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A class of two-step P-stable methods for the accurate integration of second order periodic initial value problems

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Abstract: In this paper we consider a two parameter family of two-step methods for the accurate numerical integration of second order periodic initial value problems. By applying the methods to the test equation $y''+\lambda^2 y=0$, we determine the parameters α, β so that the phase-lag (frequency distortion) of the method is minimal. The resulting method is a P-stable method with a minimal phase-lag $\lambda^6 h^6/42000$. The superiority of the method over the other P-stable methods is illustrated by a comparative study of the phase-lag errors and by illustrating with a numerical example.

1. Introduction

The development of numerical integration formulae for the direct integration of the periodic initial value problem

$$y''=f(t, y), \qquad y(t_0)=y_0, \quad y'(t_0)=y'_0 \tag{1.1}$$

has created considerable interest in the recent years. Symmetric linear two-step methods for the direct integration of (1.1) are of the form

$$y_{n+1}-2y_n+y_{n-1}=h^2\left[\beta_0 f_{n+1}+(1-2\beta_0)f_n+\beta_0 f_{n-1}\right]. \tag{1.2}$$

Lambert and Watson [8] have originated the concept of interval of periodicity of a numerical method and connected it with the symmetry property of linear multistep methods. They proposed the trapezoidal method ((1.2) with $\beta_0=\frac{1}{2}$) which is of order two and has interval of periodicity $(0, \infty)$ and defined it to be P-stable method. Dahlquist [5] proposed the unconditionally stable method ((1.2) with $\beta_0=\frac{1}{4}$) which is of order two and has shown that the order of an unconditionally stable method can not exceed two and the method must be implicit. However higher order P-stable methods have been discussed by Hairer [7], Costabile [4] and Chawla [2,3].

The P-stable Noumerov-type methods of order four presented by Chawla [2] are of the form

$$\begin{aligned}
\bar{y}_{n+1}&=y_{n+1}-\alpha h^2(f_{n+2}-2f_{n+1}+f_n),\\
\bar{f}_{n+1}&=f(t_{n+1}, \bar{y}_{n+1}),\\
y_{n+2}-2y_{n+1}+y_n&=\frac{h^2}{12}(f_{n+2}+10\bar{f}_{n+1}+f_n)
\end{aligned} \tag{1.3}$$

with the truncation error

$$\mathrm{T}E = -\frac{h^6}{240}\left[y_n^{(6)} - 200\alpha y_n^{(4)}\left(\frac{\partial f}{\partial y}\right)_n\right] + \mathrm{O}(h^8). \tag{1.4}$$

The one parameter family of methods (1.3) denoted by $M_4(\alpha)$ are P-stable for $\alpha \geqslant \frac{1}{120}$.

Costabile [4] proposed the P-stable method of order four in the form:

$$y_{n+1} - 2\bar{y}_n + y_{n-1} = \tfrac{1}{8}h^2(f_n + f_{n+1}), \qquad \bar{f}_n = f(t_n, \bar{y}_n),$$
$$y_{n+1} - 2y_n + y_{n-1} = -\tfrac{1}{12}h^2[f_{n+1} + f_{n-1} + 2f_n - 8(\bar{f}_n + \bar{f}_{n-1})]. \tag{1.5}$$

In this paper we propose a class of two parameter two-step methods of order two denoted by $M_2(\alpha, \beta)$. By applying the methods to the test equation

$$y'' + \lambda^2 y = 0, \qquad y(t_0) = y_0, \quad y'(t_0) = y_0' \tag{1.6}$$

we determine the parameters α, β so that the phase-lag (frequency distortion) for the method is minimal. The resulting method is a P-stable method with minimal phase-lag $P(H) = \lambda^6 h^6/42000$. The superiority of the method over the other P-stable methods is illustrated by a comparative study of the phase-lag errors and by illustrating with a numerical example.

2. Two-step P-stable method with minimal phase-lag

For $h > 0$, let $t_n = t_0 + nh$, $n = 0, 1, 2, \ldots$.

Set $y_n = y(t_n)$, $f_n = f(t_n, y_n)$.

Let $\bar{y}_{n+1} = y_{n+1} - \beta h^2(f_{n+1} + 2f_n + f_{n-1})$. (2.1a)

Set $\bar{f}_{n+1} = f(t_{n+1}, \bar{y}_{n+1})$.

