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Additive parameters methods for the numerical integration of $y'' = f(t, y, y')$

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Abstract

In this paper numerical methods for the initial value problems of general second order differential equations are derived. The methods depend upon the parameters p and q which are the new additional values of the coefficients of y' and y in the given differential equation. Here, we report a new two step fourth order method. As p tends to zero and $q \geqslant (2\pi/h)^2$ the method is absolutely stable. Numerical results are presented for Bessel's, Legendre's and general second order differential equations.

Keywords: General second order initial value problems; Additive parameters; Absolutely stable

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1. Introduction

We consider the general second order differential equation

$$y'' = f(t, y, y') \tag{1.1}$$

subject to the initial conditions

$$y(t_0) = y_0, \qquad y'(t_0) = y'_0 \tag{1.2}$$

For finding the numerical solution of (1.1) by finite differences, we can either reduce (1.1) to a system of two first order differential equations and then apply the standard methods for first order equations or alternatively, have direct methods without reducing (1.1) into a system. Various single step direct methods have been proposed in the literature, the most notable among them being the classical Runge–Kutta–Nyström method, the Hubolt method, the Wilson θ-method and the

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Newmark average acceleration method which are a few linear multistep methods used by structural engineers in the analysis of structure (see [3]).

The additive parameters methods are the numerical methods which contain arbitrary parameters p and q. The methods depend upon the parameters p and q which are the new additional values of the coefficients of y' and y. As p tends to zero and $q \geqslant (2\pi/h)^2$, the method is absolutely stable when applied to the test equation

$$y'' + py' + qy = 0. \tag{1.3}$$

The method has been applied to solve Bessels, Legendre's and general second order differential equations. The coefficients of Φ_{n+1}, Φ_n and Φ_{n-1} are in terms of the parameters p and q.

2. Derivation of the methods

We write (1.1) in the form

$$y'' + py' + qy = \Phi(t, y, y'), \tag{2.1}$$

where

$$\Phi(t, y, y') = f(t, y, y') + py' + qy \tag{2.2}$$

with $p > 0$, $q > 0$ and $p^2 - 4q < 0$.

Eq. (2.1) can be written as

$$y'' + py' + qy = g(t), \tag{2.3}$$

where $g(t)$ is an approximation to $\Phi(t, y, y')$.

The general solution of (2.3) is

$$y(t) = A\mathrm{e}^{\sigma_1 t} + B\mathrm{e}^{\sigma_2 t} + \frac{1}{\sigma_2 - \sigma_1}\int_{t_n}^{t} (\mathrm{e}^{\sigma_2(t-z)} - \mathrm{e}^{\sigma_1(t-z)})g(z)\,\mathrm{d}z, \tag{2.4}$$

where $\sigma_1 = u + \mathrm{i}v$ and $\sigma_2 = u - \mathrm{i}v$ are the complex roots of the characteristic equation $m^2 + pm + q = 0$; where $u = -p/2$ and $v = \sqrt{(4q - p^2)}/2$; A and B are the arbitrary constants.

Differentiating (2.4) with respect to t, we have

$$y'(t) = \sigma_1 A\mathrm{e}^{\sigma_1 t} + \sigma_2 B\mathrm{e}^{\sigma_2 t} + \frac{1}{\sigma_2 - \sigma_1}\int_{t_n}^{t} (\sigma_2\mathrm{e}^{\sigma_2(t-z)} - \sigma_1\mathrm{e}^{\sigma_1(t-z)})g(z)\,\mathrm{d}z. \tag{2.5}$$

Eliminating A and B by substituting $t = t_{n+1}$, t_n and t_{n-1} in (2.4) and (2.5) we obtain

$$\begin{aligned} &y(t_{n+1}) - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h})y(t_n) + \mathrm{e}^{-ph}y(t_{n-1}) \\ &\quad = \frac{1}{\sigma_2 - \sigma_1}\int_{t_n}^{t_{n+1}} (\mathrm{e}^{\sigma_2(t_{n+1}-z)} - \mathrm{e}^{\sigma_1(t_{n+1}-z)})(g(z) + \mathrm{e}^{-ph}g(2t_n - z))\,\mathrm{d}z, \end{aligned} \tag{2.6}$$

$$\begin{aligned} &y'(t_{n+1}) - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h})y'(t_n) + \mathrm{e}^{-ph}y'(t_{n-1}) \\ &\quad = \frac{1}{\sigma_2 - \sigma_1}\int_{t_n}^{t_{n+1}} (\sigma_2\mathrm{e}^{\sigma_2(t_{n+1}-z)} - \sigma_1\mathrm{e}^{\sigma_1(t_{n+1}-z)})(g(z) + \mathrm{e}^{-ph}g(2t_n - z))\,\mathrm{d}z. \end{aligned} \tag{2.7}$$

