If we apply the Algorithm 5.6 to each component T_i for $1 \leq i \leq d(u)-1$, then either $\gamma_{\frac{d}{2}}(T_i) = |D \cap V(T_i)|]\ \ 1$ or $\gamma_{\frac{d}{2}}(T_i) = |D \cap V(T_i)|$ for all i, $1 \leq i \leq d(u)-1$. Let $u \in C_v \subseteq V - D$, v be a child neighbor of u and T_1 be the component of $T-u$ containing vertex v. Since v is dominating its parent vertex, by the procedure of the Algorithm 5.6, vertex v can not dominate any extra child vertex. Hence, If we apply Algorithm 5.6 to the tree T_1, then $V(T_1) \cap D$ is a minimum 2-DRD set of tree T_1, where T_1 is the component of tree $T-u$ containing the vertex v.

Lemma 5.3.1. *Let T be a tree and $uv \in E(T)$. If $d(u) = 2$ and $d(v) = 1$, then $\gamma_{\frac{d}{k}}(T) = \gamma_{\frac{d}{k}}(T - \{u,v\})]\ \ 1$.*

Theorem 5.3.2. *The resultant set D of the Algorithm 5.6 is a k-DRD set.*

Proof. For every $v \in V$, if the label of v is “Required”, then $v \in D$ and by the construction of C_v, $|C_v| \leq \left\lceil \frac{d(v)}{k} \right\rceil$. If the label of v is “Bound”, then either $v \in C_u$ for some $u \in D$ or $v \in D$ with $C_v = \emptyset$. Suppose the label of v is “Free”. Then, $v \in C_u$ for some $u \in D$ with $|C_u| \leq \left\lceil \frac{d(u)}{k} \right\rceil$. Hence, D is a k-DRD set. □

Theorem 5.3.3. *The Algorithm 5.6 runs in $O(n^2)$ time.*

Proof. For any given graph T, finding degree of each vertex takes $O(n^2)$ time. To label each vertex as “Bound” will take $O(n)$ time. When Label of a vertex v is “Required”, then construction of N_v and C_v takes $O(n)$ time and for all the vertices it takes $O(n^2)$ time. Computing set D takes $O(n^2)$ time. Also, for each vertex it checks whether its label is bound or free in $O(n)$ time. Hence, the complexity of the Algorithm 5.6 is $O(n^2)$. □

Theorem 5.3.4. *The resultant set D of the Algorithm 5.6 is a minimum k-DRD set.*

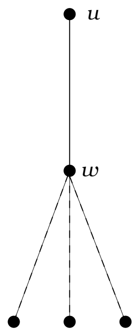


Figure 5.3 The tree T_1

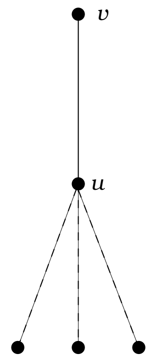


Figure 5.4 The tree T_2

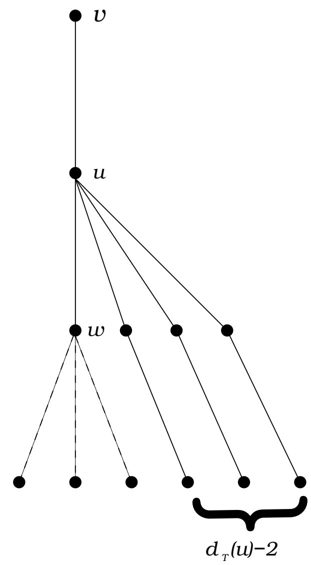


Figure 5.5 The tree T'

Figure 5.6 The construction of T_1, T_2 and T' from Tree T

Proof. We prove the result by induction on order =n+of tree T. Clearly, result holds for n [1, 2, 3, 4. Let T be a tree of order n, we know that every tree has at least two pendant vertices. Consider root of tree T as a pendant vertex v and label that vertex as "1" and continue the recursive labeling. Now from T remove v and relabel the vertex labeled as i] 1 in T as i in $T-v$ for all i, $1 \leq i \leq n-1$. Let D', D be the k-DRD sets of $T-v$, T respectively, obtained from the algorithm and u be the support vertex of v. By induction assumption D' is a $\gamma_{\frac{d}{k}}$-set of $T-v$ and we can observe that either D' [D or D [$D' \cup \{v\}$(or D [=$D' \cup \{v\}$+— $\{u\}$). If $|D'|$ [$|D|$, then D is a $\gamma_{\frac{d}{k}}$-set of T. Assume that D [$D' \cup \{v\}$. If D is not a minimum k-DRD set of T, then we can find a minimum k-DRD set D^* of T such that $u \in D^*$, u dominates v in D^* ($v \in C_u^*$) and $|D|$ [$|D^*|$] 1. Also note that if d=u+[2, then by the Lemma 5.3.1 and induction assumption result holds, hence throughout our discussion we assume that d=u+> 2. For each $w \in D$ we define C_w is the set of vertices dominated by w in D and for each $w \in D^*$, C_w^* is the set of vertices dominated by w in D^*.

Case 1: $u \in D$. Since $v \in C_u^* - C_u$ and $|C_u|$ [$\left\lceil \frac{d=u+}{k} \right\rceil$, there exists a non pendant vertex $w \in N$=u+such that $w \in C_u - C_u^*$. Then, either $w \in D^*$ or $w \notin D^*$.

Case i: $w \notin D^*$. Let $w \in C_x^*$ for some $x \in D^*$, T_1 be the component of $T - wx$ containing w, T^* be the another component of $T - wx$ and T_2 be the subgraph of T

obtained from T^* by adding a new vertex w to x (That is, $T_2 = (V(T^*) \cup \{w\}, E(T^*) \cup \{wx\})$). By the Algorithm 5.6, $\gamma_{\frac{d}{k}}(T_1) = |V(T_1) \cap D|$ and $\gamma_{\frac{d}{k}}(T_2) = |V(T_2) \cap D| + 1$. Also note that, $(V(T_1) \cap D^*) \cup \{w\}$ is a k-DRD set of T_1 and $V(T_2) \cap D^*$ is a k-DRD set of T_2, hence $|D| = |V(T_1) \cap D| + |V(T_2) \cap D| \leq |V(T_1) \cap D^*| - 1 + |V(T_2) \cap D^*| + 1 = |D^*|$, a contradiction.

Case ii: $w \in D^*$ and assume that $d(u) > 2$. Let T_1 be the component of $T - wu$ containing the vertex u, $T_1^* = T_1 - \{u, v\}$ and T_2 be the graph obtained from T by removing all the vertices in T_1^* (That is, $T_2 = T - (T_1 - \{u, v\})$). If $\left\lceil \frac{d(u)}{k} \right\rceil = \left\lceil \frac{d(u)-1}{k} \right\rceil$, then by the Algorithm 5.6 $\gamma_{\frac{d}{k}}(T_1) = |V(T_1) \cap D| - 1$ and $\gamma_{\frac{d}{k}}(T_2) = |V(T_2) \cap D|$. (Since $u \in D$ dominates v in T_1). Also we can find k-DRD sets of T_1 and T_2 of order $|V(T_1) \cap D^*|$ and $|V(T_2) \cap D^*|$ respectively. Then, $|D| = |V(T_1) \cap D| + |V(T_2) \cap D| - 2 \leq |V(T_1) \cap D^*| + |V(T_2) \cap D^*| - 1 = |D^*|$, a contradiction. Suppose $\left\lceil \frac{d(u)}{k} \right\rceil \neq \left\lceil \frac{d(u)-1}{k} \right\rceil$, then by the Algorithm 5.6 $\gamma_{\frac{d}{k}}(T_1) = |V(T_1) \cap D|$. Here, also we can find k-DRD sets of T_1 and T_2 of order $|V(T_1) \cap D^*| + 1$ and $|V(T_2) \cap D^*|$ respectively. Then, $|D| = |V(T_1) \cap D| + |V(T_2) \cap D| - 2 \leq |V(T_1) \cap D^*| + |V(T_2) \cap D^*| - 1 = |D^*|$, contradiction.

Case 2: $u \notin D$. Let $u \in C_w \subseteq V - D$, for some $w \in D$. Then, either $w \in D^*$ or $w \notin D^*$.

Case i: $w \notin D^*$. Suppose $w \in C_u^* \subseteq V - D^*$ (u is dominating w in D^*). Since $u \in C_w$ and u is the parent vertex of w, by the procedure of the Algorithm 5.6 neighbors of u other than v are not pendant vertices. Let T^* be the component of $T - wu$ containing the vertex w, T_2 is the another component of $T - wu$ and T_1 be the subgraph of T obtained from T^* by adding a new vertex u to w (That is, $T_1 = (V(T^*) \cup \{u\}, E(T^*) \cup \{wu\})$, See Figure 5.6). Then, by the Algorithm 5.6 $\gamma_{\frac{d}{k}}(T_1) = |V(T_1) \cap D|$ and $\gamma_{\frac{d}{k}}(T_2) = |V(T_2) \cap D|$. Also, $V(T_1) \cap D^*$ and $V(T_2) \cap D^*$ are k-DRD sets of T_1 and T_2 respectively. Since $\gamma_{\frac{d}{k}}(T_1) = |V(T_1) \cap D| \leq |V(T_1) \cap D^*|$, $\gamma_{\frac{d}{k}}(T_2) = |V(T_2) \cap D| \leq |V(T_2) \cap D^*|$ and $|D| = |V(T_1) \cap D| + |V(T_2) \cap D| > |V(T_1) \cap D^*| + |V(T_2) \cap D^*| - 1 = |D^*|$, $|V(T_1) \cap D| = |V(T_1) \cap D^*|$ and $|V(T_2) \cap D| = |V(T_2) \cap D^*|$. Let T' be the tree obtained from T_1 by joining $d_T(u) - 1$ new vertices to u by an edge and subdividing $d_T(u) - 2$ newly added edges (See Figure 5.6). If $T = T'$, then by Lemma 5.3.1 result holds. If not, then by th algorithm $\gamma_{\frac{d}{k}}(T') = |V(T_1) \cap D| + d_T(u) - 1$. Since $w, v \in C_u^*$ for $u \in D^*$, we can also find a k-DRD set of T' of cardinality $|V(T_1) \cap D^*| + d(u) - 2$, contradiction.

Suppose $w \in C_x^* \subseteq V - D^*$, for some $x \neq u$ (x is dominating w in D^*). Let T_1 be component of $T - uw$ containing w, $T_1^* = T_1 - \{w, x\}$ and T_2 be the graph obtained from T by removing all the vertices in T_1^* (That is, $T_2 = T - (T_1 - \{w, x\})$). If $x \in D$, then by the Algorithm 5.6 $\gamma_{\frac{d}{k}}(T_1) = |V(T_1) \cap D|$ and $\gamma_{\frac{d}{k}}(T_2) = |V(T_2) \cap D| - 1$. If $x \notin D$, then by the Algorithm 5.6 $\gamma_{\frac{d}{k}}(T_1) = |V(T_1) \cap D|$ and $\gamma_{\frac{d}{k}}(T_2) = |V(T_2) \cap D|$. Also $V(T_1) \cap D^*$ and $V(T_2) \cap D^*$ are k-DRD sets of T_1 and T_2 respectively. Hence, $|D| \leq |V(T_1) \cap D^*| + |V(T_2) \cap D^*| - 1 = |D^*|$, a contradiction.

Case ii: $w \in D^*$. In this case also we can find a subtree T^* of tree T of cardinality less than n such that $|D^*| < |D|$, where D is a minimum k-DRD set of T^* obtained from the Algorithm 5.6 and D^* is a k-DRD set of T^*, which leads to the contradiction of induction assumption. Hence, we can conclude that resultant D of the Algorithm 5.6 is a minimum k-DRD set.

