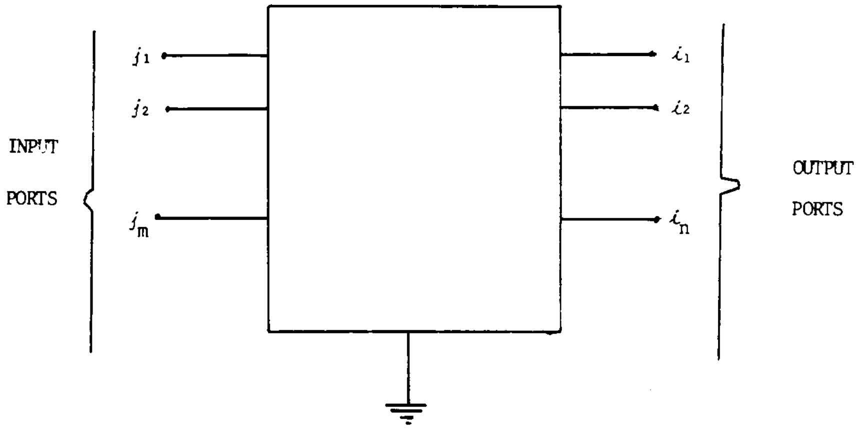

Fig. 1.

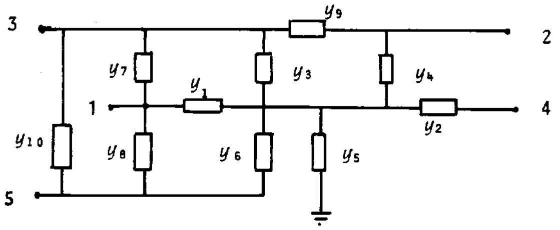

Fig. 2. Example: Input ports: 2, 4, 5; outputs: 1, 3.

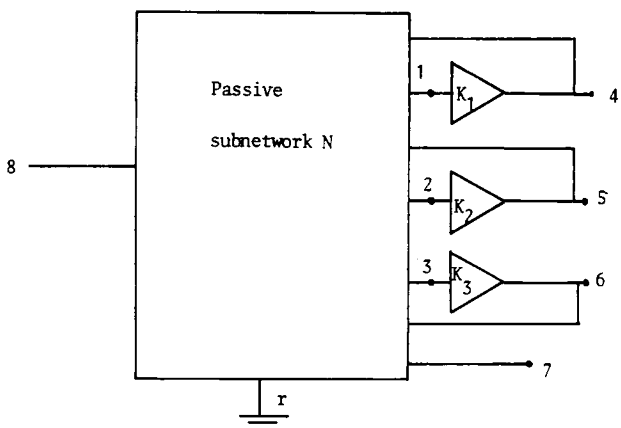

Fig. 3. Input ports for N: 4, 5, 6, 8. Output ports: 1, 2, 3, and 7.

The second factor of (5) has the same form as the left-hand side of (4). This yields the following.

Theorem 2. Let $\{\hat{i}_1, \cdots, \hat{i}_\sigma\}$ be a subset of S, disjoint of E, for the unbalance port of Fig. 1. Let $\hat{i}'_1\hat{i}'_2 \cdots \hat{i}'_\sigma$ be a permutation of the ordered sequence $\hat{i}_1\hat{i}_2 \cdots \hat{i}_\sigma$ and $\epsilon(i')$ equal to 1 for even permutations, and -1 for odd ones. Let $\{\hat{j}_1, \hat{j}_2, \cdots, \hat{j}_\sigma\}$ be a subset of E and $\{\hat{j}_{\sigma+1}, \cdots, \hat{j}_m\}$ its complement. The determinant (2) is given by

$$\mathrm{T}^{[\hat{i}_1\hat{i}_2\cdots\hat{i}_\sigma]}_{\hat{j}_1\hat{j}_2\cdots\hat{j}_\sigma} = \frac{\sum_{i'} \epsilon(i')\, \mathrm{W}_{r,\hat{j}_1\hat{i}'_1,\hat{j}_2\hat{i}'_2,\cdots,\hat{j}_\sigma\hat{i}'_\sigma,\hat{j}_{\sigma+1},\hat{j}_{\sigma+2},\hat{j}_m}}{\mathrm{W}_{r,j_1,j_2,\cdots,j_m}} \tag{6}$$

the sum being taken over all the permutations of $\hat{i}_1\hat{i}_2 \cdots \hat{i}_\sigma$.