Implicit Multistep Method

We replace $\Phi(t, y, y')$ with the Newton Backward difference interpolation polynomial of degree k at the points $t_{n+1}, t_n, t_{n-1}, \dots, t_{n-k+1}$

$$g(t) = \phi_{n+1} + \frac{(t - t_{n+1})}{h} \nabla\phi_{n+1} + \frac{(t - t_{n+1})(t - t_n)}{2!h^2} \nabla^2\phi_{n+1} + \cdots$$
$$+ \frac{(t - t_{n+1})(t - t_n)\cdots(t - t_{n-k+2})}{k!h^k} \nabla^k\phi_{n+1}$$
$$+ \frac{(t - t_{n+1})(t - t_n)\cdots(t - t_{n-k+1})}{(k+1)!} \phi^{(k+1)}(\xi); \quad t_n < \xi < t_{n+1}. \tag{2.8}$$

Neglecting the error term in (2.8) and substituting $g(t)$ in (2.6) we obtain

$$y_{n+1} - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h})y_n + \mathrm{e}^{-ph}y_{n-1}$$
$$= \frac{h}{\sigma_1 - \sigma_2}\int_0^1 [\mathrm{e}^{\sigma_2(1-s)h} - \mathrm{e}^{\sigma_1(1-s)h}] \sum_{m=0}^{k} (-1)^m \left[\binom{-s}{m} + \binom{s}{m}\mathrm{e}^{-ph}\right] \nabla^m\phi_{n+1}\mathrm{d}s, \tag{2.9}$$

where $(t - t_n)/h = s$.

When $m = 2$ this can be written in the form

$$y_{n+1} - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h})y_n + \mathrm{e}^{-ph}y_{n-1} = a_0\phi_{n+1} + a_1\phi_n + a_2\phi_{n-1}, \tag{2.10}$$

where $\phi_n = py'_n + qy_n + f(t_n, y_n, y'_n)$.

The coefficients a_0, a_1 and a_2 can be determined using the undetermined coefficients method. Putting $y_n = 1$, $y_n = t_n$, $y_n = t_n^2$ and $t_n = 0$, $t_n^2 = 0$, we get

$$a_0 + a_1 + a_2 = \frac{1}{q}[1 - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h}) + \mathrm{e}^{-ph}], \tag{2.11a}$$

$$(p + qh)a_0 + pa_1 + (p - qh)a_2 = h(1 - \mathrm{e}^{-ph}), \tag{2.11b}$$

$$(2hp + 2 + qh^2)a_0 + 2a_1 + (-2hp + 2 + qh^2)a_2 = h^2(1 + \mathrm{e}^{-ph}). \tag{2.11c}$$

Solving a_0, a_1 and a_2 we obtain

$$a_0 = \frac{1}{2q^3h^2}[2(1 - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h}) + \mathrm{e}^{-ph})(p^2 - q)$$
$$- pqh(3 - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h}) - \mathrm{e}^{-ph}) + 2q^2h^2],$$

$$a_1 = -\frac{1}{2q^3h^2}[4(1 - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h}) + \mathrm{e}^{-ph})(p^2 - q)$$
$$- 4pqh(1 - \mathrm{e}^{-ph}) + 2q^2h^2(\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h})],$$

$$a_2 = \frac{1}{2q^3h^2}[2(1 - (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h}) + \mathrm{e}^{-ph})(p^2 - q)$$
$$- pqh(1 + (\mathrm{e}^{\sigma_1 h} + \mathrm{e}^{\sigma_2 h}) - 3\mathrm{e}^{-ph}) + 2q^2h^2\mathrm{e}^{-ph}].$$

Similarly, replace $g(t)$ using Newton Backward difference polynomial (2.8) in (2.7) and neglecting the error term we get

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1}$$
$$= \frac{h}{\sigma_1 - \sigma_2}\int_0^1 [\sigma_2 e^{\sigma_2(1-s)h} - \sigma_1 e^{\sigma_1(1-s)h}] \sum_{m=0}^{k} (-1)^m\left[\binom{-s}{m} + \binom{s}{m}e^{-ph}\right] \nabla^m \phi_{n+1}\,ds. \tag{2.12}$$

When $m = 2$ using multistep implicit form this can be written as

$$y'_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y'_n + e^{-ph}y'_{n-1} = b_0\phi_{n+1} + b_1\phi_n + b_2\phi_{n-1}. \tag{2.13}$$