Let $\bar{\bar{y}}_{n+1} = y_{n+1} - \alpha h^2(\bar{f}_{n+1} - 22f_n + f_{n-1})$. (2.1b)

Set $\bar{\bar{f}}_{n+1} = f(t_{n+1}, \bar{\bar{y}}_{n+1})$.

Now we consider the modification of the method (1.2) with $\beta_0 = \frac{1}{20}$ as given by

$$y_{n+1} - 2y_n + y_{n-1} = \frac{h^2}{20}(\bar{\bar{f}}_{n+1} + 18f_n + f_{n-1}) \tag{2.1c}$$

with the truncation error

$$\mathrm{TE} = \frac{h^4}{30}\left[y_n^{(4)} - 30\alpha y_n''\left(\frac{\partial f}{\partial y}\right)_{n+1}\right] - \frac{h^6}{720}\left[y_n^{(6)} - 36\alpha y_n^{(4)}\left(\frac{\partial f}{\partial y}\right)_{n+1} + 144\alpha\beta y_n''\left(\frac{\partial f}{\partial y}\right)^2_{n+1}\right] + \mathrm{O}(h^8). \tag{2.2}$$

Applying the method (2.1) to the test equation (1.6) we obtain the stability polynomial

$$p(\xi) = A(H)\xi^2 - 2B(H)\xi + A(H) \tag{2.3}$$

where

$$A(H)=1+\frac{H^2}{20}+\frac{\alpha H^4}{20}+\frac{\alpha\beta H^6}{20}, \tag{2.4}$$

$$B(H)=1-\frac{9H^2}{20}+\frac{11\alpha H^4}{20}-\frac{\alpha\beta H^6}{20} \tag{2.5}$$

with $H^2=\lambda^2h^2$.

The roots of (2.3) are complex and of modulus one if

$$|B(H)/A(H)|<1. \tag{2.6}$$

Let the roots of (2.3) be

$$\xi_{1,2}=e^{\pm i\theta(H)}$$

when (2.6) is satisfied. We define the phase-lag of a numerical method as the leading coefficient of

$$\left|\frac{\theta(H)-H}{H}\right| \tag{2.7}$$

and denote it by $P(H)$.

For the methods $M_2(\alpha, \beta)$ given by (2.1) we obtain

$$\tan\theta(H)=\left[A^2(H)-B^2(H)\right]^{1/2}/B(H). \tag{2.8}$$

Substituting for $A(H)$, $B(H)$ from (2.4), (2.5) and simplifying we obtain

$$\tan\theta(H)=H+\frac{(7-10\alpha)}{20}H^3+\frac{(61-230\alpha+40\alpha\beta-50\alpha^2)}{400}H^5$$
$$+\frac{(545-3470\alpha+1250\alpha^2-500\alpha^3+600\alpha\beta+400\alpha^2\beta)}{8000}H^7. \tag{2.9}$$

Using the expansion

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$$

we obtain the phase-lag of the methods $M_2(\alpha, \beta)$ given by (2.1) as

$$P(H)=\left|\frac{\theta(H)-H}{H}\right|$$
$$=\left|\frac{(1-30\alpha)H^2}{60}+\frac{[1-30\alpha+10\alpha(4\beta-5\alpha)]H^4}{400}\right.$$
$$\left.+\frac{[15-490\alpha+350\alpha(2\alpha-1)(4\beta-5\alpha)]H^6}{56000}\right| \tag{2.10}$$

When $\alpha=\frac{1}{30}$, $\beta=\frac{1}{24}$ we obtain the phase-lag of the method (2.1) as $P(H)=H^6/42000$, which is minimal. Also we note that for $\alpha=\frac{1}{30}$, $\beta=\frac{1}{24}$ the P-stability condition (2.6) is satisfied. The method $M_2(\frac{1}{30}, \frac{1}{24})$ is a P-stable method with minimal phase-lag $P(H)=H^6/42000$. For $\alpha\geqslant\frac{1}{30}$, $\beta\geqslant 5\alpha/4$, the condition (2.6) is satisfied and hence the two parameter family of methods $M_2(\alpha, \beta)$ are P-stable methods.