Example: For the unbalanced port of Fig. 2, with input ports 2, 4 and 5 and output ports 1 and 3, one has $\mathrm{T}^{13}_{24} = (\mathrm{W}_{r,21,34,5} - \mathrm{W}_{r,23,14,5})/\mathrm{W}_{r,2,4,5} = -y_1y_2y_9/\{(y_A + y_B + y_8)(y_1y_3 + y_1y_7 + y_3y_7) + y_By_A(y_1 + y_7 + y_8) + y_3y_8(y_B + y_A) + y_7y_8y_A\}$ with $y_A = y_2 + y_4 + y_5 + y_6$ $y_B = y_9 + y_{10}$. This determinant is evaluated without the need to know the transfer functions involved.

The formula (6) is in general of the minimum effort type, except for the case when the short-circuiting of the input ports not involved in the determinant yield separable networks, in which case a common factor arise if the input ports involved are all in one of the parts. This common factor is the sum of the trees-admittance products of the other part, and hence is easily recognized.

An example of application of the preceding determinants arises in the calculation of transfer functions for active circuits with voltage amplifiers [4]. The case of Fig. 3 is limited to three amplifiers for the sake of brevity, here we have

$$\frac{V_7}{V_8} = \frac{t^7_8 - \sum K_i \mathrm{T}^{[7\ \ i]}_{8\ (3+i)} + \sum_{i<j} K_iK_j \mathrm{T}^{[7\ \ i\ \ j]}_{8\ (3+i)(3+j)} - K_1K_2K_3 \mathrm{T}^{[7123]}_{8456}}{1 - \sum K_i t^{\ i}_{(3+i)} + \sum_{i<j} K_iK_j \mathrm{T}^{[\ i\ \ \ j\ ]}_{(3+i)\ (3+j)} - K_1K_2K_3 \mathrm{T}^{[123]}_{456}}$$

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An Improved Algorithm for Inverting Cauer I and II Continued Fractions

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Abstract—**This letter presents a one-shot algorithm, for inverting both Cauer I and II forms of continued fraction. The algorithm, which is amenable to digital computation, proceeds in the forward direction yielding at every stage the corresponding transfer function.**

Manuscript received August 19, 1980; revised February 4, 1981.

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0018-9219/81/0700-0830$00.75 © 1981 IEEE

I. Introduction

Consider the Cauer I and II continued fractions in the generalized form

$$g(s) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}} \tag{1}$$

where for Cauer I

$$a_i = \begin{cases} h_i s, & \text{for '}i\text{' odd} \\ h_i, & \text{for '}i\text{' even} \end{cases}$$

and for Cauer II

$$a_i = \begin{cases} h_i, & \text{for '}i\text{' odd} \\ h_i/s, & \text{for '}i\text{' even.} \end{cases}$$

The inversion of (1) to a rational transfer function is a problem that has attracted the attention of several research workers. A number of algorithms based on Routh-type array is available in the literature [2].

Alok Kumar and Vimal Singh [3] presented a single algorithm where they unified the method for inverting both Cauer I and Cauer II forms. This was extended in [5] to handle the general case where the continued fraction is terminated in a rational function. Recently, Rathore *et al.* [4] also proposed a single approach for the inversion problem.

The object of this letter is to develop an alternative procedure to [3], exploiting the technique of Chen and Chang [1]. The algorithm we present here begins with the first quotient h_1 and progresses in the forward direction, as successive quotients are added. At every stage the corresponding transfer function can be directly written from the respective rows of the inversion table. This is a built-in feature of the algorithm, which is the basis for its flexibility and power to generate a number of functional approximations of different orders.

II. Proof of the Algorithm

Let us define

$$g_{i,j}(s) = \frac{q_{i,j}(s)}{p_{i,j}(s)} = \cfrac{1}{a_i + \cfrac{1}{a_{i+1} + \cfrac{1}{\ddots + \cfrac{1}{a_j}}}}. \tag{2}$$

Then, obviously

$$g_{i,j}(s) = \frac{1}{a_i + g_{i+1,j}(s)}. \tag{3}$$

From (3),

$$\frac{q_{i,j}(s)}{p_{i,j}(s)} = \frac{p_{i+1,j}(s)}{a_i p_{i+1,j}(s) + q_{i+1,j}(s)}. \tag{4}$$

