Projection Scheme for Newton-Type Iterative Method for Lavrentiev Regularization

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Abstract. In this paper we consider the finite dimensional realization of a Newton-type iterative method for obtaining an approximate solution to the nonlinear ill-posed operator equation $F(x) = f$, where $F : D(F) \subseteq X \to X$ is a nonlinear monotone operator defined on a real Hilbert space X. It is assumed that $F(\hat{x}) = f$ and that the only available data are f^δ with $\|f - f^\delta\| \leq \delta$. It is proved that the proposed method has a local convergence of order three. The regularization parameter α is chosen according to the balancing principle considered by Perverzev and Schock (2005) and obtained an optimal order error bounds under a general source condition on $x_0 - \hat{x}$ (here x_0 is the initial approximation). The test example provided endorses the reliability and effectiveness of our method.

Keywords: Newton Lavrentiev method, nonlinear ill-posed operator equation, nonlinear monotone operator, balancing principle, finite dimensional.

1 Introduction

Throughout this paper X is a real Hilbert space and $F : D(F) \subseteq X \to X$ is a monotone operator, i.e.,

$$\langle F(x) - F(y), x - y \rangle \geq 0, \qquad \forall x, y \in D(F).$$

The inner product and the norm in X are denoted by $\langle ., . \rangle$ and $\|.\|$ respectively. We consider the problem of approximately solving the ill-posed operator equation

$$F(x) = f \tag{1}$$

in the finite dimensional setting.

Let $S := \{x : F(x) = f\}$. Then S is closed and convex if F is monotone and continuous (see, e.g., [10]) and hence has a unique element of minimal norm, denoted by $\hat{x}$ such that $F(\hat{x}) = f$.

We assume that F possesses a locally uniformly bounded, self adjoint Fréchet derivative $F'(.)$ (i.e., there exists some constant C_F such that $\|F'(x)\| \leq C_F$) in the domain $D(F)$ of F. Note that since F is monotone, $F'(.) \geq 0$, i.e., $F'(.)$ is

J. Mathew et al. (Eds.): ICECCS 2012, CCIS 305, pp. 302–310, 2012.

a positive self adjoint operator and hence $(F'(.) + \alpha I)^{-1}$ exists for any $\alpha > 0$. In application, usually only noisy data f^δ are available, such that $\|f - f^\delta\| \leq \delta$. Since (1) is ill-posed, the regularization methods are used to obtain a stable approximate solution for (1).

The Lavrentiev regularization method (see [1,6,11,12]) is used for appropriately solving (1) when F is monotone. In this method the regularized approximation x_α^δ is obtained by solving the operator equation

$$F(x) + \alpha(x - x_0) = f^\delta \tag{2}$$

where $\alpha > 0$ is the regularization parameter and $x_0 \in D(F)$ is a known initial approximation of the solution $\hat{x}$. From the general regularization theory it is known that the equation (2) has a unique solution x_α^δ for any $\alpha > 0$ and $x_\alpha^\delta \to \hat{x}$ as $\alpha \to 0, \delta \to 0$ provided α is chosen appropriately (see, [9] and [12]).

In [3], the authors considered a Two Step Newton Lavrentiev Method (TSNLM) for approximating the solution x_α^δ of (2). In this paper we consider the finite dimensional realization of the method considered in [3].

This paper is organized as follows. In section 2, we set up the method and analyze its convergence. The error analysis under a general source condition is considered in Section 3. The numerical example and the computational results are presented in section 4. Finally a conclusion is made in section 5.

2 The Method and Its Convergence

The purpose of this section is to obtain an approximate solution for the equation (2), in the finite dimensional subspace of X. Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X.

Let $\varepsilon_h := \|F'(x)(I - P_h)\|, \quad \forall x \in D(F)$ and $\{b_h : h > 0\}$ be such that $lim_{h\to 0}\frac{\|(I-P_h)x_0\|}{b_h} = 0$ and $lim_{h\to 0} b_h = 0$. We assume that $\varepsilon_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ pointwise and if $F'(x)$ is a compact operator. Further we assume that $\varepsilon_h \leq \varepsilon_0$, $b_h \leq b_0$ and $\delta \in (0, \delta_0]$.

2.1 Projection Method

Let $x_{0,\alpha}^{h,\delta} := P_h x_0$ be the projection of the initial guess x_0 on to $R(P_h)$, the range of P_h and let $R_\alpha(x) := P_h F'(x) P_h + \alpha P_h$ with $\alpha > \alpha_0 > 0$. We define the iterative sequence as:

$$y_{n,\alpha}^{h,\delta} = x_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{n,\alpha}^{h,\delta}) P_h [F(x_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(x_{n,\alpha}^{h,\delta} - x_0)] \tag{3}$$

and

$$x_{n+1,\alpha}^{h,\delta} = y_{n,\alpha}^{h,\delta} - R_\alpha^{-1}(x_{n,\alpha}^{h,\delta}) P_h [F(y_{n,\alpha}^{h,\delta}) - f^\delta + \alpha(y_{n,\alpha}^{h,\delta} - x_0)]. \tag{4}$$

Note that the iteration (3) and (4) are the finite dimensional realization of the iteration (3) and (4) in [3]. We will be selecting the parameter $\alpha = \alpha_i$ from some finite set $D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N\}$ using the adaptive method considered by Perverzev and Schock in [9].

The following assumptions and Lemmas are used for proving our results.