□

5.4 ALGORITHM TO FIND A MINIMUM 2-DRD SET OF AN INTERVAL GRAPH

A graph $G = (V, E)$ is an interval graph, if every vertex in the graph can be associated with an interval in the real line so that two vertices are adjacent in the graph if and only if the two corresponding intervals intersects. That is, interval graphs are the intersection graphs of sets of intervals on the real line. Most of the domination related problems have linear time algorithms when restricted to interval graphs, but NP-complete when restricted to chordal graphs.

Theorem 5.4.1. *Ramalingam and Pandu Rangan (1988) A graph $G = (V, E)$ of order n is an interval graph if and only if its vertices can be numbered from 1 to n such that, for $i < j < k$, (i,k) is an edge in the graph only if (j,k) is an edge in the graph.*

Some notations, terminologies and definitions

Ramalingam and Pandu Rangan (1988) identified some properties of interval graph and they proposed linear time algorithm for weighted version of various domination problems like independent domination, connected domination and total domination . In this section, using some notations, terminologies and a similar approach considered by Ramalingam and Pandu Rangan (1988), we obtain an algorithm to find a minimum 2-DRD set of an interval graph.

Theorem 5.4.1 implies that, in the interval graph vertices can be numbered $1, 2, \ldots, n$. Let $V_i = \{1, 2, 3, \ldots, i\}$ and $G_i = \langle V_i \rangle$ be the subgraph of G induced by the vertices labeled $1, 2, 3, \ldots, i$. Notice that the graph G_i is obtained from G_{i-1} by adding a vertex i and joining it to zero or more consecutive vertices at the right end of the sequence $1, 2, 3, \ldots, i-1$. For each vertex i, LowNbr(i) is the smallest index of a vertex adjacent to i, if vertex i is not adjacent to any vertices to its left, then LowNbr(i) $= i$. Also, vertex i is not adjacent to vertices $1, 2, \ldots LowNbr(i) - 1$, but it is adjacent to every vertex between LowNbr(i) and i. For each vertex i, $MaxLow(i)$ and two more sets of vertices are defined as follows:

$$\begin{aligned} MaxLow(i) &= max\{LowNbr(s) : LowNbr(i) \leq s \leq i\}. \\ L(i) &= \{MaxLow(i), \ldots, i\}. \\ M(i) &= \{j : j > i \text{ and } j \in N(i)\}. \end{aligned}$$

Observe that the vertices in $L(i)$ forms a clique in G.

2-part degree restricted domination number of an interval graph

In order to compute a minimum 2-DRD set of an interval graph, let D_i be a 2-DRD set of G_i and $MinsetDRD(i)$ denote a collection of 2-DRD sets of G_i. In this case, we will permit a vertex not in V_i to be an element of D_i. As in the case of dominating set (1-DRD set) of an interval graph for every i, $1 \leq i \leq n$, there is a vertex in $L(i)$, say k, such that $N[k] \subseteq L(i) \cup M(i)$. Thus, any dominating set (1-DRD) D of G_i must include at least one vertex in $L(i) \cup M(i)$. Since every 2-DRD set is a dominating set, a 2-DRD set D_i of G_i must include at least one vertex, say j in $L(i) \cup M(i)$ for each i, $1 \leq i \leq n$. It is necessary and sufficient that D_i is a 2-DRD set of G_i if and only if $D_i - \{j\}$ dominates vertices in $D_{(LowNbr(j)-1)} \cup ((N_j \cap V_i) - C_j)$ as per the definition of 2-DRD domination. Here, we prove that if set $D_i \subseteq V$ is a 2-DRD set of G_i, then it is of the following form for some $j \in L(i) \cup M(i)$:

Let $MaxLow(i) = q+1$ and for every k, $1 \leq k \leq n$, $D'_k \subseteq V - V_k$ is the set of vertices dominated by D_k as per the definition of 2-DRD set.

Case 1: $j \in D_q$.

Let $D'_{q_1} = D_q - \{j\}$. If $|V_i - (V_q \cup D'_{q_1} \cup D_q)| \leq \left\lceil \frac{d(j)}{2} \right\rceil$, then D_q is a 2-DRD set of G_i. Suppose $|V_i - (V_q \cup D'_{q_1} \cup D_q)| > \left\lceil \frac{d(j)}{2} \right\rceil$. Since the vertices in $L(i)$ form a clique in G, $|V_i - (V_q \cup D'_{q_1} \cup D_q)| - 2 \leq \left\lceil \frac{d(j)}{2} \right\rceil + \left\lceil \frac{d(p)}{2} \right\rceil + \left\lceil \frac{d(r)}{2} \right\rceil$, for $p, r \in L(i) \cup M(i) - D_q$. Hence, either $D_q \cup \{p\}$ or $D_q \cup \{p, r\}$ is a 2-DRD set of G_i, for $p, r \in L(i) \cup M(i) - D_q$. Therefore, $j \in S_1 \cup S_2 \cup S_3$, whenever $j \in D_q$, where

$$\begin{aligned} S_1 &= \left\{ f \in L(i) \cup M(i) \cap D_q : |V_i - (V_q \cup D'_{q_1} \cup D_q)| \leq \left\lceil \frac{d(f)}{2} \right\rceil \right\} \\ S_2 &= \Big\{ f \in L(i) \cup M(i) \cap D_q : |V_i - (V_q \cup D'_{q_1} \cup D_q)| - 1 \leq \left\lceil \frac{d(f)}{2} \right\rceil + \left\lceil \frac{d(p)}{2} \right\rceil \\ &\qquad p \in L(i) \cup M(i) - D_q \Big\} \\ S_3 &= \Big\{ f \in L(i) \cup M(i) \cap D_q : |V_i - (V_q \cup D'_{q_1} \cup D_q)| - 2 \leq \left\lceil \frac{d(f)}{2} \right\rceil + \left\lceil \frac{d(p)}{2} \right\rceil + \left\lceil \frac{d(r)}{2} \right\rceil \end{aligned}$$

$$p, r \in L(i) \cup M(i) - D_q \Big\}$$

We observe that, $S_g \cap S_h = \emptyset$ for $g \neq h$.

Case 2: $j \notin D_q$ and $LowNbr(j) \leq q+1$.

If $|V_i - (V_q \cup D'_q \cup D_q)| - 1 \leq \left\lceil \frac{d(j)}{2} \right\rceil$, then $D_q \cup \{j\}$ is a 2-DRD set of G_i. If not, since the vertices in $L(i)$ form a clique in G, $|V_i - (V_q \cup D'_q \cup D_q)| - 2 \leq \left\lceil \frac{d(j)}{2} \right\rceil + \left\lceil \frac{d(p)}{2} \right\rceil$, for $p \in L(i) \cup M(i) - D_q$.

Case 3: $LowNbr(j) > q+1$. Then, clearly $j \notin D_q$ and $j \in M(i)$.

If $|(V_i \cap N(j)) - (D'_{LowNbr(j)-1} \cup D_{LowNbr(j)-1})| \leq \left\lceil \frac{d(j)}{2} \right\rceil$, then $D_{LowNbr(j)-1} \cup \{j\}$ is a 2-DRD set of G_i. Otherwise, $D_{LowNbr(j)-1} \cup \{j, p\}$ is a 2-DRD set of G_i, for $p \in L(i) \cup M(i) - D_{LowNbr(j)-1}$. Hence, if $j \in L(i) \cup M(i) - D_q$, then $j \in S_4 \cup S_5 \cup S_6 \cup S_7$, where

$$\begin{aligned}
S_4 &= \Big\{ f \in L(i) \cup M(i) - D_q : LowNbr(f) \leq q+1, |V_i - (V_q \cup D'_q \cup \{D_q\})| - 1 \leq \left\lceil \frac{d(f)}{2} \right\rceil \Big\} \\
S_5 &= \Big\{ f \in L(i) \cup M(i) - D_q : LowNbr(f) \leq q+1, \\
&\qquad |V_i - (V_q \cup D'_q \cup \{D_q\})| - 2 \leq \left\lceil \frac{d(f)}{2} \right\rceil + \left\lceil \frac{d(p)}{2} \right\rceil, p \in L(i) \cup M(i) - D_q \Big\} \\
S_6 &= \Big\{ f \in M(i) - D_q : LowNbr(f) > q+1, \\
&\qquad |(V_i \cap N(j)) - (D'_{LowNbr(f)-1} \cup D_{LowNbr(f)-1})| \leq \left\lceil \frac{d(f)}{2} \right\rceil \Big\} \\
S_7 &= \Big\{ f \in M(i) - D_q : LowNbr(f) > q+1, \\
&\qquad |(V_i \cap N(j)) - (D'_{LowNbr(f)-1} \cup D_{LowNbr(f)-1})| \leq \left\lceil \frac{d(f)}{2} \right\rceil + \left\lceil \frac{d(p)}{2} \right\rceil \\
&\qquad p \in L(i) \cup M(i) - D_{LowNbr(j)-1} \Big\}
\end{aligned}$$

We observe that, $S_g \cap S_h = \emptyset$ for $g \neq h$. Conversely, suppose D_q is a 2-DRD set of G_q and $j \in S_1$. Then, D_q is a 2-DRD set of G_i. The rest of the cases can be proved similarly.

Algorithm 5.7: $\gamma_{\frac{d}{2}}$-set of an interval graph

```
Input: An interval graph G = (V,E),
Output: minimum 2-DRD set D
MinsetDRD(0) = {∅}
for l=1 to n do
    i = l and S = ∅
    for j ∈ L(i) ∪ M(i) do
        Find LowNbr(j), S_j = ∅
        for D ∈ MinsetDRD(LowNbr(j) − 1) do
            for u ∈ D ∩ N((N(j) ∩ V_i)) ∪ (D ∩ N(j) ∩ V_i) ≠ ∅ do
                while |C_u| < ⌈d(u)/2⌉ and (N(u) ∩ N(j) ∩ V_i) − D ≠ ∅ do
                    Choose a vertex v ∈ (N(u) ∩ N(j) ∩ V_i), C_u = C_u ∪ {v} and
                    N(j) = N(j) − {v}
                end
            end
            if |(N(j) ∩ V_i) − D| ≤ ⌈d(j)/2⌉ then
                D' = D ∪ {j}, C_j = (N(j) ∩ V_i) − D and S_j = S_j ∪ {D'}
            end
        end
        if S_j ≠ ∅ then
            S = S ∪ S_j
        end
        else
            Find maxLowNbr(i) = q + 1
            Call Procedure 2-DRDMinD_(j≤q)(j)
        end
    end
    l = min{|D'| : D' ∈ S} and MinsetDRD(i) = {D' ∈ S : |D'| = l}
end
Return MinsetDRD(n)
```

Table 5.7 Algorithm to find minimum 2-DRD set of an interval graph

Algorithm 5.8: 2-DRDMin$D_{(j \leq q)}(j)$

Input: Vertex j
Output: Set S
if $LowNbr(j) \leq maxLowNbr(i)$ **then**
  for $D \in MinsetDRD(maxLowNbr(i) - 1)$ **do**
    $N'(j) = N(j) - D$
    if $j \in D$ **then**
      for $u \in (D \cap (N(N'(j) \cap (V_i - V_q))) \cup (D \cap N[j]) \neq \emptyset$ **do**
        while $|C_u| < \lceil \frac{d(u)}{2} \rceil$ *and* $(N(u) \cap N'(j) \cap (V_i - V_q)) \neq \emptyset$ **do**
          Choose a vertex $v \in (N(u) \cap N'(j) \cap (V_i - V_q))$, $C_u = C_u \cup \{v\}$ and $N(j) = (N'(j) \cap (V_i - V_q)) - \{v\}$
        end
      end
      for *each* $p \in L(i) \cup M(i) - D$ **do**
        if $|N(j) \cap (V_i - V_q)| - 1 \leq \left\lceil \frac{d(p)}{2} \right\rceil$ **then**
          $D' = D \cup \{p\}$, $C_p = N(j) \cap (V_i - V_q)$ and $S = S \cup \{D'\}$
        end
        else
          for *each* $q \in L(i) \cup M(i) - D \cup \{p\}$ **do**
            $D' = D \cup \{p, q\}$, $C_q = N(j) \cap (V_i - V_q) - C_p$ and $S = S \cup \{D'\}$
          end
        end
      end
    end
    else
      for $u \in (D \cap (N(N(j) \cap (V_i - V_q))) \cup (D \cap N(j)) \neq \emptyset$ **do**
        while $|C_u| < \left\lceil \frac{d(u)}{2} \right\rceil$ *and* $(N(u) \cap N'(j) \cap (V_i - V_q)) \neq \emptyset$ **do**
          Choose a vertex $v \in N(u) \cap N'(j) \cap (V_i - V_q)$, $C_u = C_u \cup \{v\}$ and $N(j) = (N'(j) \cap (V_i - V_q)) - \{v\}$
        end
      end
      if $|N(j) \cap (V_i - V_q)| \leq \left\lceil \frac{d(j)}{2} \right\rceil$ **then**
        $D' = D \cup \{j\}$, $C_j = N(j) \cap (V_i - V_q)$ and $S = S \cup \{D'\}$
      end
      else
        for *each* $p \in L(i) \cup M(i) - D \cup \{j\}$ **do**
          $D' = D \cup \{j, p\}$, $C_p = (N(j) \cap (V_i - V_q)) - C_j$ and $S = S \cup \{D'\}$
        end
      end
    end
  end
end
else
  Call Procedure 2-DRDMin$D_{(j>q)}(j)$
end
Return