3. Comparative study of phase-lag errors

The frequency distortion of the trapezoidal method and the Dahlquist method is quite substantial being of the size $H^2/12$ while for the Noumerov's method ($\beta_0 = \frac{1}{12}$ in (1.2)) the phase-lag is $H^4/480$ (see [1,6]). The trapezoidal and Dahlquist methods are P-stable second order methods where as Noumerov method is a fourth order method with interval of periodicity (0, 2.449). The two-step P-stable method (1.5) of order four proposed by Costabile [5] has the phase-lag $P(H) = H^2/12$. The P-stable methods $M_4(\alpha)$ of order four given by (1.3) for $\alpha \geqslant \frac{1}{120}$ have the phase-lag $P(H) = |(200\alpha - 1)/480|H^4$. When $\alpha = \frac{1}{120}$, the method (1.3) has the phase-lag $P(H) = H^4/720$ which is equivalent to the phase-lag error of the P-stable fourth order Hairer's method [7]. When $\alpha = \frac{1}{200}$, the method $M_4(\frac{1}{200})$ given by (1.3) has the phase-lag $P(H) = H^6/12096$ and the interval of periodicity is (0, 2.71), (see [3]). The new method $M_2(\frac{1}{30}, \frac{1}{24})$ defined by (2.1) is a P-stable method with minimal phase-lag $P(H) = H^6/42000$.

4. Polynomial order of the methods

A numerical method is said to be of polynomial order p if it integrates $y(t) = t^k$, $k = 0, 1, 2, \ldots, p$ exactly. We find that the trapezoidal method and Dahlquist method are P-stable methods of polynomial order two. The methods proposed by Costabile (1.5) and Hairer [7] are also P-stable methods of polynomial order four. The family $M_4(\alpha)$ for $\alpha \geqslant \frac{1}{120}$ proposed by Chawla [2,3] are P-stable methods of polynomial order four with phase-lag $P(H) = |(200\alpha - 1)/480|H^4$. For $\alpha = \frac{1}{200}$, it becomes a method of polynomial order six with phase error $P(H) = H^6/12096$ and interval of periodicity (0, 2.71), (see [3]). The method $M_2(\frac{1}{30}, \frac{1}{24})$ is a P-stable method of polynomial order six. Thus M_2 $(\frac{1}{30}, \frac{1}{24})$ has the special characteristics of being P-stable with minimal phase-lag $P(H) = H^6/42000$ and integrating polynomials of order six.

5. Numerical example

We consider the test problem

$$y'' + \lambda^2 y = 0, \qquad y(0) = 1, \quad y'(0) = 0 \tag{5.1}$$

with $\lambda^2 = 25$. The problem is solved by using the methods

(i) $M_2(\frac{1}{30}, \frac{1}{24})$ given by (2.1);
(ii) $M_4(\frac{1}{120})$ given by (1.3);
(iii) $M_4(\frac{1}{200})$ given by (1.3) and;
(iv) Costabile method given by (1.5).

Using the steplength $h = \frac{1}{12}\pi$, the absolute errors in the solution $y(t)$ are tabulated for $t = 2\pi(2\pi)10\pi$, in Table 1.

6. Conclusions

The methods $M_2(\alpha, \beta)$ are P-stable methods of order two for $\alpha > \frac{1}{30}$ and $\beta > 5\alpha/4$. The method $M_2(\frac{1}{30}, \frac{1}{24})$ is a P-stable method with minimal phase-lag $P(H) = H^6/42000$ of poly-

Table 1
Absolute errors in $y(t)$ for the problem (5.1)

t	New method (2.1) $M_2(\frac{1}{30}, \frac{1}{24})$	Hairer's method (1.3) $M_4(\frac{1}{120})$	Chawla's method (1.3) $M_4(\frac{1}{200})$	Costabile method (1.5)
π	2.23 (−07)	1.38 (−03)	2.07 (−05)	1.62 (00)
2π	9.87 (−07)	6.07 (−03)	9.12 (−05)	1.07 (00)
4π	4.11 (−06)	2.53 (−02)	3.81 (−04)	2.02 (00)
6π	9.39 (−06)	5.75 (−02)	8.70 (−04)	1.22 (00)
8π	1.68 (−05)	1.02 (−01)	1.56 (−03)	4.08 (−02)
10π	2.64 (−05)	1.59 (−01)	2.44 (−03)	5.17 (−01)

nomial order six. The method remains to be a two-step method with increased stability facilities and polynomial orders, with minimal phase-lag.

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