This can be written as the matrix product, following the method in [1]:

$$\begin{bmatrix} p_{i,j}(s) \\ q_{i,j}(s) \end{bmatrix} = [M_i] \begin{bmatrix} p_{i+1,j}(s) \\ q_{i+1,j}(s) \end{bmatrix} \tag{5}$$

where

$$[M_i] = \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

From (5) we obtain

$$\begin{aligned} \begin{bmatrix} p_{1,m}(s) \\ q_{1,m}(s) \end{bmatrix} &= [M_1] \begin{bmatrix} p_{2,m}(s) \\ q_{2,m}(s) \end{bmatrix} \\ &= [M_1]\,[M_2] \cdots [M_m] \begin{bmatrix} p_{m+1,m}(s) \\ q_{m+1,m}(s) \end{bmatrix} \\ &= [M^{(m)}] \begin{bmatrix} p_{m+1,m}(s) \\ q_{m+1,m}(s) \end{bmatrix} \end{aligned} \tag{7}$$

where

$$[M^{(m)}] = [M_1]\,[M_2] \cdots [M_m] = \begin{bmatrix} a^{(m)} & b^{(m)} \\ c^{(m)} & d^{(m)} \end{bmatrix}. \tag{8}$$

When the continued fraction is truncated after 'm' quotients, that is when,

$$p_{m+1,m}(s) = 1 \quad \text{and} \quad q_{m+1,m}(s) = 0$$

we have

$$g_{1,m}(s) = \frac{q_{1,m}(s)}{p_{1,m}(s)} = \frac{c^{(m)}(s)}{a^{(m)}(s)}. \tag{9}$$

Thus $g_{1,m}(s)$ can be evaluated from $a^{(m)}$ and $c^{(m)}$ for $m = 1, 2, \cdots$

III. The Algorithm

In this section, the inversion algorithm is developed. From (8) it follows that

$$[M^{(m)}] = [M^{(m-1)}] \begin{bmatrix} a_m & 1 \\ 1 & 0 \end{bmatrix} \tag{10}$$

which leads to the following relations:

$$\begin{aligned} a^{(m)} &= a^{(m-1)} a_m + a^{(m-2)} \\ c^{(m)} &= c^{(m-1)} a_m + c^{(m-2)}. \end{aligned} \tag{11}$$

(i) Cauer I

The polynomials in 's' $a^{(m)}$ and $c^{(m)}$ when evaluated recursively for $m = 1, 2, \cdots$ can be arranged in the form of an M-table (Table I).

We start with the entries of $[M^{(1)}]$ to form the first two rows as $a_{01} = 1$, $a_{11} = h_1$, $a_{12} = 0$, $c_{01} = 0$ and $c_{11} = 1$. The remaining rows are evaluated using the following relations for

$$i = 2, 3, \cdots$$

$$\begin{array}{ll} a_{i,0} = 0 & c_{i,0} = 0 \\ a_{i,(i+3)/2} = 0, \quad \text{for '}i\text{' odd} & c_{i,(i+1)/2} = 1, \quad \text{for '}i\text{' odd} \\ a_{i,j} = a_{i-1,j} h_i + a_{i-2,j-1} & c_{i,j} = c_{i-1,j} h_i + c_{i-2,j-1} \\ j = 2, 3, \cdots, \frac{i}{2} + 1, & j = 2, 3, \cdots, \frac{i}{2}, \\ \text{for '}i\text{' even} & \text{for '}i\text{' even} \\ = 2, 3, \cdots \frac{i+1}{2}, & = 2, 3, \cdots, \frac{i-1}{2}, \\ \text{for '}i\text{' odd} & \text{for '}i\text{' odd.} \end{array} \tag{12}$$

Once the table is formed the transfer function corresponding to the continued fraction with 'm' quotients, $m = 1, 2, \cdots$ can be directly written from the entries in the $(m+1)$th row as

$$g_{1,m}(s) = \frac{\sum_{j=1}^{n} c_{m,j} s^{n-j}}{\sum_{j=1}^{n+1} a_{m,j} s^{n-j+1}} \tag{13}$$

TABLE I
M-TABLE

'a' Rows				'c' Rows		
1				0		
a_{11}	0			1		
a_{21}	1			c_{21}		
a_{31}	a_{32}	0		c_{31}	1	
a_{41}	a_{42}	1		c_{41}	c_{42}	
a_{51}	a_{52}	a_{53}	0	c_{51}	c_{52}	1
a_{61}	a_{62}	a_{63}	1	c_{61}	c_{62}	c_{63}
	⋮				⋮	

where

$$n = \begin{cases} m/2, & \text{for '}m\text{' even} \\ (m+1)/2, & \text{for '}m\text{' odd.} \end{cases}$$