Table 5.8 Algorithm to find all $\gamma_{\frac{d}{2}}$-sets of graph G_i containing vertex j with LowNbr$(j) \leq$ maxLowNbr(i) for some $j \in L(i) \cup M(i)$

Algorithm 5.9: 2-DRDMin$D_{(j>q)}(j)$

Input: Vertex j
Output: Set S
for $D \in MinsetDRD(LowNbr(j)-1)$ **do**
  $N'(j) = (N(j) \cap V_i) - D$
  for $u \in (D \cap (N(N'(j) \cap V_i))) \cup (D \cap N(j)) \neq \emptyset$ **do**
    while $|C_u| < \left\lceil \frac{d(u)}{2} \right\rceil$ *and* $N(u) \cap N'(j) \cap V_i \neq \emptyset$ **do**
      Choose a vertex $v \in (N(u) \cap N'(j) \cap V_i)$, $C_u = C_u \cup \{v\}$ and $N(j) = N'(j) - \{v\}$
    end
  end
  $C_j = \emptyset$
  while $|C_j| < \left\lceil \frac{d(j)}{2} \right\rceil$ *and* $N'(j) \neq \emptyset$ **do**
    Choose a vertex $v \in N'(j)$, $C_j = C_j \cup \{v\}$ and $N(j) = N'(j) - \{v\}$
  end
  for *each* $p \in L(i) \cup M(i) - D$ **do**
    $D' = D \cup \{p, j\}$, $C_p = N'(j) - C_j$, $S = S \cup \{D'\}$
  end
end
Return

Table 5.9 Algorithm to find all $\gamma_{\frac{d}{2}}$-sets of graph G_i containing vertex j with LowNbr$(j) >$ maxLowNbr(i) for some $j \in L(i) \cup M(i)$

Theorem 5.4.2. *Algorithm 5.7 runs in* $O\left(n^6 \binom{n}{\lfloor \frac{n}{2} \rfloor}\right)$ *time.*

Proof. For a given graph G of order n, computing degree and neighborhood of all the vertices using adjacency matrix takes $O(n^2)$ time. So, it takes $O(n^2)$ time to compute the set $L(i) \cup M(i)$ for any vertex i. Now, MinDRDset(i) is the collection of all $\gamma_{\frac{d}{2}}$ sets of G_i which are of same cardinality for any vertex i, hence the cardinality of MinDRDset(i) is at most $\frac{n!}{(n-\lfloor \frac{n}{2} \rfloor)! \lfloor \frac{n}{2} \rfloor !}$. The second for loop in the Algorithm 5.8 takes $O(n^3)$ time to check whether the vertices in V_i can be dominated by some 2-DRD set D_q of G_q for any i, $q \leq i$. Now, the third for loop in the Algorithm 5.8 takes $O(n^4)$ time to check whether $D \cup \{p\}$ is a 2-DRD set of G_i for each p in $L(i) \cup M(i)$. If not, then $D \cup \{p, q\}$ is a 2-DRD set of G_i for each q in $L(i) \cup M(i)$. So the Algorithm 5.8 takes $O\left(n^4 \left(\frac{n!}{(n-\lfloor \frac{n}{2} \rfloor)! \lfloor \frac{n}{2} \rfloor !}\right)\right)$ time. Similar steps are followed in the Algorithm 5.7 and in the Algorithm 5.9. Hence, the running time of Algorithm 5.7 is $O\left(n^6 \binom{n}{\lfloor \frac{n}{2} \rfloor}\right)$. □

In this Chapter, we showed that the k-part degree restricted domination problem is *NP*-complete for bipartite graphs, chordal graphs, undirected path graphs, chordal bipartite graphs, circle graphs, planar graphs and split graphs. We also proved that the k-part degree restricted domination problem is polynomial time solvable for trees and we provided a polynomial time algorithm to find a minimal k-part degree restricted dominating set of a graph. As we know interval graphs $\subseteq$ directed path graphs $\subseteq$ undirected path graphs $\subseteq$ chordal graphs, the complexity status of the k-part degree restricted domination problem is still unknown for graph classes interval graphs, directed path graphs and block graphs.

CHAPTER 6

CRITICAL ASPECTS OF 2-PART DEGREE RESTRICTED DOMINATION NUMBER

In the social or computer network, a set of people or a set of nodes are selected, as per certain criteria. Similarly a dominating set acts as a virtual backbone in any network. Suppose in a dominating set one person or one node is inactive, how does it affect the network? What is the impact of this on the entire network's working ability? This problem motivated the mathematicians to explore the level at which the dominating property is suddenly changing. This concept is studied as the critical aspects of the domination number. Some mathematicians approached this problem independently and studied this concept as changing and unchanging domination. Here, we present a study of critical aspects of 2-part degree restricted domination number of a graph.

6.1 SOME BASIC DEFINITIONS AND OBSERVATIONS

Definition 6.1.1. *Let G be a graph and let x be any element of the graph G. Then the element x is said to be*

1. $\gamma_{\frac{d}{2}}$*-critical if* $\gamma_{\frac{d}{2}}(G-x) \neq \gamma_{\frac{d}{2}}(G)$.
2. $\gamma^{+}_{\frac{d}{2}}$*-critical if* $\gamma_{\frac{d}{2}}(G-x) > \gamma_{\frac{d}{2}}(G)$.
3. $\gamma^{-}_{\frac{d}{2}}$*-critical if* $\gamma_{\frac{d}{2}}(G-x) < \gamma_{\frac{d}{2}}(G)$.
4. $\gamma_{\frac{d}{2}}$*-redundant if* $\gamma_{\frac{d}{2}}(G-x) = \gamma_{\frac{d}{2}}(G)$.
5. $\gamma_{\frac{d}{2}}$*-fixed if x belongs to every* $\gamma_{\frac{d}{2}}$*-set.*
6. $\gamma_{\frac{d}{2}}$*-free if x belongs to some* $\gamma_{\frac{d}{2}}$*-sets but not all* $\gamma_{\frac{d}{2}}$*-sets.*
7. $\gamma_{\frac{d}{2}}$*-totally free if x belongs to no* $\gamma_{\frac{d}{2}}$*-set.*

For example, consider the graph G in Figure 6.1. The sets $D_1 = \{v_3, v_4, v_5\}$, $D_2 = \{v_3, v_6, v_7\}$, $D_3 = \{v_3, v_5, v_6\}$, $D_4 = \{v_3, v_4, v_7\}$ are the $\gamma_{\frac{d}{2}}$-sets of graph G and we can observe the following:

The vertex v_3 is a $\gamma_{\frac{d}{2}}$-critical vertex of graph, since $\gamma_{\frac{d}{2}}(G - v_3) = 4 > 3 = \gamma_{\frac{d}{2}}(G)$ and vertex v_3 is the $\gamma_{\frac{d}{2}}^{+}$-critical vertex of G. The vertex v_2 is a $\gamma_{\frac{d}{2}}$-redundant vertex of graph, since $\gamma_{\frac{d}{2}}(G - v_2) = 3 = \gamma_{\frac{d}{2}}(G)$. Vertex v_7 is a $\gamma_{\frac{d}{2}}^{-}$-critical vertex of G, since $\gamma_{\frac{d}{2}}(G - v_7) = 2 < 3 = \gamma_{\frac{d}{2}}(G)$. The vertex v_3 is $\gamma_{\frac{d}{2}}$-fixed vertex of graph, since v_3 lies in every $\gamma_{\frac{d}{2}}$-set of G. The vertices v_4, v_5, v_6, v_7 are $\gamma_{\frac{d}{2}}$-free vertices of graph, since these vertices lie in some $\gamma_{\frac{d}{2}}$-sets of G. The vertices v_1 and v_2 are $\gamma_{\frac{d}{2}}$-totally free vertices of graph, since v_1 and v_2 are not in any of the $\gamma_{\frac{d}{2}}$-set of G.

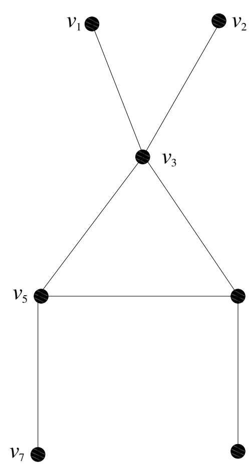


Figure 6.1 The graph G

6.2 CHANGE IN THE 2-PART DEGREE RESTRICTED DOMINATION NUMBER UPON VERTEX REMOVAL

Theorem 6.2.1. *For any connected graph G of order n and $v \in V(G)$,*

$$\gamma_{\frac{d}{2}}(G) - 1 \leq \gamma_{\frac{d}{2}}(G - v) \leq \gamma_{\frac{d}{2}}(G) + d(v) - 1.$$

Proof. Let D be a $\gamma_{\frac{d}{2}}$-set of $G - v$. Then, $D \cup \{v\}$ is a 2-DRD set of G. Hence, $\gamma_{\frac{d}{2}}(G) - 1 \leq \gamma_{\frac{d}{2}}(G - v)$. Let D' be a $\gamma_{\frac{d}{2}}$-set of G with $\bigcup\limits_{u \in D'} C_u = V - D'$. Since $\left\lceil \frac{d_G(u)}{2} \right\rceil -$

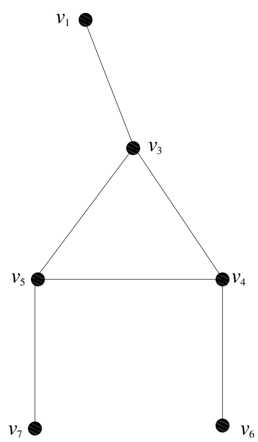


Figure 6.2 The graph $G-v_2$

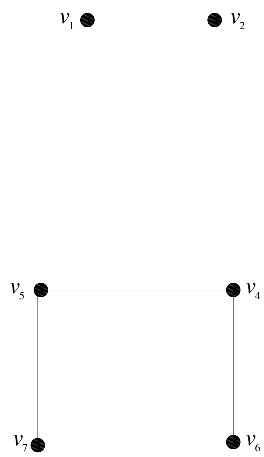


Figure 6.3 The graph $G-v_3$

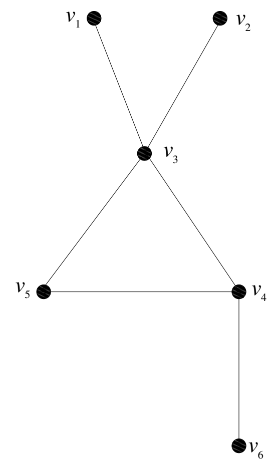


Figure 6.4 The graph $G-v_7$

$1 \le \left\lceil \frac{d_G(u)-1}{2} \right\rceil$ for $u \in V(G)$, vertex $u \in N(v) \cap D'$ may fail to dominate at most one vertex belongs to C_u in graph $G-v$. Suppose $|C_u| = \left\lceil \frac{d_G(u)}{2} \right\rceil > \left\lceil \frac{d_G(u)-1}{2} \right\rceil$ for every $u \in N(v) \cap D'$. Let $N(v) \cap D' = \{v_1, v_2, \ldots, v_m\}$. Since $C_{v_i} \neq \emptyset$ for $v_i \in N(v) \cap D'$, we consider $u_i \in C_{v_i}$ for $1 \le i \le m$. If $v \in C_{v_j}$ for some j, $1 \le j \le m$, then consider $u_j = v$. Then, all the vertices in $C_{v_j} - \{v\}$ can be dominated by v_j in the graph $G-v$ and $D' \cup \{u_1, u_2, \ldots, u_{j-1}, u_{j+1}, \ldots, u_m\}$ is a 2-DRD set of $G-v$. Similarly, if $v \in D'$, then also there exists a 2-DRD set of G of cardinality at most $|D' - \{v\}| + d(v)$ and $\gamma_{\frac{d}{2}}(G-v) \le \gamma_{\frac{d}{2}}(G) + d(v) - 1$. □