The coefficients b_0, b_1 and b_2 can be found out using the undetermined coefficients method

$$b_0 + b_1 + b_2 = 0, \tag{2.14a}$$

$$(p + qh)b_0 + pb_1 + (p - qh)b_2 = (1 - (e^{\sigma_1 h} + e^{\sigma_2 h}) + e^{-ph}), \tag{2.14b}$$

$$(2hp + qh^2 + 2)b_0 + 2b_1 + (-2hp + qh^2 + 2)b_2 = 2h(1 - e^{-ph}). \tag{2.14c}$$

Solving b_0, b_1 and b_2 we get

$$b_0 = \frac{1}{2q^2h^2}[(1 - (e^{\sigma_1 h} + e^{\sigma_2 h}) + e^{-ph})(qh - 2p) + 2hq(1 - e^{-ph})],$$

$$b_1 = \frac{1}{2q^2h^2}[(1 - (e^{\sigma_1 h} + e^{\sigma_2 h}) + e^{-ph})4p - 4hq(1 - e^{-ph})],$$

$$b_2 = -\frac{1}{2q^2h^2}[(1 - (e^{\sigma_1 h} + e^{\sigma_2 h}) + e^{-ph})(qh + 2p) - 2hq(1 - e^{-ph})].$$

3. Truncation error and order of the methods

The linear difference operator L is defined by

$$L[y(t_n), h] = C_0 y(t_n) + C_1 h y'(t_n) + C_2 h^2 y''(t_n) + \cdots$$

We get $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0$ and $C_6 = (49/23232)$ when $p = 0$ and $q = (2\pi/h)^2$;

$$\text{truncation error} = \frac{49}{23232}h^6 y^6(t_n) + O(h^8). \tag{3.1}$$

4. Stability

Applying the method to the test equation (1.3), we obtain

$$y_{n+1} - (e^{\sigma_1 h} + e^{\sigma_2 h})y_n + e^{(\sigma_1 + \sigma_2)h}y_{n-1} = 0. \tag{4.1}$$

Definition. The linear multistep method is said to be *absolutely stable* if the roots of the characteristic equation is of modulus less than one for all values of the step length h.

The characteristic root ξ of the test equation (1.3) is given by

$$\xi^2 - (e^{\sigma_1 h} + e^{\sigma_2 h})\xi + e^{(\sigma_1+\sigma_2)h} = 0. \tag{4.2}$$

Since σ_1 and σ_2 are complex roots (i.e., $\sigma_{1,2} = u \pm iv$) we obtain

$$\xi^2 - 2e^{uh}(\cos vh)\xi + e^{2uh} = 0. \tag{4.3}$$

From this, we find the roots $\xi_{1,2} = e^{uh}(\cos vh \pm i \sin vh)$ are of modulus less than one since $e^{uh} = e^{-ph/2} \leqslant 1$.

5. Numerical results

Numerical results are presented for the following initial value problems to illustrate the order, accuracy and implementational aspects of the methods (2.10) and (2.13).

Problem 1. Consider Legendre's equation

$$(1 - t^2)y'' - 2ty' + n(n+1)y = 0, \tag{5.1}$$

when $n = 4$, $y(t) = p_4(t) = (35t^4 - 30t^2 + 3)/8$.

The problem is solved using the methods (2.10) and (2.13). The absolute errors in the numerical solution for the problem (5.1) are tabulated in Table 1.

Problem 2. Consider Bessel's differential equation

$$t^2y'' + ty' + (t^2 - 0.25)y = 0, \tag{5.2}$$

with $y(t) = j_{1/2}(t) = \sqrt{(2/\pi t)}\sin t$ as the exact solution.

The problem is solved using the methods (2.10) and (2.13). The absolute errors in $y(t)$ are presented in Table 1.

Table 1

h	Error at $t = 8.0$ when $p = 0.1$ and $q = (100\pi/h)^2$		
	Problem 1 $O(h^4)$	Problem 2 $O(h^4)$	Problem 3 $O(h^4)$
2^{-3}	0.66865323(−06)	0.69488559(−09)	0.63807596(−06)
2^{-4}	0.42011379(−07)	0.43222648(−10)	0.42313559(−07)
2^{-5}	0.26266207(−08)	0.26905145(−11)	0.27246756(−08)
2^{-6}	0.16370905(−09)	0.16792123(−12)	0.17303137(−09)
2^{-7}	0.18189894(−10)	0.10491608(−13)	0.10857093(−10)

Problem 3. We consider the general second order differential equation

$$(1 + t)y'' + 2y' - (1 + t)y = 0 \tag{5.3}$$

with exact solution $y(t) = e^t/(1 + t)$.

The problem is solved using the methods (2.10) and (2.13). The absolute errors in the solution for the problem (5.3) are given in Table 1, which shows that they are of order four.

6. Conclusion

The numerical results presented for linear problems show that the methods are of order four and absolutely stable when the parameters p and q are chosen as the coefficients of y' and y in the given differential equation.

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