It will be noted that

$$a_{n,n+1} = \begin{cases} 0, & \text{for '}m\text{' odd} \\ 1, & \text{for '}m\text{' even} \end{cases}$$

and

$$c_{n,n} = 1, \quad \text{for '}m\text{' odd.}$$

(ii) Cauer II

The transfer function for Cauer II can be obtained from Cauer I through the relation

$$g_2(s) = s g_1(1/s) \tag{14}$$

Thus the *M*-Table is equally applicable to Cauer II form and the transfer function can be written from (13) as

$$g_{1,m}(s) = \frac{\sum_{j=1}^{n} c_{m,j} s^{j-1}}{\sum_{j=1}^{n+1} a_{m,j} s^{j-1}}. \tag{15}$$

IV. CONTINUED FRACTION WITH TERMINATION $g_0(s)$

In this section, we extend the algorithm to the general case, namely, when the continued fraction is terminated in a rational function $g_0(s)$.

Here $g_{1,m}(s)$ represents a general transfer function of the form given by (2) with quotients a_i to a_m with a termination

$$g_0(s) = \frac{q_0(s)}{p_0(s)}.$$

Now to evaluate $g_{1,m}(s)$, $p_{1,m}(s)$ and $q_{1,m}(s)$ can be derived from (7) by substituting $p_{m+1,m}(s) = p_0(s)$ and $q_{m+1,m}(s) = q_0(s)$.

Thus we have

$$\begin{bmatrix} p_{1,m}(s) \\ q_{1,m}(s) \end{bmatrix} = \begin{bmatrix} a^{(m)} & b^{(m)} \\ c^{(m)} & d^{(m)} \end{bmatrix} \begin{bmatrix} p_0(s) \\ q_0(s) \end{bmatrix}$$

Then

$$g_{1,m}(s) = \frac{q_{1,m}(s)}{p_{1,m}(s)} = \frac{c^{(m)} p_0(s) + d^{(m)} q_0(s)}{a^{(m)} p_0(s) + b^{(m)} q_0(s)} = \frac{c^{(m)} + d^{(m)} g_0(s)}{a^{(m)} + b^{(m)} g_0(s)}.$$

From (10)

$$d^{(m)} = c^{(m-1)} \quad \text{and} \quad b^{(m)} = a^{(m-1)}.$$

Finally we have

$$g_{1,m}(s) = \frac{c^{(m)} + c^{(m-1)} g_0(s)}{a^{(m)} + a^{(m-1)} g_0(s)}. \tag{16}$$

V. CONCLUSION

An algorithm which is superior to [3] computationally is presented for inverting both Cauer I and Cauer II continued fractions. A proof, in the light of the matrix method of Chen and Chang [1], is given and the structure of the inversion table has been derived in Table I. It will be further observed that the *M*-table once constructed will be applicable for any termination $g_0(s)$, whereas the procedure in [5] requires constructing the table individually for each termination.

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Determination of Quantization Error in Two-Dimensional Digital Filters

L. M. ROYTMAN AND M.N.S. SWAMY

Abstract—**The evaluation of the quantization error in two-dimensional (2-D) digital filters involves the following computation $J = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y^2(m,n)$. In this paper general method for the evaluation of J based on Laurent expansion of the integrand is presented. Illustrative example for such a computation is given.**

Notations: **We denote with $\overline{V}^2 = \{(z_1, z_2): |z_1| \leq 1, |z_2| \leq 1\}$ the closed unit bidisc, with $V^2 = \{(z_1, z_2): |z_1| < 1, |z_2| < 1\}$ the open unit bidisc, and with $T_2 = \{(z_1, z_2): |z_1| = 1, |z_2| = 1\}$ the distinguished boundary of the unit bidisc.**

I. INTRODUCTION

In the implementation of the transfer function the quantization error is of great importance. The value of the quantization error can be related to a complex integral by use of Parseval's theorem

$$J = \sum_{m=0}^{\infty} \sum_{n-0}^{\infty} y^2(m,n) = \frac{1}{(2\pi j)^2} \oint\oint_{T^2} Y(z_1, z_2)\, Y(z_1^{-1}, z_2^{-1}) \frac{dz_1\, dz_2}{z_1 z_2} \tag{1}$$

Recently Jury *et al.* [1] had attacked the problem of the evaluation of (1) using the residue method. This approach involves the problem of the determination of the zeros of a polynomial in one variable whose coefficients are polynomials of the other variable and it is obvious that this problem does not have an explicit solution in general [2]. As a result, [1] fails to give general formulas for the evaluation of (1).

Manuscript received November 13, 1980.
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0018-9219/81/0700-0832$00.75 © 1981 IEEE