Corollary 6.2.2. *For any pendant vertex v of graph G, $\gamma_{\frac{d}{2}}(G-v) \le \gamma_{\frac{d}{2}}(G)$.*

Corollary 6.2.3. *Every vertex in a tree T can not be $\gamma_{\frac{d}{2}}$-critical.*

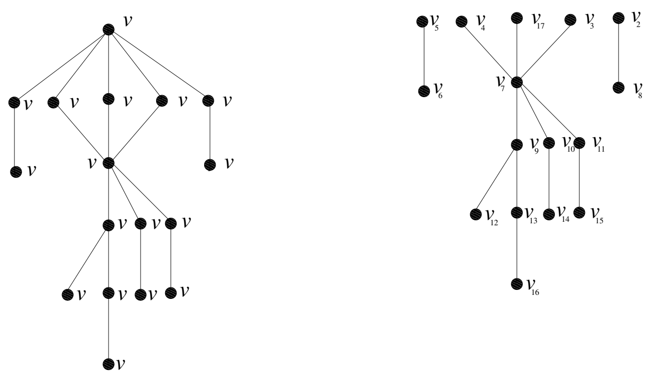
Theorem 6.2.4. *For any graph G, a vertex v is $\gamma^{-}_{\frac{d}{2}}$-critical if and only if G has a $\gamma_{\frac{d}{2}}$-set D satisfying the following conditions:*

1. *$v \in D$ with $C_v = \emptyset$.*

2. *Vertices in $D \cap N(v)$ are of even degree in G.*

Proof. Assume that a vertex v in G is $\gamma^{-}_{\frac{d}{2}}$-critical and D' is a $\gamma_{\frac{d}{2}}$-set of $G-v$ with $\bigcup_{u \in D'} C'_u = V(G-v) - D'$. Then, $D = D' \cup \{v\}$ is a $\gamma_{\frac{d}{2}}$-set of G, which satisfies the first condition of the hypothesis. Suppose the degree of u is odd in G for some $u \in D \cap N(v)$. Then, $|C'_u \cup \{v\}| \le \left\lceil \frac{d_G(u)}{2} \right\rceil$ and u can dominate v in G. Hence, D' is a 2-DRD set of G, a contradiction. Conversely, assume that G has a $\gamma_{\frac{d}{2}}$-set D satisfying above conditions. Then, clearly $D-\{v\}$ is a 2-DRD set of $G-v$ and $\gamma_{\frac{d}{2}}\ \ G-v\ \ \leq |D-v| <$ $|D| \quad \gamma_{\frac{d}{2}}\ \ G$. □

Remark 6.2.5. *For any graph G, if a vertex v is $\gamma_{\frac{d}{2}}$-redundant, then it is not necessary that there exists a $\gamma_{\frac{d}{2}}$-set of $G-v$, which is a $\gamma_{\frac{d}{2}}$-set of G. For example consider graph G in Figure 6.5a. The set $\{v_1, v_2, v_5, v_9, v_{10}, v_{11}, v_{16}\}$ is a $\gamma_{\frac{d}{2}}$-set of G, $\gamma_{\frac{d}{2}}(G) = 7$ and set $\{v_2, v_5, v_7, v_9, v_{10}, v_{11}, v_{16}\}$ is a $\gamma_{\frac{d}{2}}$-set of $G-v_1$. Although, v_1 is a $\gamma_{\frac{d}{2}}$-redundant vertex of graph G, the graph $G-v_1$ has no $\gamma_{\frac{d}{2}}$-set which is a $\gamma_{\frac{d}{2}}$-set of G.*



(a) The graph G (b) The graph $G-v_1$

Figure 6.5 Example in reference to Remark 6.2.5

Proposition 6.2.6. *For any connected graph G, if a vertex v is $\gamma^{-}_{\frac{d}{2}}$-critical, then v is $\gamma_{\frac{d}{2}}$-free.*

Proof. If $\gamma_{\frac{d}{2}}\ \ G-v\ \ < \gamma_{\frac{d}{2}}\ \ G$, then G has a $\gamma_{\frac{d}{2}}$-set D with $C_v \quad \emptyset$. Hence, v is not $\gamma_{\frac{d}{2}}$-totally free. Let $u \in N\ \ v$. If $u \in V-D$, then $D \cup \{u\} - \{v\}$ is a $\gamma_{\frac{d}{2}}$-set of G. Suppose $u \in D$. Then, since $C_v \quad \emptyset$, we get $C_u\ /\ \emptyset$. Then, $D \cup \{w\} - \{v\}$ is a $\gamma_{\frac{d}{2}}$-set of G, for some $w \in C_u$ and v is not $\gamma_{\frac{d}{2}}$-fixed. Hence, v is $\gamma_{\frac{d}{2}}$- free. □

Lemma 6.2.7. *For any graph G, a pendant vertex v is $\gamma_{\frac{d}{2}}$-redundant if and only if there exists a $\gamma_{\frac{d}{2}}$-set of $G-v$, which is a $\gamma_{\frac{d}{2}}$-set of G.*

Proof. The converse part of the statement is trivial. Let D be $\gamma_{\frac{d}{2}}$-set of G and u be the support vertex of v. Then, at least on of u, v is in D. If both $u, v \in D$, then $C_v = \emptyset$, the degree of u is odd and $|C_u| = \left\lceil \frac{d(u)}{2} \right\rceil \geq 1$. Let $w \in C_u$. Then, $(D-\{v\}) \cup \{w\}$ is a $\gamma_{\frac{d}{2}}$-set of G and $G-v$. Suppose $u \in D$ and $v \notin D$. Then, clearly D is a $\gamma_{\frac{d}{2}}$-set of G and $G-v$. Suppose $v \in D$ and $u \notin D$. Then, since v is $\gamma_{\frac{d}{2}}$-redundant, $C_v = \{u\}$ and $(D-\{v\}) \cup \{u\}$ is a $\gamma_{\frac{d}{2}}$-set of G and $G-v$. □

Lemma 6.2.8. *For any tree T, v is $\gamma_{\frac{d}{2}}$-redundant if and only if there exists a $\gamma_{\frac{d}{2}}$-set of $G-v$, which is a $\gamma_{\frac{d}{2}}$-set of G.*

Proof. Clearly, the converse part holds. Let v be a $\gamma_{\frac{d}{2}}$-redundant vertex, T be a rooted tree having v as a root and D be a $\gamma_{\frac{d}{2}}$-set of tree T obtained from the Algorithm 5.6 in Chapter 5. Let $T_1, T_1, \ldots, T_{d\ v}$ be the components of $T-v$. Let D_i be a $\gamma_{\frac{d}{2}}$-set of tree T_i, $1 \leq i \leq d\ v$ obtained from the Algorithm 5.6, considering the vertex u_i as a root, where $N\ v\ \cap V\ T_i \quad \{u_i\}$. Then, $\cup_{j\ \ 1}^{d\ v} D_j$ is a minimum 2-DRD set of $T-v$ and by the Algorithm 5.6 we can observe that $D^* \quad \cup_{j\ \ 1}^{d\ u} D_i$ is a 2-DRD set of T.

□

Theorem 6.2.9. *Suppose a vertex $v \in V$ is $\gamma_{\frac{d}{2}}$-critical. Then, for any $\gamma_{\frac{d}{2}}$-set D of G with $\bigcup_{u \in D} C_u \quad V-D$,*

- *if $v \in D$, then $|C \cup C_v| > 1$,*
- *if $v \in C_w \subseteq V-D$ for some $w \in D$, then $|C-\{w\}| \geq 1$, where $C \quad \{u \in N\ v\ : d\ u\ > 2$ and $|C_u| \quad \frac{d\ u\quad 1}{2}\}$.*

Proof. Let vertex v be $\gamma_{\frac{d}{2}}$-critical and D be a $\gamma_{\frac{d}{2}}$-set of G. Assume that $v \in D$. If C $C_v \quad \emptyset$, then $D-\{v\}$ is a 2-DRD set of $G-v$ and $\gamma_{\frac{d}{2}}\ G-v\ \leq |D-\{v\}| < |D| \quad \gamma_{\frac{d}{2}}\ G$, a contradiction. If $C \cup C_v \quad \{w\}$, $C_v \quad \{w\}$ and $C \quad \emptyset$, then $D \cup \{w\}\ -\{v\}$ is a 2-DRD set of $G-v$, a contradiction. If $C_v \quad \emptyset$ and $C \quad \{w\}$, then $D \cup \{u\}\ -\{v\}$ is a 2-DRD set of $G-v$, for some $u \in C_w$, a contradiction. Similarly, we can prove $|C| \geq 1$, if $v \notin D$. □

Corollary 6.2.10. *For any Eulerian graph G, if a vertex v is $\gamma_{\frac{d}{2}}$-critical, then v is $\gamma_{\frac{d}{2}}$-fixed.*

Proof. Let a vertex v be $\gamma_{\frac{d}{2}}$-critical. Since G is Eulerian for any $u \in V$, $\left\lceil \frac{d\ u}{2} \right\rceil\ /\ \frac{d\ u\quad 1}{2}$. Hence, $C \quad \emptyset$. Then, Theorem 6.2.9 implies $v \in D$ for any $\gamma_{\frac{d}{2}}$-set D. □

Corollary 6.2.11. *Let G be an Eulerian graph and D be a $\gamma_{\frac{d}{2}}$-set of G such that each vertex in D is $\gamma_{\frac{d}{2}}$-critical. Then, D is a unique $\gamma_{\frac{d}{2}}$-set of G.*

Proof. Assume that there exists two $\gamma_{\frac{d}{2}}$-sets D' and D of G. Let $u \in D-D'$. Since G is Eulerian and u is $\gamma_{\frac{d}{2}}$-critical, u is $\gamma_{\frac{d}{2}}$-fixed, a contradiction. Hence, D is unique. □

Theorem 6.2.12. *Let v be a vertex in graph G such that for any $\gamma_{\frac{d}{2}}$-set D of G, if $v \in D$, then $|C \cup C_v| > 1$, where $C \quad \{u \in N\ v\ : |C_u| \quad \frac{d\ u\quad 1}{2}\}$. Then, either v is $\gamma_{\frac{d}{2}}$-critical or $\gamma_{\frac{d}{2}}$-redundant.*

Proof. Let $v \in V$ and D^* be a $\gamma_{\frac{d}{2}}$-set of $G-v$. Then, $D^* \cup \{v\}$ is a 2-DRD set of G and by the above condition $D^* \cup \{v\}$ is not a $\gamma_{\frac{d}{2}}$-set of G. Hence, $\gamma_{\frac{d}{2}}(G) < |D^*| + 1$, which implies $\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(G-v)$. Therefore, either v is $\gamma_{\frac{d}{2}}$-critical or $\gamma_{\frac{d}{2}}$-redundant. □

Proposition 6.2.13. *For any non trivial tree T, if $uv \in E(T)$, then both u and v are not $\gamma^-_{\frac{d}{2}}$-critical.*

Proof. Let $uv \in E(T)$, vertex v be $\gamma^-_{\frac{d}{2}}$-critical and D be a $\gamma_{\frac{d}{2}}$-set of $T-v$. Then, $D^* = D \cup \{v\}$ is a $\gamma_{\frac{d}{2}}$-set of T with $C_v = \emptyset$. Let $T'_1, T'_2, \ldots, T'_{d(u)}$ be the components of $T-u$ and $T'_{d(u)}$ be the component containing v. Then, $D_u = D^* \cap V(T'_{d(u)})$ is a $\gamma_{\frac{d}{2}}$-set of $T'_{d(u)}$ with $C_v = \emptyset$. Let D' be the union of $\gamma_{\frac{d}{2}}$-sets of each T'_i, $1 \leq i < d(u)$. Then, $D' \cup D_u$ is a $\gamma_{\frac{d}{2}}$-set of $T-u$ with $C_v = \emptyset$. Since $C_v = \emptyset$, v can dominate u in T and $D' \cup D_u$ is a 2-DRD set of T. Hence, $\gamma_{\frac{d}{2}}(T) \leq \gamma_{\frac{d}{2}}(T-u)$ and u is not $\gamma^-_{\frac{d}{2}}$-critical. □

Corollary 6.2.14. *Every vertex in a non trivial tree T can not be $\gamma^-_{\frac{d}{2}}$-critical.*

Proposition 6.2.15. *Every vertex in a tree T is $\gamma_{\frac{d}{2}}$-redundant if and only if T is an even path.*

Proof. Let T be a rooted tree with m levels having v as a root. Since every vertex in tree T is $\gamma_{\frac{d}{2}}$-redundant, the degree of each vertex in $(m-1)^{\text{th}}$ level is less than or equal to two. Let u be the vertex of degree greater than two in the $(m-j)^{\text{th}}$, $(2 \leq j < m)$ level such that the vertices in $(m-l)^{\text{th}}$ level for $0 \leq l < j$ are of degree less than or equal to two. Let u_i, $1 \leq i \leq d(u)-1$ be the child neighbor of u and v_i, $1 \leq i \leq d(u)-1$ be the pendant vertex in the succeeding levels (That is, levels $(m-l)$ for $0 \leq l < j$) lies in the unique $v_i - v$ path $P(v_i)$. Since $d(u) > 2$, u has at least two child neighbor say u_1, u_2. Since $\gamma_{\frac{d}{2}}(T-u) = \gamma_{\frac{d}{2}}(T)$, Lemma 6.2.8 implies that there exists a $\gamma_{\frac{d}{2}}$-set D of T which is $\gamma_{\frac{d}{2}}$-set of $T-u$. Suppose both u_1, u_2 are at even distance from v_1, v_2 respectively. Then, $u_1, u_2 \in D$ with $C_{u_1} = C_{u_2} = \emptyset$ (Since $u \in V-D$). Since $d(u) > 2$, u can dominate both u_1, u_2 in D. Then, $(D - \{u_1, u_2\}) \cup \{u\}$ is a 2-DRD set of T, a contradiction. Suppose u_1 is at even distance from v_1 and u_2 is at odd distance from v_2. Then, $u_1 \in D$ such that $C_{u_1} = \{u\}$ with respect to $\gamma_{\frac{d}{2}}$-set of T. Then, $\gamma_{\frac{d}{2}}(T-v_2) < \gamma_{\frac{d}{2}}(T)$, a contradiction. Suppose both u_1, u_2 are at odd distance from v_1, v_2, respectively. Let w be the parent vertex of u. Since both u_1 and u_2 are at odd distance from $v_1$and v_2, there exists a $\gamma_{\frac{d}{2}}$-set D^* of T, which is a $\gamma_{\frac{d}{2}}$-set of $T-w$ such that $u \in D^*$ with $C_u = \emptyset$. Then, $\gamma_{\frac{d}{2}}(T-v_1) < \gamma_{\frac{d}{2}}(T)$, a contradiction. Hence, $d(u) \leq 2$ and T is a path. If T is an odd path, then $\gamma_{\frac{d}{2}}(T-v) < \gamma_{\frac{d}{2}}(T)$, for any pendant vertex v of T. Conversely, for any vertex v of degree two in path P_{2m}, $P_{2m} - v = P_{2n} \cup P_{2n^*+1}$ such that $2n + 2n^* + 1 = 2m-1$. Then, $\gamma_{\frac{d}{2}}(P_{2m}-v) = \gamma_{\frac{d}{2}}(P_{2n}) + \gamma_{\frac{d}{2}}(P_{2n^*+1}) = \lceil \frac{2n}{2} \rceil + \left\lceil \frac{2n^*+1}{2} \right\rceil = \lceil \frac{2m}{2} \rceil = \gamma_{\frac{d}{2}}(P_{2m})$. If the degree of v is one, then $P_{2m} - v = P_{2m-1}$ and $\gamma_{\frac{d}{2}}(P_{2m}) = \gamma_{\frac{d}{2}}(P_{2m-1})$. □

Proposition 6.2.16. *For any Eulerian graph G,*

1. *If a vertex v is free, then $\gamma_{\frac{d}{2}}(G-v) \leq \gamma_{\frac{d}{2}}(G)$.*

2. *If a vertex v is totally free, then vertex v is $\gamma_{\frac{d}{2}}$-redundant.*

Proof. Let D be $\gamma_{\frac{d}{2}}$-set of G. Assume that vertex v is $\gamma_{\frac{d}{2}}$-free or $\gamma_{\frac{d}{2}}$-totally free. Since G is Eulerian, for any $u \in V$ $\left\lceil \frac{d(u)}{2} \right\rceil \neq \frac{d(u)-1}{2}$. Hence, D is a 2-DRD set of $G-v$ and $\gamma_{\frac{d}{2}}(G-v) \leq \gamma_{\frac{d}{2}}(G)$. If v is totally free and $\gamma_{\frac{d}{2}}(G-v) < \gamma_{\frac{d}{2}}(G)$, then for any $\gamma_{\frac{d}{2}}$-set D' of $G-v$, $D' \cup \{v\}$ is $\gamma_{\frac{d}{2}}$-set of G, a contradiction to the fact that v is $\gamma_{\frac{d}{2}}$-totally free. Hence, $\gamma_{\frac{d}{2}}(G-v) = \gamma_{\frac{d}{2}}(G)$ and v is $\gamma_{\frac{d}{2}}$-redundant. □

Proposition 6.2.17. *For any graph G, D is the unique $\gamma_{\frac{d}{2}}$-set of G if and only if G has no free vertices.*

Proof. If D is the unique $\gamma_{\frac{d}{2}}$-set of G, then all the vertices in D will be fixed and all the vertices in $V-D$ is totally free. Conversely, assume that G has no free vertices. If G has two $\gamma_{\frac{d}{2}}$-sets say D_1 and D_2, then vertex $v \in D_1 - D_2$ is a free vertex, a contradiction. □

Theorem 6.2.18. *Let T be a rooted tree having vertex x as a root and $\gamma_{\frac{d}{2}}(T) \neq \gamma_{\frac{d}{2}}(T-v)$ for any $v \in V$. Then, there exists a $\gamma_{\frac{d}{2}}$-set D of T satisfying the following conditions.*

1. *For any $v \in V-D$, $\gamma_{\frac{d}{2}}(T-v) < \gamma_{\frac{d}{2}}(T)$.*

2. *If u is a parent vertex of $v \in D$, then $u \notin C_v$ (or vertices in D is not dominating its parent vertex).*

3. *If $u \in D$ with $|C_u| > 1$, then the degree of u is even and there exist a vertex $v \in D \cap N(u)$ such that $C_v = \emptyset$.*

Proof. Let D be a $\gamma_{\frac{d}{2}}$-set of tree T obtained from the Algorithm 5.6 in Chapter 5, $v \in V-D$ and assume that $\gamma_{\frac{d}{2}}(T-v) > \gamma_{\frac{d}{2}}(T)$. Then, by Theorem 6.2.9, there exists a vertex $v_1 \in N(v) \cap D$ such that the degree of v_1 is odd and $|C_{v_1}| = \left\lceil \frac{d(v_1)}{2} \right\rceil$. Suppose v_1 has a pendant neighbor w. Since $d(v_1)$ is odd and $|C_{v_1}| = \left\lceil \frac{d(v_1)}{2} \right\rceil > \left\lceil \frac{d(v_1)-1}{2} \right\rceil$, we get $\gamma_{\frac{d}{2}}(T-w) = \gamma_{\frac{d}{2}}(T)$, a contradiction.

Claim: Suppose v_1 do not have any pendant neighbor. Then, for any child neighbor v_2 of v_1 in C_{v_1}, $\gamma_{\frac{d}{2}}(T-v_2) > \gamma_{\frac{d}{2}}(T)$.

Since the degree of v_1 is odd and $|C_{v_1}| = \left\lceil \frac{d(v_1)}{2} \right\rceil > 0$, v_1 dominates at least one vertex. Therefore, v_1 is not a pendant vertex and $d(v_1) \geq 3$. Since $|C_{v_1}| = \left\lceil \frac{d(v_1)}{2} \right\rceil \geq \left\lceil \frac{3}{2} \right\rceil = 2$, vertex v_1 dominates at least one child neighbor. Let $v_2 \in C_{v_1}$ be a child neighbor of v_1 (see Figure 6.6), $T_1, T_2, \ldots, T_{d(v_2)}$ be components of $T-v_2$. By the Algorithm

5.6, $\gamma_{\frac{d}{2}}\ T_i\quad |D \cap V\ T_i\ |\quad 1$ or $\gamma_{\frac{d}{2}}\ T_i\quad |D \cap V\ T_i\ |$ for all i, $1 \leq i \leq d\ v_2$. (That is, $\gamma_{\frac{d}{2}}\ T_i\ \geq |D \cap V\ T_i\ |$.) Hence, $\gamma_{\frac{d}{2}}\ T - v_2 \quad \sum_{1 \leq i \leq d\ v_2} \gamma_{\frac{d}{2}}\ T_i\ \geq \gamma_{\frac{d}{2}}\ T$, which implies $\gamma_{\frac{d}{2}}\ T - v_2\ > \gamma_{\frac{d}{2}}\ T$ and the claim holds.

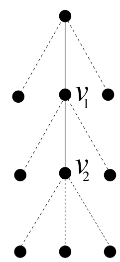


Figure 6.6 Tree T having v_2 as a child neighbor of vertex of v_1

Then, there exists a child neighbor $v_4 \in\ N\ v_2\ \cap D\ - \{v_1\}$ such that the degree of v_4 is odd and $|C_{v_4}|\quad \left\lceil \frac{d\ v_4}{2} \right\rceil > 1$. If v_4 has a pendant neighbor w_1, then $\gamma_{\frac{d}{2}}\ T - w_1$ $\gamma_{\frac{d}{2}}\ T$, a contradiction. If not, then for any child neighbor v_5 of v_4 in C_{v_4}, $\gamma_{\frac{d}{2}}\ T - v_2\ >$ $\gamma_{\frac{d}{2}}\ T$. Since T is a rooted tree, by continuing the above procedure we can find a vertex $v_l \in D$ of odd degree having pendant neighbor w_l. Then, $\gamma_{\frac{d}{2}}\ T - w_l \quad \gamma_{\frac{d}{2}}\ T$, a contradiction. Hence, $\gamma_{\frac{d}{2}}\ T - v\ < \gamma_{\frac{d}{2}}\ T$.

Suppose u is the parent vertex of $v \in D$ and $u \in C_v$. Let $T_1, T_2, \ldots, T_{d\ u}$ be the components of $T - u$. If $u\quad x$, then from the Algorithm 5.6, $\gamma_{\frac{d}{2}}\ T_i\quad |D \cap V\ T_i\ |\quad 1$ or $\gamma_{\frac{d}{2}}\ T_i\quad |D \cap V\ T_i\ |$ for all i, $1 \leq i \leq d\ v_2$. Suppose $u\ /\ x$. Let u' be the parent vertex of u and T_1 be the component of $T - u$ containing u'. Note that label of u' does not change (That is, label "Bound" or "Required" of u' in the Algorithm 5.6 does not depend on u). Therefore, either $D \cap V\ T_1$ is a minimum 2-DRD set of T_1 or $\gamma_{\frac{d}{2}}\ T_1\quad |D \cap V\ T_1\ |\quad 1$. Clearly, $\gamma_{\frac{d}{2}}\ T_i\quad |D \cap V\ T_i\ |\quad 1$ or $\gamma_{\frac{d}{2}}\ T_i\quad |D \cap V\ T_i\ |$ for all i, $2 \leq i \leq d\ v_2$. Hence, $\gamma_{\frac{d}{2}}\ T - u \quad \sum_{1 \leq i \leq d\ u} \gamma_{\frac{d}{2}}\ T_i\ \geq \gamma_{\frac{d}{2}}\ T$, which implies $\gamma_{\frac{d}{2}}\ T - u\ > \gamma_{\frac{d}{2}}\ T$, a contradiction to the first statement.

Let $u \in D$ with $|C_u| > 1$, $w \in C_u$, $C\quad \{v \in D \cap N\ u\ : C_v\quad \emptyset\}$, $T_{w'_1}, T_{w'_2}, \ldots, T_{w'_{d\ w\ -1}}$ be the components of $T - w$ and T_u be the component of $T - w$ containing u. By the second statement, it is clear that w is not a parent vertex of u. Now, $|C_u| > 1$ and u is not dominating its parent vertex. If the degree of u is odd, then $\left\lceil \frac{d\ u}{2} \right\rceil$ $1\quad \left\lceil \frac{d\ u\ -1}{2} \right\rceil$. Then, by applying the Algorithm 5.6 to tree T_u, u can dominate same vertices in T_u other than w, as dominating in tree T and $\gamma_{\frac{d}{2}}\ T_u\quad D \cap V\ T_u$. Also, $\gamma_{\frac{d}{2}}\ T_{w'_{d\ w\ -1}}\ \geq |D \cap V\ T_{w'_{d\ w\ -1}}\ |$ for all $1 \leq i \leq d\ w\ - 1$. Hence, $\gamma_{\frac{d}{2}}\ T - w\ > \gamma_{\frac{d}{2}}\ T$, a contradiction to the first statement. Therefore, $d\ u$ is even. Suppose $C\quad \emptyset$. Then, for each $v \in N(u)$ either $v \in D$ with $C_v \neq \emptyset$ or $v \in V - D$. Then also, by applying the Algorithm 5.6 to the tree T_u, u can dominate same vertices in T_u other than w, as in the case of tree T and $\gamma_{\frac{d}{2}}(T_u) = D \cap V(T_u)$. (That is, u can not dominate any extra vertex in T_u even $\left\lceil \frac{d(u)}{2} \right\rceil = \left\lceil \frac{d(u)-1}{2} \right\rceil$ and u is not dominating w in T_u.) Also, $\gamma_{\frac{d}{2}}(T_{w'_{d(w)-1}}) \geq |D \cap V(T_{w'_{d(w)-1}})|$ for all $1 \leq i \leq d(w)-1$. Hence, $\gamma_{\frac{d}{2}}(T - w) > \gamma_{\frac{d}{2}}(T)$, a contradiction to the first statement. Hence, $C \neq \emptyset$ □

In order to characterize the trees T such that $\gamma_{\frac{d}{2}}(T - v) \neq \gamma_{\frac{d}{2}}(T)$ for any $v \in V(T)$, we define the family ψ of trees T that can be obtained recursively from a sequence $T_0, T_1, T_2, \ldots, T_j (j \geq 1)$ of trees such that T_0 is a star $K_{1,2m}$, $m > 1$. If $T = T_i$ and $i \geq 2$, then T_i can be obtained recursively from T_{i-1} as defined follows:

} For any positive integer $m > 1$, $K_{1,2m}$ is a tree such that $\gamma_{\frac{d}{2}}(T - v) \neq \gamma_{\frac{d}{2}}(T)$ for all $v \in V(T)$.

} Let $T_0 = K_{1,2m}$ and vertex of maximum degree (r_0) be the root of T_0. Now, we construct a new tree T_1 using T_0.

} Take a new vertex r_1, l copies of T_0 for different integer m. Join edges from r_1 to root (r_0) of each copy of T_0. Make the degree of r_0 even in the new graph by joining a new vertex (pendant) to r_0 or removing a pendant vertex adjacent to r_0. While removing the pendant vertices adjacent to r_0 (in the new graph), the degree of r_0 should be greater than 2 in the new graph. Name the resultant graph as T_1 having r_1 as the root.

} Add some vertices (pendant) to r_1 in T_1 by an edge such that number of pendant vertices adjacent to r_1 should be greater than $\left\lceil \frac{d(r_1)}{2} \right\rceil$ and the degree of r_1 is even in the resultant graph. Name the resultant graph as T_1'.

} Now by applying the Algorithm 5.6 to the graph T_0, T_1, T_1', we obtain the minimum 2-DRD sets D_0, D_1, D_2 respectively, such that $r_0 \in D_0$ with $C_{r_0} \neq \emptyset$, $r_1 \in D_1$ with $C_{r_1} = \emptyset$ and $r_1 \in D_2$ with $C_{r_1} \neq \emptyset$.

} Trees obtained from the above constructions are classified into two classes called ψ_0, ψ_1 as follows:

} Let T_i be the tree obtained from above construction having r_i as a root. Then clearly r_i is in the minimum 2-DRD set obtained from the Algorithm 5.6. If $C_{r_i} = \emptyset$, then $T_i \in \psi_0$ and if $C_{r_i} \neq \emptyset$, then $T_i \in \psi_1$ and $K_1 \in \psi_0 \cap \psi_1$.

} Consider a new vertex r_j and some copies of trees from ψ_1. Join edges from r_j to root of each copy of tree chosen from ψ_1 other than K_1. Make the degree of root vertex r_i of each tree chosen from ψ_1 as even in the new graph by joining a new edge from r_i to K_1 or by removing some neighbor of r_i. Suppose while choosing graph from ψ_1 the graph K_1 is also chosen. Then, the number of K_1 joined to r_j should be more than $\left\lceil \frac{d(r_j)}{2} \right\rceil$ and the degree of r_j is even in the resultant tree.

} Choose some copies of trees from ψ_0. Join edges from r_j to root of each copy of tree chosen from ψ_0 such that number of trees chosen from ψ_0 to join to r_j should be more than $\left\lceil \frac{d(r_j)}{2} \right\rceil$ in the resultant tree and the degree of r_j is even.

Theorem 6.2.19. *For any $v \in V(T)$ of tree T, $\gamma_{\frac{d}{2}}(T) \neq \gamma_{\frac{d}{2}}(T-v)$ if and only if $T \in \psi = \psi_0 \cup \psi_1$.*

Proof. Assume that T is a tree such that $\gamma_{\frac{d}{2}}(T) \neq \gamma_{\frac{d}{2}}(T-v)$ for any $v \in V(T)$. We prove that $T \in \psi$. We prove the result by induction on n. Suppose we consider all the trees of order $n \leq 5$. Then, $\gamma_{\frac{d}{2}}(T) \neq \gamma_{\frac{d}{2}}(T-v)$ for any $v \in V(T)$ if and only if $T = K_{1,4}$. Assume that the result holds for all the trees of order n. Let T be a rooted tree of order $n+1$ $(n \geq 5)$ having m levels. Suppose the degree of a vertex v in $(m-1)^{\text{th}}$ level is odd and greater than 1, then $\gamma_{\frac{d}{2}}(T-u) = \gamma_{\frac{d}{2}}(T)$ for any pendant vertex u adjacent to v. Suppose $d(v) = 2$ and u is the pendant neighbor of v. Then, either $\gamma_{\frac{d}{2}}(T-u) = \gamma_{\frac{d}{2}}(T)$ or $\gamma_{\frac{d}{2}}(T-v) = \gamma_{\frac{d}{2}}(T)$. Hence, degree of a vertex in $(m-1)^{\text{th}}$ level is either 1 or even greater than 2.

Case 1: Suppose there exists a vertex v in $(m-1)^{\text{th}}$ level such that $d(v) \geq 6$. Let T^* be the tree obtained from T by removing two pendant vertices adjacent to v. Since $d(v) \geq 6$ and v lies in $(m-1)^{\text{th}}$ level, at least two pendant neighbors of v should be in any $\gamma_{\frac{d}{2}}$-set of T. Also, for any $\gamma_{\frac{d}{2}}$-set D of T, $D - \bullet w\{$ is a $\gamma_{\frac{d}{2}}$-set of T^*, where w is a pendant neighbor of v in D.

Claim: For every $u \in V(T^*)$, $\gamma_{\frac{d}{2}}(T^*-u) \neq \gamma_{\frac{d}{2}}(T^*)$.

Clearly, $\gamma_{\frac{d}{2}}(T^*-v) > \gamma_{\frac{d}{2}}(T^*)$ and for any pendant neighbor w of vertex v in T^*, $\gamma_{\frac{d}{2}}(T^*-w) < \gamma_{\frac{d}{2}}(T^*)$. Now consider a vertex $u \in V(T^*) - \bullet v\{$, other than the child neighbor of v. Let $T_1^*, T_2^*, \ldots, T_{d(u)}^*$ be the components of T^*-u, T_1^* be the component of T^*-u containing the vertex v, $T_1, T_2, \ldots, T_{d(u)}$ be the components of $T-u$ and T_1 be the component of $T-u$ containing the vertex v. Then, for $2 \leq i \leq d(u)$, $T_i^* \cong T_i$ and any $\gamma_{\frac{d}{2}}$-set of T_i, is a $\gamma_{\frac{d}{2}}$-set of T_i^*. Also, for any $\gamma_{\frac{d}{2}}$-set D' of T_1, $D' - \bullet v'\{$ is a $\gamma_{\frac{d}{2}}$-set of T_1^*, where v' is a pendant neighbor of v in D'. Hence, $\gamma_{\frac{d}{2}}(T^*)+1 = \gamma_{\frac{d}{2}}(T) \neq \gamma_{\frac{d}{2}}(T-u) = \sum_{i=2}^{d(u)} \gamma_{\frac{d}{2}}(T_i) + \gamma_{\frac{d}{2}}(T_1) = \sum_{i=2}^{d(u)} \gamma_{\frac{d}{2}}(T_i^*) + \gamma_{\frac{d}{2}}(T_1^*) + 1$ and $\gamma_{\frac{d}{2}}(T^*) \neq \gamma_{\frac{d}{2}}(T^*-u)$. Therefore, by induction assumption $T^* \in \psi$. Now, T can be constructed by joining 2 new vertices to v by an edge and by the properties of tree in ψ one can observe that $T \in \psi$.

Case 2: Suppose the degree of each vertex in $(m-1)^{\text{th}}$ level is less than 6. Then, the degree of each vertex that lies in $(m-1)^{\text{th}}$ level is either 4 or 1. Since tree T is having m levels, at least one vertex v in $(m-1)^{\text{th}}$ level is of degree 4. Let w' be the parent vertex of v in T. Then, either w' has pendant child neighbors or neighbors of w' is of degree 4 (see Figure 6.7).

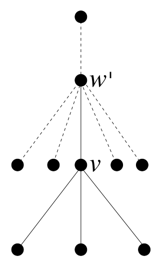


Figure 6.7 Rooted tree T having vertex v in $(m-1)^{\text{th}}$ level

Case i Suppose w' has a child pendant neighbor. If w' has exactly one child pendant neighbor u', then either $\gamma_{\frac{d}{2}}(T-u') = \gamma_{\frac{d}{2}}(T)$ or $\gamma_{\frac{d}{2}}(T-w') = \gamma_{\frac{d}{2}}(T)$, not possible. Let $N_1(w')$ be the set of all pendant neighbors of w' in T. Then, by the above argument $|N_1(w')| \geq 2$. If $2 \leq |N_1(w')| \leq \left\lceil \frac{d(w')}{2} \right\rceil$, then $\gamma_{\frac{d}{2}}(T) = \gamma_{\frac{d}{2}}(T-u')$ for any $u' \in N_1(w')$. Hence, $|N_1(w')| > \left\lceil \frac{d(w')}{2} \right\rceil$ and the degree of w' is even. Let T^* be the tree obtained from T by removing all the child neighbors of v.

Claim: For any $u \in V(T^*)$, $\gamma_{\frac{d}{2}}(T^*-u) \neq \gamma_{\frac{d}{2}}(T^*)$.

Since vertex v has 3 pendant neighbors in T, v and one child neighbor of v is in every $\gamma_{\frac{d}{2}}$-set of T. Since $|N_1(w')| > \left\lceil \frac{d(w')}{2} \right\rceil$, for any $\gamma_{\frac{d}{2}}$-set D of T, $D - \{w\}$ is a $\gamma_{\frac{d}{2}}$-set of T^*, where w is a pendant neighbor of v in D. Clearly, $\gamma_{\frac{d}{2}}(T^*-v) \neq \gamma_{\frac{d}{2}}(T^*)$. Now consider a vertex $u \in V(T^*) - \{v\}$. Let $T_1^*, T_2^*, \ldots, T_{d(u)}^*$ be the components of T^*-u, T_1^* be the component of T^*-u containing the vertex v, $T_1, T_2, \ldots, T_{d(u)}$ be the components of $T-u$ and T_1 be the component of $T-u$ containing the vertex v. Then, for all $2 \leq i \leq d(u)$, $T_i^* \cong T_i$ and any $\gamma_{\frac{d}{2}}$-set of T_i is a $\gamma_{\frac{d}{2}}$-set of T_i^*. Also, for any $\gamma_{\frac{d}{2}}$-set D' of T_1, $D' - \{v'\}$ is a $\gamma_{\frac{d}{2}}$-set of T_1^*, where v' is a pendant neighbor of v in D'. Hence, $\gamma_{\frac{d}{2}}(T^*)+1 = \gamma_{\frac{d}{2}}(T) \neq \gamma_{\frac{d}{2}}(T-u) = \sum_{i=2}^{d(u)} \gamma_{\frac{d}{2}}(T_i) + \gamma_{\frac{d}{2}}(T_1) = \sum_{i=2}^{d(u)} \gamma_{\frac{d}{2}}(T_i^*) + \gamma_{\frac{d}{2}}(T_1^*) + 1$ and $\gamma_{\frac{d}{2}}(T^*) \neq \gamma_{\frac{d}{2}}(T^*-u)$. Therefore, by induction assumption $T^* \in \psi$. Now, T can be constructed by joining three new vertices to v by the edge and by the properties of tree in ψ one can observe that $T \in \psi$.

Case ii Suppose w' has no child pendant neighbors. Then, each child neighbor of w' is of degree 4. Hence, for any $\gamma_{\frac{d}{2}}$-set D of T either $w' \in D$ with $|C_{w'}| \leq 1$ or $w' \in V - D$. Since vertex v has three pendant neighbors in T, v and one child neighbor of v is in every $\gamma_{\frac{d}{2}}$-set of T. Let $e = w'v$ and T^* be the component of $T - e$ containing vertex w'. Also, for any $\gamma_{\frac{d}{2}}$-set D of T, $D - \{w, v\}$ is a $\gamma_{\frac{d}{2}}$-set of T^*, where w is a pendant neighbor of v in D. Then, by the similar argument as in case 1 we can prove that, for any $u \in V(T^*)$, $\gamma_{\frac{d}{2}}(T^* - u) \neq \gamma_{\frac{d}{2}}(T^*)$ and hence $T \in \psi$.

Conversely, $T \in \psi = \psi_0 \cup \psi_1$. Let $v \in V$, u be the parent vertex of v, $T_1, T_2, \dots T_{d(v)}$ be the components of $T - v$ and T_1 be the component of $T - v$ containing u and D be a $\gamma_{\frac{d}{2}}$-set of T obtained from the Algorithm 5.6 in Chapter 5.

If $v \in V - D$, then by the construction of tree T in ψ, we can observe that $v \in C_u$, degree of u is even, u has a neighbor $v_1 \neq v$ such that $v_1 \in D$ with $C_{v_1} = \emptyset$. Suppose v has a child neighbor v_2. Then, $v_2 \in D$ (By the properties of tree in ψ, for any vertex $w \in V - D$, w is dominated by its parent vertex.) with $C_{v_2} \neq \emptyset$, $|C_{v_2}| = \left\lceil \frac{d(v_2)}{2} \right\rceil$, the degree of v_2 is even and v_2 has a child neighbor in D dominating itself. Therefore, if we apply the Algorithm 5.6 to each component of $T - v$, then $\gamma_{\frac{d}{2}}(T_i) = |D \cap V(T_i)|$ for all $2 \leq i \leq d(v)$ and $\gamma_{\frac{d}{2}}(T_1) < |D \cap V(T_1)|$. Hence, $\gamma_{\frac{d}{2}}(T - v) < \gamma_{\frac{d}{2}}(T)$.

If $v \in D$ and $C_v = \emptyset$, then by construction of tree T, $u \in D$ and u has a child neighbor in D which dominate itself. If v has a child neighbor v_1, then $v_1 \in D$ with $C_{v_1} \neq \emptyset$ and degree of v_1 is even. If we apply the Algorithm 5.6 to each component of $T - v$, then $\gamma_{\frac{d}{2}}(T_i) = |D \cap V(T_i)|$ for all $2 \leq i \leq d(v)$ and $\gamma_{\frac{d}{2}}(T_1) < |D \cap V(T_1)|$. Hence, $\gamma_{\frac{d}{2}}(T - v) < \gamma_{\frac{d}{2}}(T)$.

Suppose $v \in D$ and $C_v \neq \emptyset$. Let $T'_1, T'_2, \dots T'_{|C_v|}$ be the components of $T - v$ containing vertices of C_v. If we apply the Algorithm 5.6 to each component of $T - v$, then $\gamma_{\frac{d}{2}}(T'_i) > |D \cap V(T'_i)|$ for all $1 \leq i \leq |C_v|$ and $\gamma_{\frac{d}{2}}(T_x) = |D \cap V(T_x)|$ for all other components of $T - v$. Hence, $\gamma_{\frac{d}{2}}(T - v) > \gamma_{\frac{d}{2}}(T)$.

□

Theorem 6.2.20. *Let T be a tree of order n other than P_3, if $\gamma_{\frac{d}{2}}(T) \geq \gamma_{\frac{d}{2}}(T - v)$ for every vertex $v \in V$, then there exists a $\gamma_{\frac{d}{2}}$-set D of tree T satisfies the following conditions.*

1. *No two pendant vertices adjacent to any vertex in V.*
2. *$|C_u| \leq 2$ for any $u \in D$.*
3. *If $|C_u| = 2$, then one is child vertex of u and other one is parent vertex of u.*

Proof. Let T be a tree of order n other than P_3, if $\gamma_{\frac{d}{2}}(T) \geq \gamma_{\frac{d}{2}}(T - v)$ for every vertex $v \in V$. Since $T \neq P_3$, if two pendant vertices are adjacent to any vertex v, then $d(v) > 2$

and v can dominate both pendant neighbors. Hence, $\gamma_{\frac{d}{2}}(T) < \gamma_{\frac{d}{2}}(T-v)$. Let D be a $\gamma_{\frac{d}{2}}$-set of T obtained from Algorithm 5.6 and $u \in D$. If $|C_u| > 2$, then C_u has at least two child neighbors $\{u_1, u_2\}$ of u. Let $T_1, T_2, \ldots, T_{d(u)}$ be the components of $T-u$ and T_1, T_2 be the components of $T-u$ containing u_1, u_2, respectively. Then, by the Algorithm 5.6, $\gamma_{\frac{d}{2}}(T_i) > |D \cap V(T_i)|$ for $1 \leq i \leq 2$ and $\gamma_{\frac{d}{2}}(T_j) \geq |D \cap V(T_j)|$ for $3 \leq j \leq d(u)$. Hence, $\gamma_{\frac{d}{2}}(T) < \gamma_{\frac{d}{2}}(T-v)$, a contradiction. Similarly, we can prove the third statement of the hypothesis. □

Corollary 6.2.21. *For any tree T, if $\gamma_{\frac{d}{2}}(T) \geq \gamma_{\frac{d}{2}}(T-v)$ for every vertex $v \in V$, then $\gamma_{\frac{d}{2}}(T) \geq \frac{n}{3}$.*

Proof. Let D be $\gamma_{\frac{d}{2}}$-set of T. Then, $|C_u| \leq 2$ for any $u \in D$, which implies

$$2|D| \geq |V-D| = n - |D|$$

$$\Longrightarrow \gamma_{\frac{d}{2}}(T) \geq \frac{n}{3}$$

. □

6.3 CHANGE IN THE 2-PART DEGREE RESTRICTED DOMINATION NUMBER UPON EDGE REMOVAL

Theorem 6.3.1. *For any graph G and an edge $e \in E(G)$,*

$$\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(G-e) \leq \gamma_{\frac{d}{2}}(G)+2$$

.

Proof. Let $e = uv$ and D be a $\gamma_{\frac{d}{2}}$-set of $G-e$. Note that $\left\lceil \frac{d_{G-e}(u)}{2} \right\rceil = \left\lceil \frac{d_G(u)-1}{2} \right\rceil \leq \left\lceil \frac{d_G(u)}{2} \right\rceil$ and similarly this condition holds for vertex v. Hence, D is a 2-DRD set of G and $\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(G-e)$. Let D' be a $\gamma_{\frac{d}{2}}$-set of G. If $u, v \in V-D'$, then D' is a 2-DRD set of $G-v$. Since $\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(G-e)$, D' is a $\gamma_{\frac{d}{2}}$-set of $G-e$ and $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$. Suppose $u \in D$ and $v \in V-D$ (or $v \in D$ and $u \in V-D$). If $C_u = \emptyset$, then D' is a $\gamma_{\frac{d}{2}}$-set of $G-e$ and $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$. If not, since $\left\lceil \frac{d_G(u)}{2} \right\rceil \leq \left\lceil \frac{d_{G-e}(u)}{2} \right\rceil + 1$, $D \cup \bullet w\{$ is a 2-DRD set of $G-e$, for some $w \in C_u$ and $\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(G-e) \leq |D|+1 \leq \gamma_{\frac{d}{2}}(G)+2$. Suppose both $u, v \in D$. If $C_u = \emptyset$ (or $C_v = \emptyset$), then $C_v \neq \emptyset$ ($C_u \neq \emptyset$) and $D \cup \bullet w\{$ is a 2-DRD set of $G-e$, for some $w \in C_v$. Suppose $C_u \neq \emptyset$ and $C_v \neq \emptyset$. Then, since $\left\lceil \frac{d_G(u')}{2} \right\rceil \leq \left\lceil \frac{d_{G-e}(u')}{2} \right\rceil + 1$ for $u' \in V$, $D \cup \bullet w, w'\{$ is a 2-DRD set of $G-e$ for some $w \in C_v$ and $w' \in C_u$ and $\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(G-e) \leq |D|+2 = \gamma_{\frac{d}{2}}(G)+2$. □

Theorem 6.3.2. *Let G be a graph and $e = uv \in E(G)$ be an edge. Then, $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$ if and only if there exists a $\gamma_{\frac{d}{2}}$-set D of G satisfying at least one of the following conditions.*

1. *$u, v \in V - D$.*

2. *The degree of both vertices u and v are even.*

3. *If $u \in D$ and $v \in V - D$, then $v \notin C_u$ and $|C_u| \leq \left\lceil \frac{d_G(u)-1}{2} \right\rceil$.*

4. *If $u, v \in D$, then $|C_u| \leq \left\lceil \frac{d_G(u)-1}{2} \right\rceil$ and $|C_v| \leq \left\lceil \frac{d_G(v)-1}{2} \right\rceil$.*

Proof. Let $e = uv \in E(G)$ and D be a $\gamma_{\frac{d}{2}}$-set of G. Suppose $u, v \in V - D$. Then, clearly D is a 2-DRD set of $G-e$. Since $\gamma_{\frac{d}{2}}(G) \leq \gamma_{\frac{d}{2}}(G-e)$, D is a $\gamma_{\frac{d}{2}}$-set of $G-e$ and $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$. If the degree of both vertices u and v are even, then $\left\lceil \frac{d_G(w)}{2} \right\rceil = \left\lceil \frac{d_{G-e}(w)}{2} \right\rceil$ for every $w \in V$ and D is a $\gamma_{\frac{d}{2}}$-set of $G-e$ and $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$. Suppose D satisfies the third condition in the hypothesis. Then, since $v \notin C_u$ and $|C_u| \leq \left\lceil \frac{d_G(u)-1}{2} \right\rceil = \left\lceil \frac{d_{G-e}(u)}{2} \right\rceil$, D is a $\gamma_{\frac{d}{2}}$-set of $G-e$. Similarly, if fourth condition of the hypothesis holds, then also D is a $\gamma_{\frac{d}{2}}$-set of $G-e$ and $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$.

Assume that $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$. Let D be a $\gamma_{\frac{d}{2}}$-set of $G-e$. Then, D is a 2-DRD set of G. Since $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$, D is a $\gamma_{\frac{d}{2}}$-set of G. If $u, v \in V - D$, then the result holds. Suppose $u \in D$ and $v \in V - D$. Since $uv \notin E(G-e)$, $v \notin C_u$. Also, $|C_u| \leq \left\lceil \frac{d_{G-e}(u)}{2} \right\rceil = \left\lceil \frac{d_G(u)-1}{2} \right\rceil$. Similarly, if $u, v \in D$, then $|C_u| \leq \left\lceil \frac{d_{G-e}(u)}{2} \right\rceil = \left\lceil \frac{d_G(u)-1}{2} \right\rceil$ and $|C_v| \leq \left\lceil \frac{d_{G-e}(v)}{2} \right\rceil = \left\lceil \frac{d_G(v)-1}{2} \right\rceil$. □

Theorem 6.3.3. *For any graph G and an edge $e = uv \in E(G)$, if $\gamma_{\frac{d}{2}}(G-e) = \gamma_{\frac{d}{2}}(G)+2$, then vertices u and v satisfies the following conditions.*

1. *Vertices u, v are fixed with respect to graph G.*

2. *The degree of both vertices u and v are odd.*

3. *For any $\gamma_{\frac{d}{2}}$-set D of G such that $V - D = \cup_{w \in D} C_w$, $|C_u| = \left\lceil \frac{d_G(u)}{2} \right\rceil$ and $|C_v| = \left\lceil \frac{d_G(v)}{2} \right\rceil$.*

Proof. Assume that there exists a $\gamma_{\frac{d}{2}}$ set D' of G such that $u \in V - D'$. If $v \in V - D'$, then D' is a 2-DRD set of $G-e$, a contradiction. If $v \in D'$ and $C_v = \emptyset$, then D' is a 2-DRD set of $G-e$. If $C_v \neq \emptyset$, then $D' \cup \bullet w\{$ is a 2-DRD set of $G-e$ for some $w \in C_v$, a contradiction to the fact that $\gamma_{\frac{d}{2}}(G)+2 = \gamma_{\frac{d}{2}}(G-e)$. Hence, vertices u, v is in every $\gamma_{\frac{d}{2}}$-set of G.

Suppose the degree of u (or v) is even and D_1 is a $\gamma_{\frac{d}{2}}$-set of G. Then, by the first statement of the hypothesis $u, v \in D_1$. Since the degree of u is even, $\left\lceil \frac{d_G(u)}{2} \right\rceil = \left\lceil \frac{d_G(u)-1}{2} \right\rceil = \left\lceil \frac{d_{G-e}(u)}{2} \right\rceil$ and all the vertices dominated by u in G can be dominated by u in $G-e$. If $C_v \neq \emptyset$, then $D_1 \cup \bullet w\{$ is a 2-DRD set of $G-e$ for some $w \in C_v$, a contradiction. If $C_v = \emptyset$, then D_1 is a 2-DRD set of $G-e$, a contradiction. Therefore, the degree of both the vertices u, v are odd.

Suppose $|C_u| > \left\lceil \frac{d_G(u)}{2} \right\rceil$. Then, also $|C_u| \leq \left\lceil \frac{d_G(u)-1}{2} \right\rceil = \left\lceil \frac{d_{G-e}(u)}{2} \right\rceil$ and all the vertices dominated by u in G can be dominated by u in $G-e$. If all the vertices dominated by v in G is dominate by v in $G-v$, then D is a 2-DRD set of $G-e$. If not, then $D \cup \bullet w\{$ is a 2-DRD set of $G-e$ for some $w \in C_v$. Therefore, $\gamma_{\frac{d}{2}}(G-v) \leq \gamma_{\frac{d}{2}}(G)+1 > \gamma_{\frac{d}{2}}(G)+2$, a contradiction. Hence, the third statement of the hypothesis holds. □

Remark 6.3.4. *The converse of the Theorem 6.3.3 need not be true always. For example in Figure 6.8, it can be observed the degree of both the vertices u and v are odd. Since vertices u and v have two pendant neighbors, u and v are in every $\gamma_{\frac{d}{2}}$-set of G. Also, for any $\gamma_{\frac{d}{2}}$ set D of G such that $V-D = \cup_{w \in D} C_w$, $|C_u| = \left\lceil \frac{d_G(u)}{2} \right\rceil$ and $|C_v| = \left\lceil \frac{d_G(v)}{2} \right\rceil$. But $\gamma_{\frac{d}{2}}(G) = 3$ and $\gamma_{\frac{d}{2}}(G-e) = 4 = \gamma_{\frac{d}{2}}(G)+1$, where $e = uv \in E(G)$.*

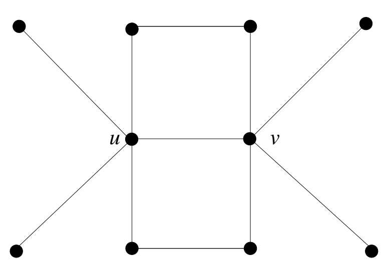


Figure 6.8 Graph G, a counter example for the converse of Theorem 6.3.3

Theorem 6.3.5. *Let G be a graph and $e = uv \in E(G)$ be an edge. Suppose vertex u and v satisfies the following conditions,*

1. *vertices u, v are fixed with respect to graph G,*
2. *for any $\gamma_{\frac{d}{2}}$ set D of G such that $V-D = \cup_{w \in D} C_w$, $|C_u| = \left\lceil \frac{d_G(u)}{2} \right\rceil$ and $|C_v| = \left\lceil \frac{d_G(v)}{2} \right\rceil$,*
3. *the degree of both vertices u and v are odd.*

Then, either $\gamma_{\frac{d}{2}}(G-e) = \gamma_{\frac{d}{2}}(G)+1$ *or* $\gamma_{\frac{d}{2}}(G-e) = \gamma_{\frac{d}{2}}(G)+2$.

Proof. Let G be a graph and $e = uv \in E(G)$ be an edge satisfying the all three conditions in the hypothesis. From Theorem 6.3.1 $\gamma_{\frac{d}{2}}(G-e) \leq \gamma_{\frac{d}{2}}(G)+2$. By Theorem 6.3.2 $\gamma_{\frac{d}{2}}(G-e) \neq \gamma_{\frac{d}{2}}(G)$. Hence, either $\gamma_{\frac{d}{2}}(G-e) = \gamma_{\frac{d}{2}}(G)+1$ or $\gamma_{\frac{d}{2}}(G-e) = \gamma_{\frac{d}{2}}(G)+2$. □

In this chapter, change in the 2-part degree restricted domination number of a graph by removing any vertex and edge are discussed. We characterized the class of trees for which $\gamma_{\frac{d}{2}}(T) \neq \gamma_{\frac{d}{2}}(T-v)$ for every $v \in V(T)$. We have given the necessary and sufficient conditions, when $\gamma_{\frac{d}{2}}(G) = \gamma_{\frac{d}{2}}(G-e)$, for any graph G and edge e . In addition to that some properties of fixed, free and totally free vertices are discussed.

CHAPTER 7

CONCLUSIONS AND FUTURE SCOPE

The thesis is mainly about a new domination parameter, k-part degree restricted domination, a new generalization of the domination problem.

Initially, in Chapter 2 and Chapter 3, a new parameter on dominating set, 2-part degree restricted domination is introduced and this concept has been generalized to k-part degree restricted domination for any positive integer k. Some bounds on $\gamma_{\frac{d}{k}}$ are found. As future work, it is planned to study the behavior of k-DRD set of graphs obtained by some other graph operators, such as cartesian product of two graphs.

In Chapter 4, the relationships between k-DRD set and some other domination invariants, such as domination, k-domination and efficient domination are studied. An algorithm to verify whether the given dominating set is a k-DRD set or not is also discussed. As future work, a study of relationship between k-DRD set and some other new domination invariants is being considered.

It is well known and generally accepted that the problem of determining the domination number of an arbitrary graph is difficult and this problem is NP-complete. In Chapter 5, it is shown that problem of finding the k-part degree restricted domination number of an arbitrary graph is NP-complete and an algorithm to find a minimal k-DRD set of a general graph is given. Also, a study of the classes of graphs for which the problem can be solved in polynomial time has been given attention and proved that k-part degree restricted domination is NP-complete for bipartite graphs, chordal graphs, undirected path graphs, chordal bipartite graphs, circle graphs, planar graphs and split graphs. An exponential time algorithm to find 2-part degree restricted domination number of interval graphs and a polynomial time algorithm to find 2-part degree restricted domination number of trees are discussed. As future work, it is planned to obtain a polynomial time algorithm to find k-part degree restricted domination number of interval graphs, directed path graphs and block graphs and other smaller classes of graphs. The k-part degree restricted domination is further explored to study the critical aspects. In Chapter 6, the variation or the change in the 2-part degree restricted domination number upon the removal of any vertex and edge are discussed. As a future work, change in the 2-part degree restricted domination number by adding a new edge can be studied. Also, as defined in the Chapter 1 Introduction, one can define the six classes of graphs CVR, CER, UVR, UER, CEA, UEA depending on the change in the 2-part degree restricted domination number by removing a vertex or an edge or by adding an edge and characterize the graphs among these six classes. The study of this concept can be extended to any positive integer $k > 2$.

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LIST OF PUBLICATIONS AND CONFERENCE PAPERS

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1. S. S. Kamath, A. Senthil Thilak and Rashmi M, *2-Part Degree Restricted Domination in Graphs*, In: proceedings of the International Workshop and Conference on Analysis and Applied Mathematics, IWCAAM 16, 211 – 221.

2. S. S. Kamath, A. Senthil Thilak and Rashmi M, Relation between k-DRD set and Dominating set, In: Applied Mathematics and Scientific Computing, Trends in Mathematics, 2019, 563–572. Springer. https://doi.org/10.1007/978-3-030-01123-9_56

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1. S. S. Kamath, A. Senthil Thilak and Rashmi M. *2-Part Degree Restricted Domination in Graphs*, International Workshop and Conference on Analysis and Applied Mathematics, NIT Tiruchirappalli, June 06-10, 2016.

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3. S. S. Kamath, A. Senthil Thilak and Rashmi M. *Some Bounds on 2-Part Degree Restricted Domination Number of a Graph*, 1^{st} International Conference on "Discrete Mathematics and Data Sciences, SASTRA Deemed to be University, Thanjavur, Tamil Nadu, September, 28-29, 2018.

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